A Review/Tour of Formalism

CMPT 886
Automated Software Analysis & Security
Nick Sumner

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  - High school algebra
  - Classic formal logic
  - Euclidean geometry

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  - Enable automated techniques for finding solutions
- Choosing the *right* tool for the job can be hard

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How to compare elements of a set

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  - Formal Grammars & Automata

Use structure to constrain
the elements of a set

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How and when to infer facts

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  - These techniques are critical for *static program analysis*

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- $\leq$ is a *total order* on $\mathbb{Z}$.
  - Intuitively, $\forall$ a, b $\in \mathbb{Z}$, either a $\leq$ b or b $\leq$ a

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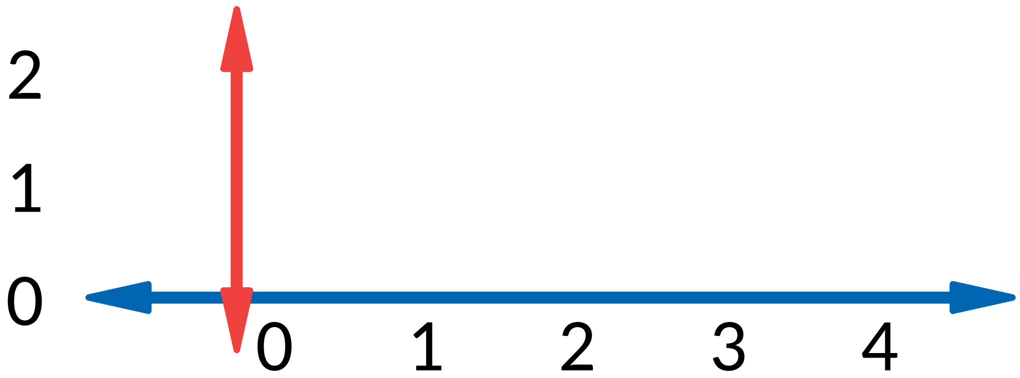

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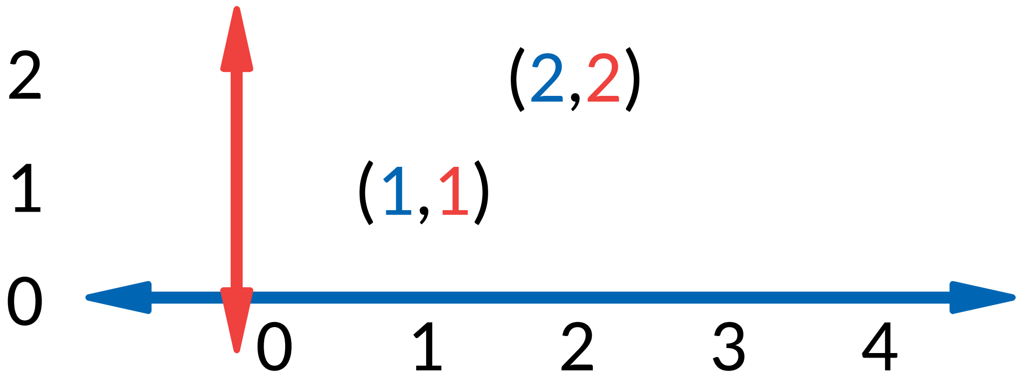

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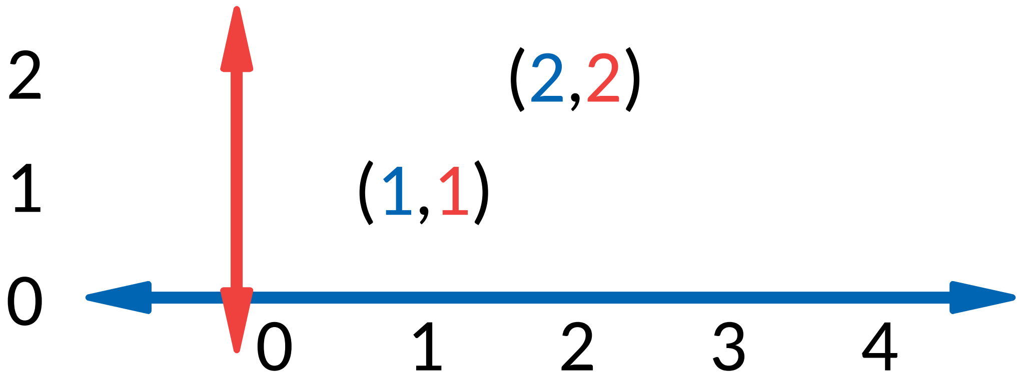

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(1,1) ≤ (2,2)?

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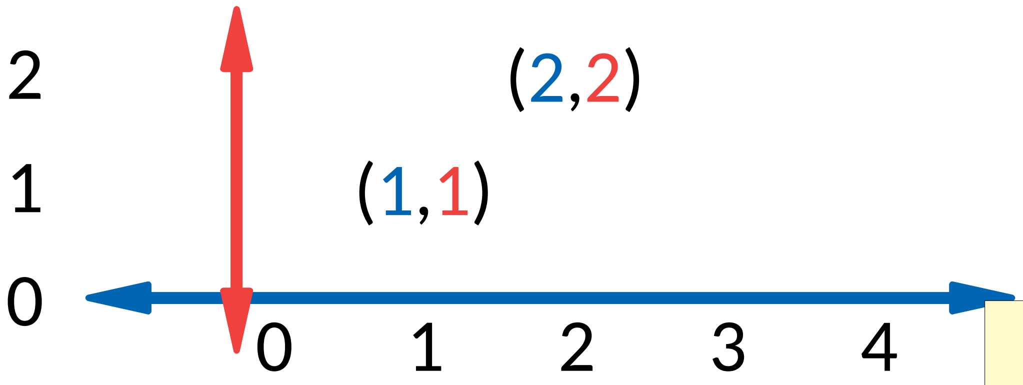

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We can take ≤ to be componentwise comparison.

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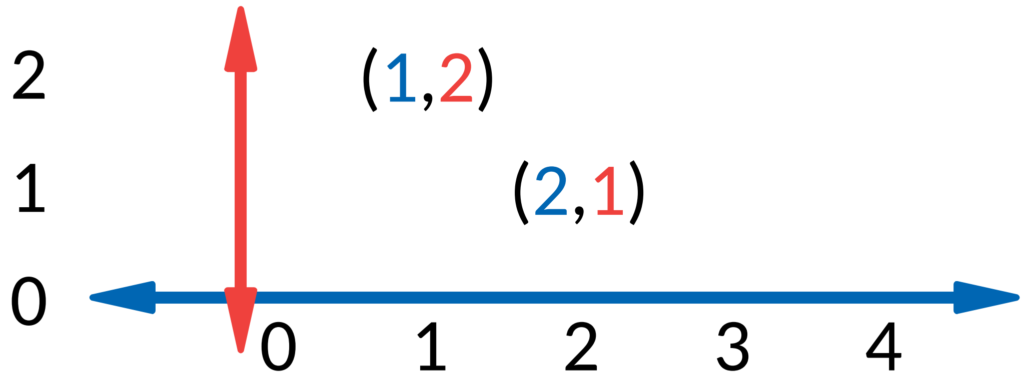

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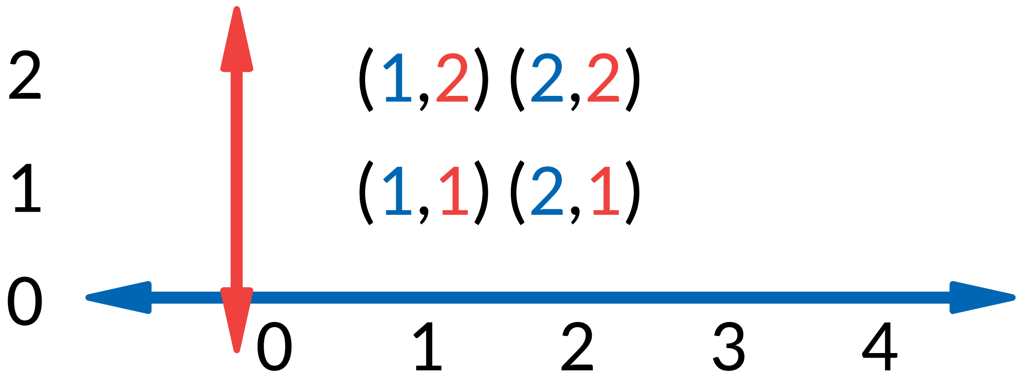

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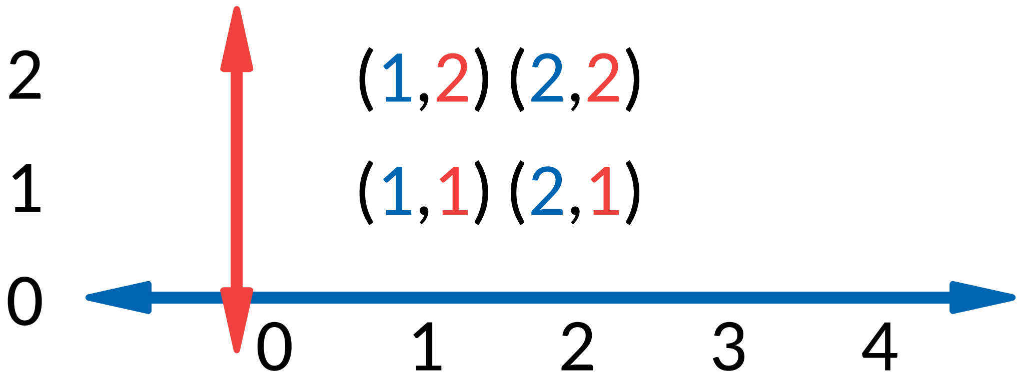

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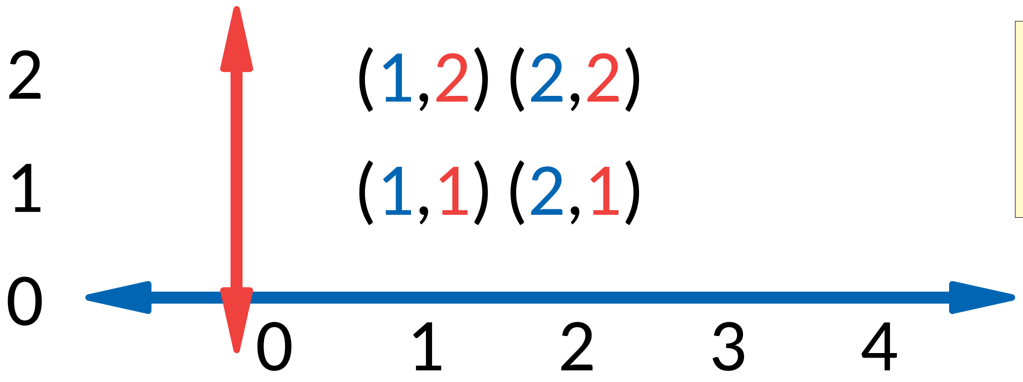

Which of these 4 elements are comparable?

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Partial Orders

- A relation $\leq$ is a *partial order* on a set S if $\forall$ a,b,c $\in$ S
  - Reflexive: $a \leq a$
  - Antisymmetric: $a \leq b \;\&\; b \leq a \Rightarrow a = b$
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How does a total order compare?

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$$ab \preceq_{str} xabyz$$
$$ab \preceq_{seq} xaybz$$
$$\{a,b\} \subseteq \{a,b,x,y,z\}$$

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$$(1,1) \sqsubseteq (1,2)$$
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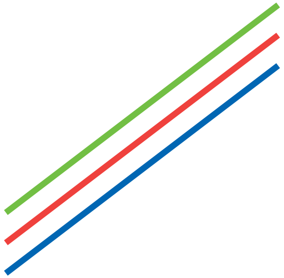
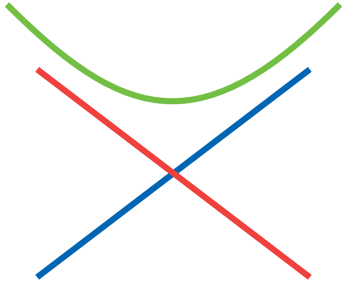


$f(x) = x + 1 \sqsubseteq g(x) = x + 2$

$h(x) = x \not\sqsubseteq i(x) = -x$



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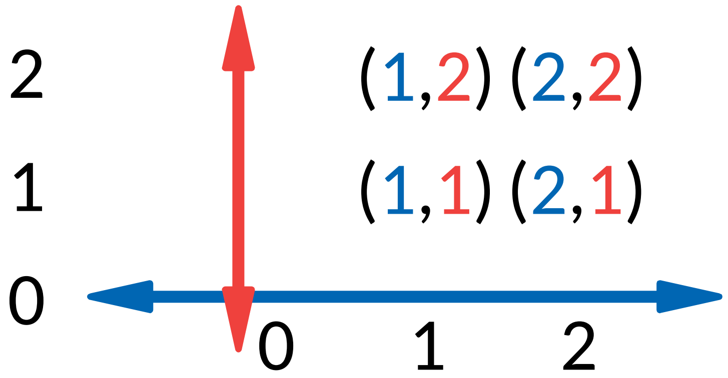

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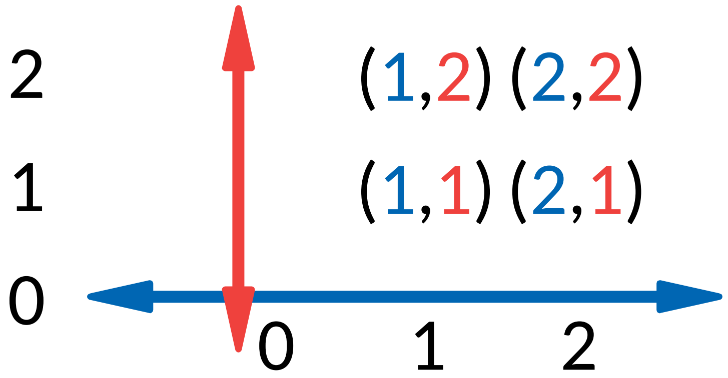

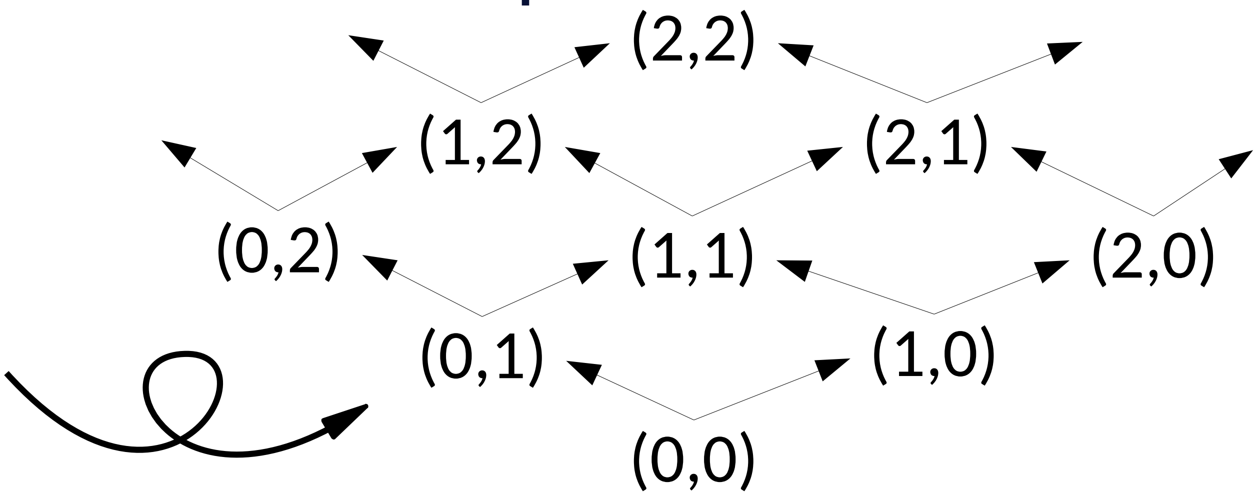

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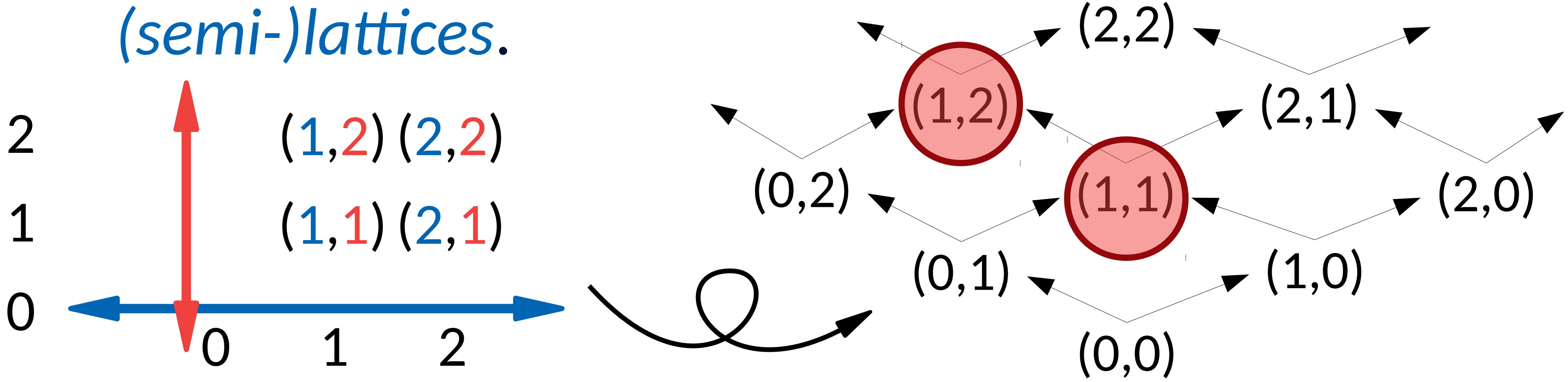

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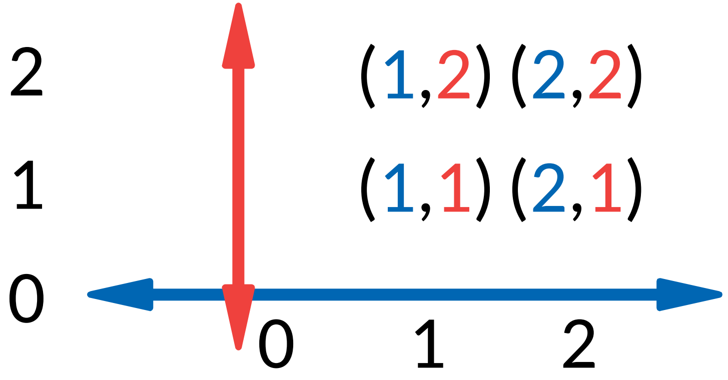

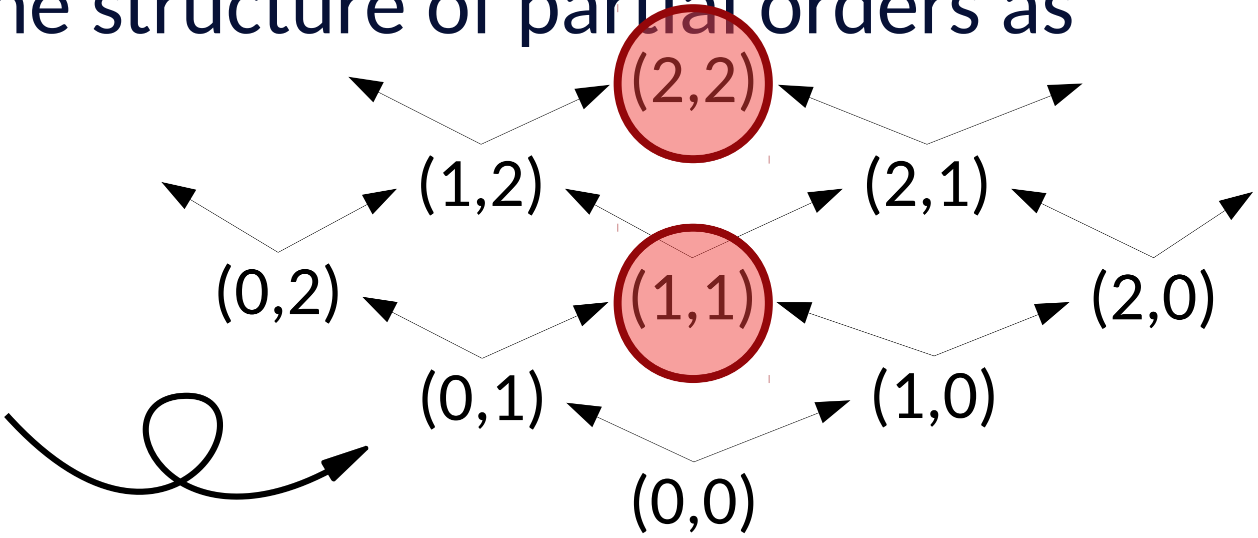

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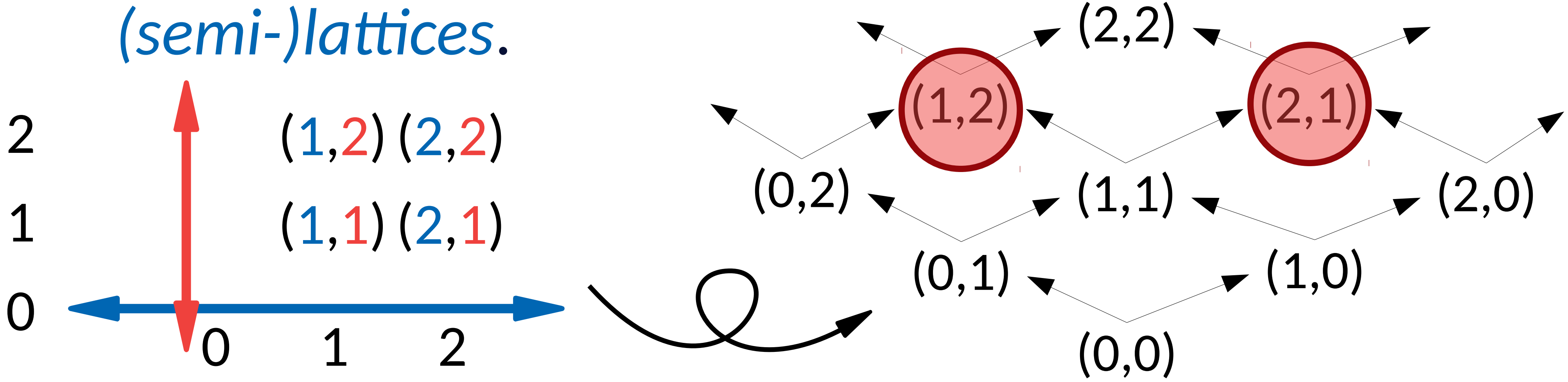

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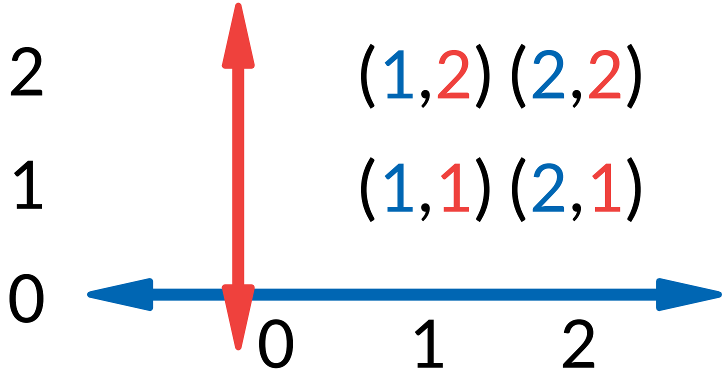

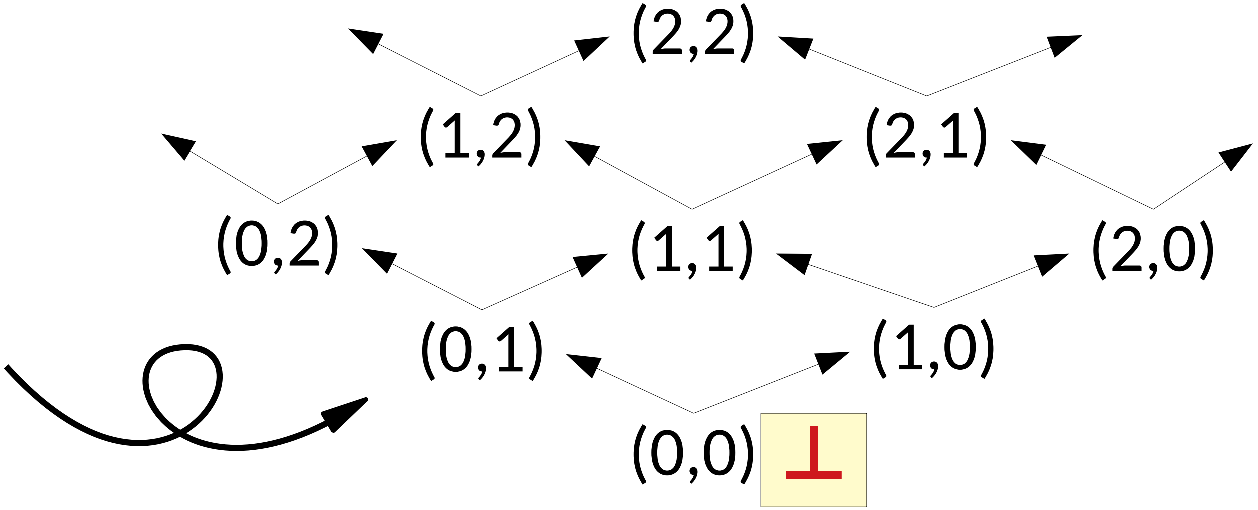

- If unique least/greatest elements exist, we call them $\perp$(bottom)/$\top$(top)

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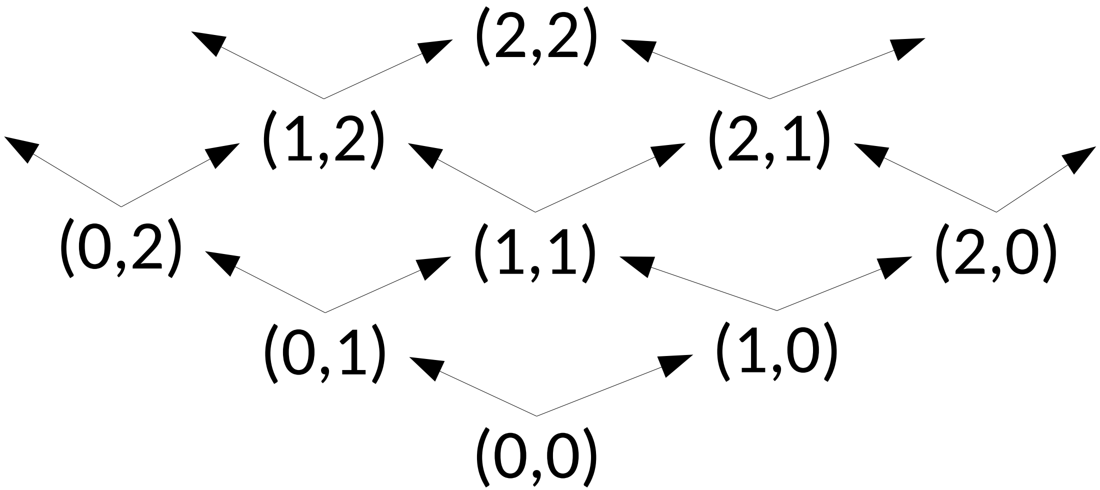

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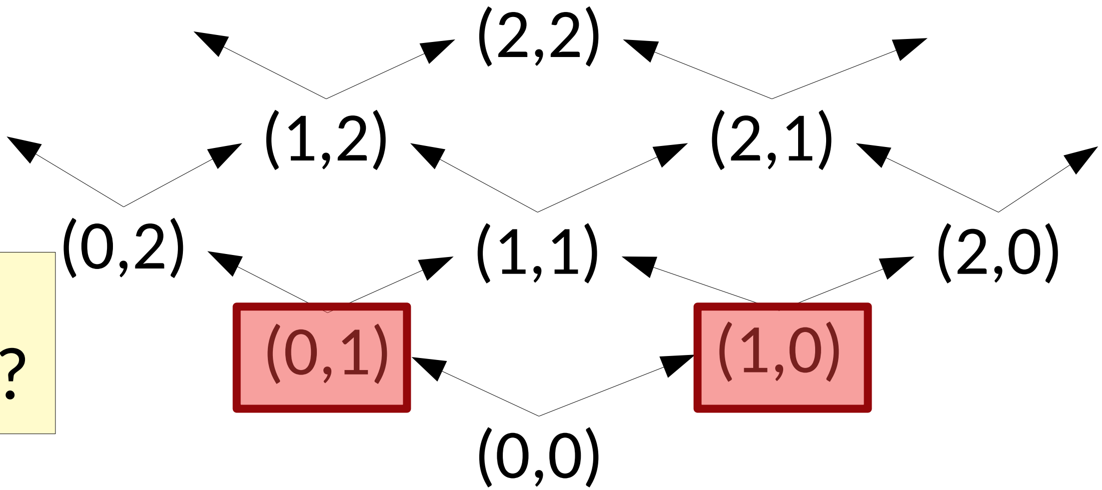

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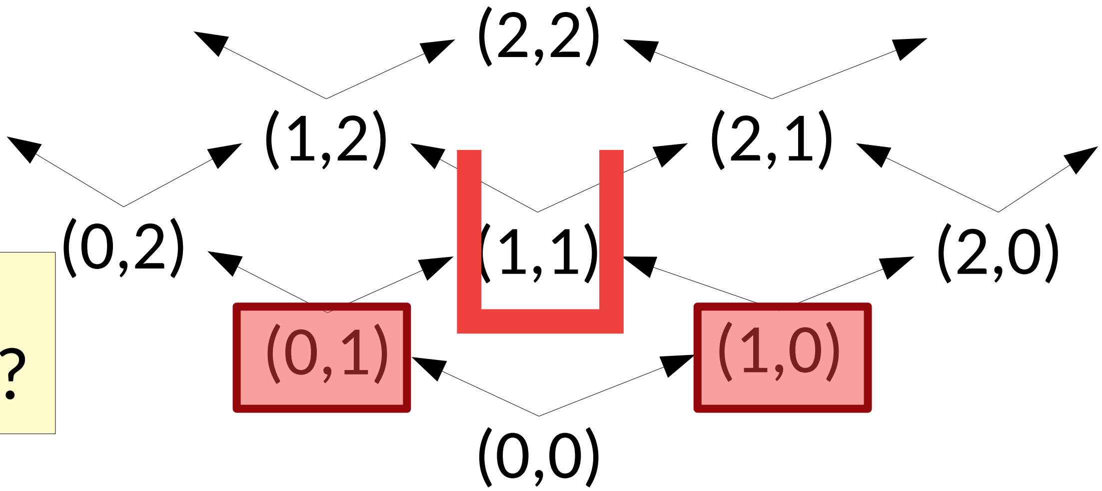

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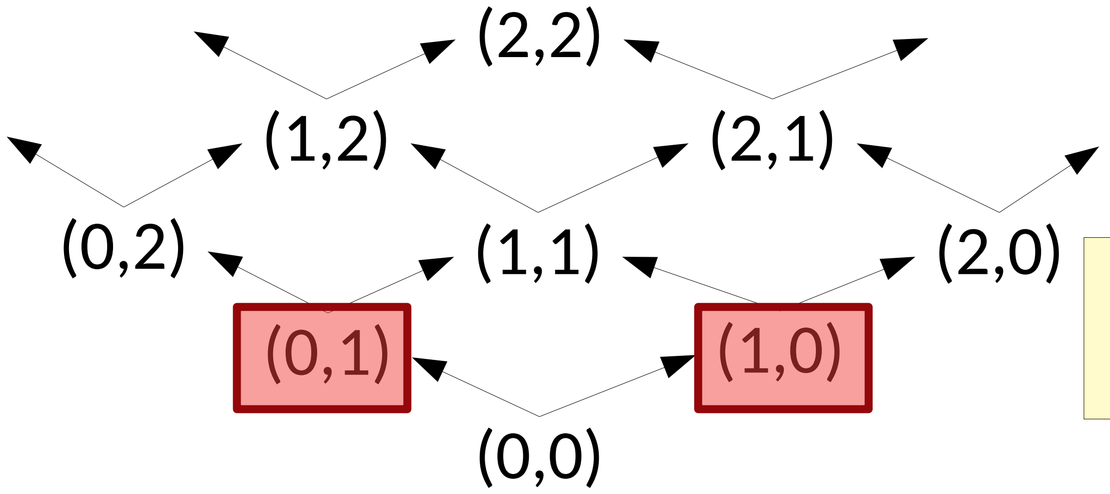

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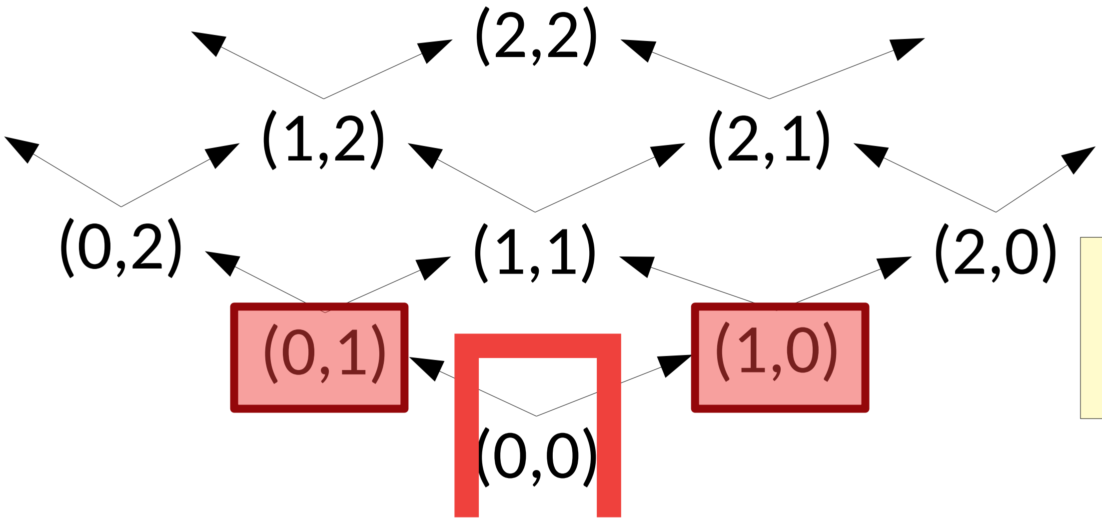

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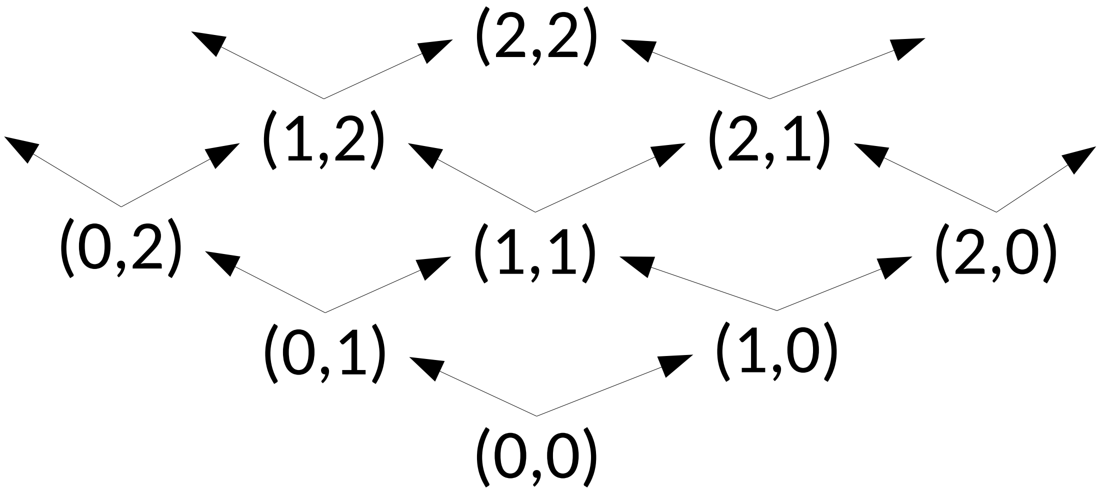

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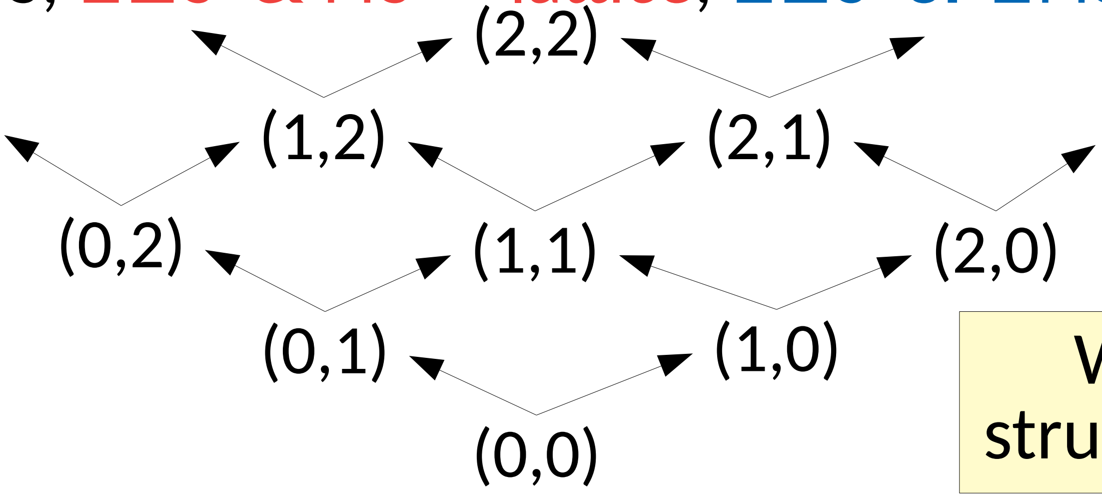
  - $\forall S' \subseteq S$, $\exists \sqcup S'$ **&** $\sqcap S' \Rightarrow$ lattice, $\exists \sqcup S'$ **or** $\exists \sqcap S' \Rightarrow$ semilattice



What is the structure shown?

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  - dataflow analysis & proving program properties

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- They commonly define *formal languages*
  - Sets of strings over a defined alphabet
- They are effective at constraining a search space

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       | `(` regex `)`
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e.g. a(bc | cd)*e defines L containing abccdbce

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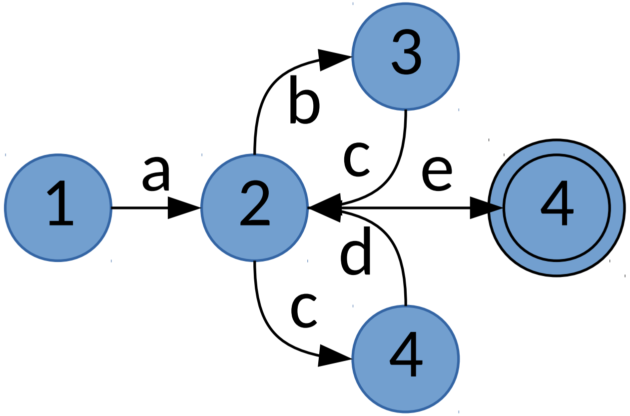

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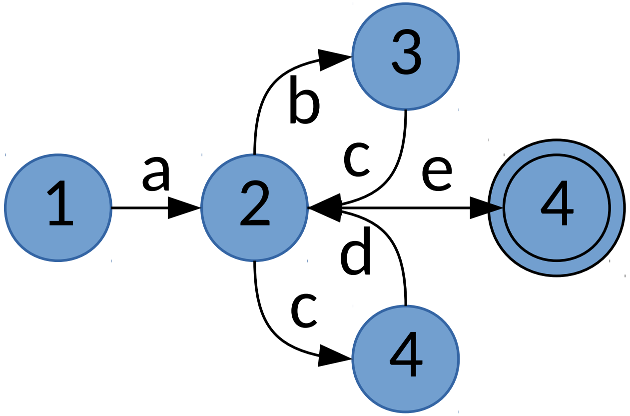

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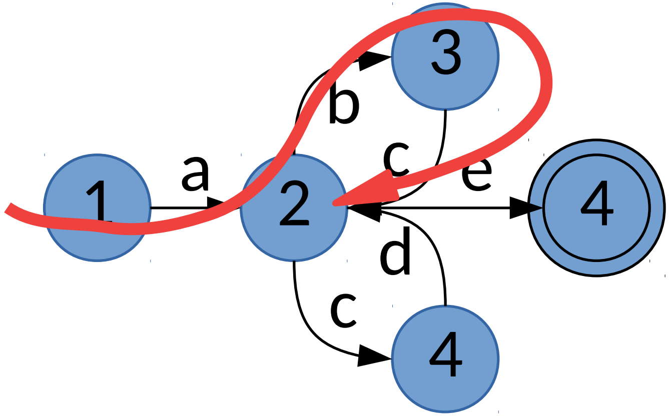

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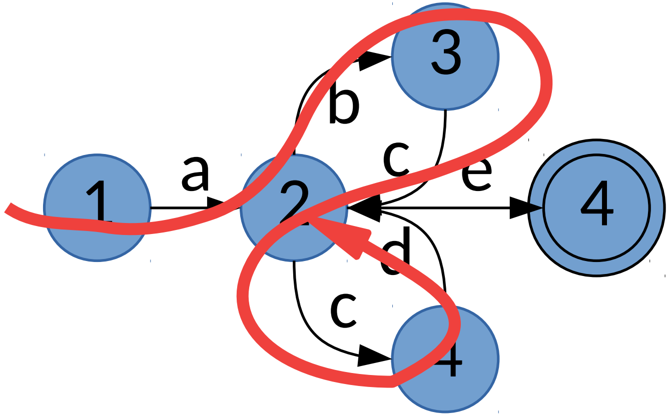

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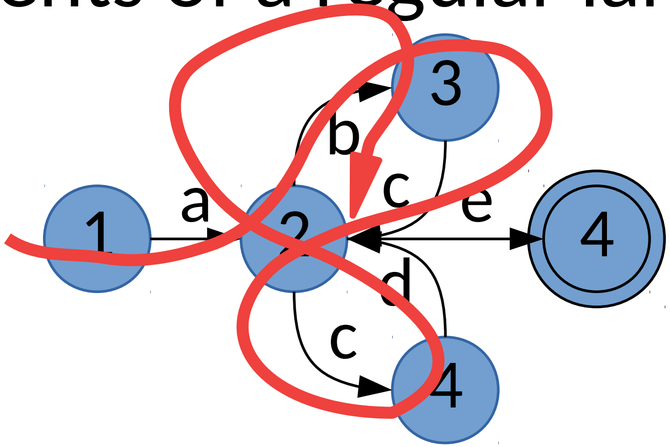

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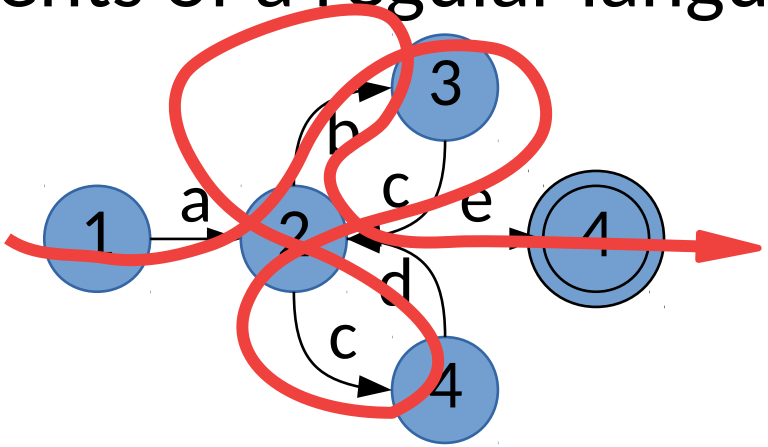

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- Recall, regular languages cannot express matched parentheses (Dyck languages)

$a^n b^n$

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Start = A
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| g

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$$
\begin{array}{lcl}
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$ce^n g f^n d$

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Start = A
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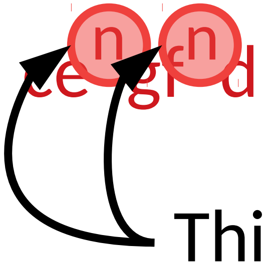
B → eBf
  | g



This requires some kind of *memory*

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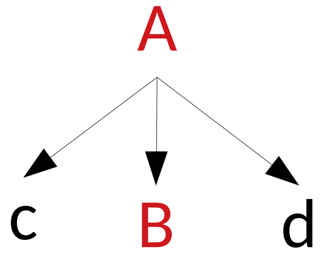

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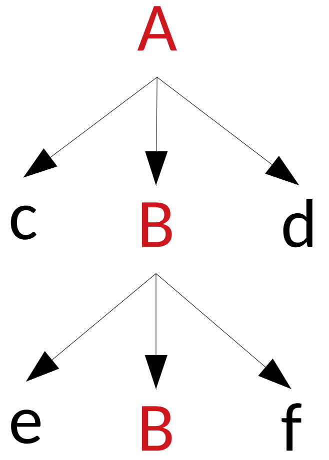

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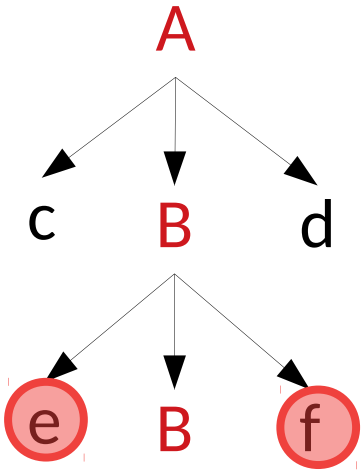

Generating symbols out of order acts as a form of memory.

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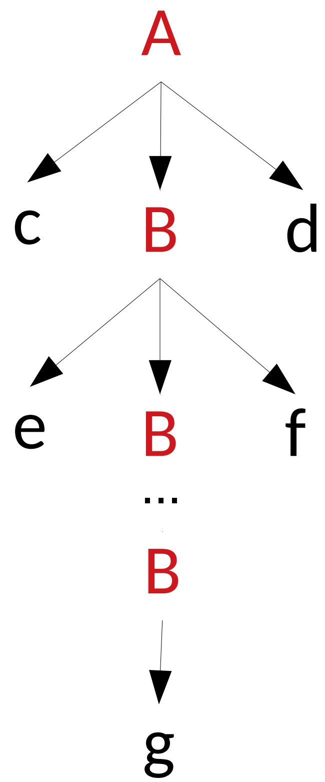

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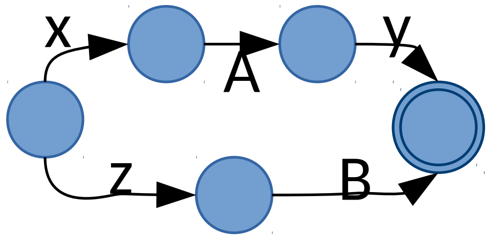

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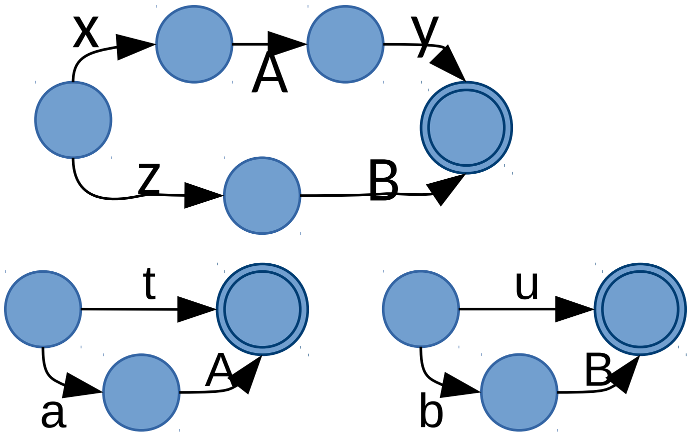

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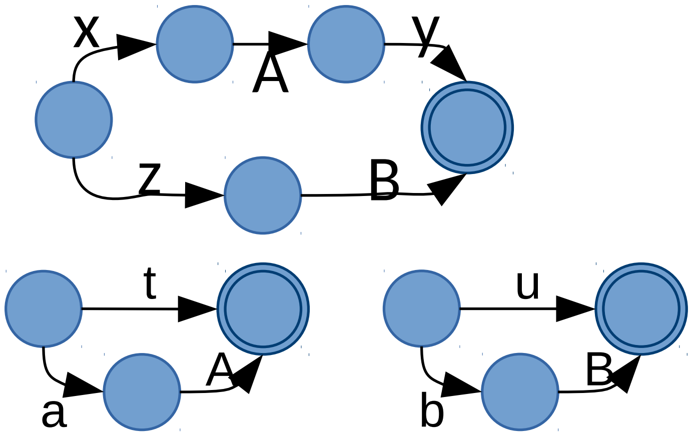

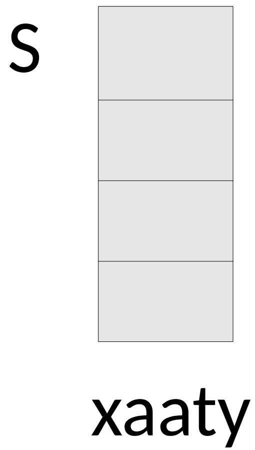

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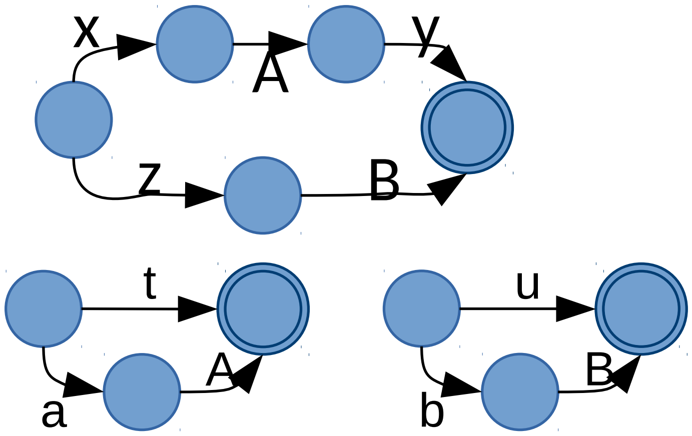

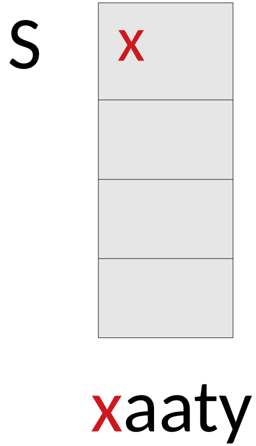

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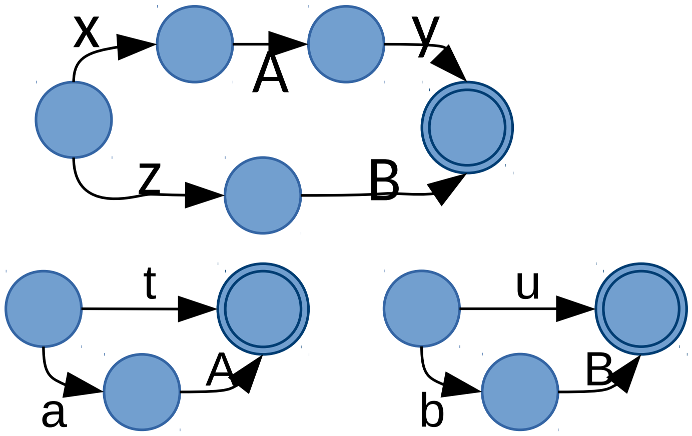

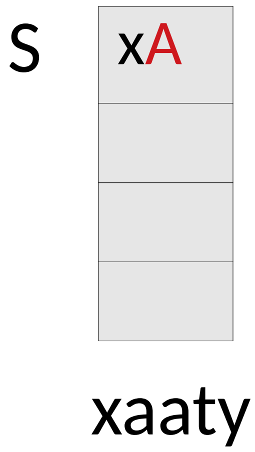

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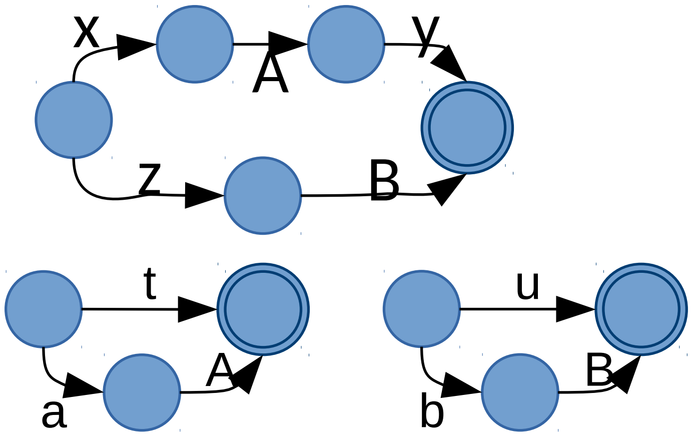

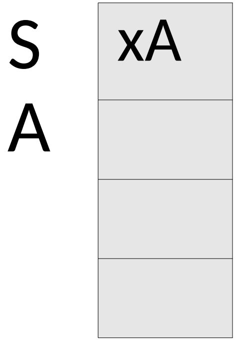

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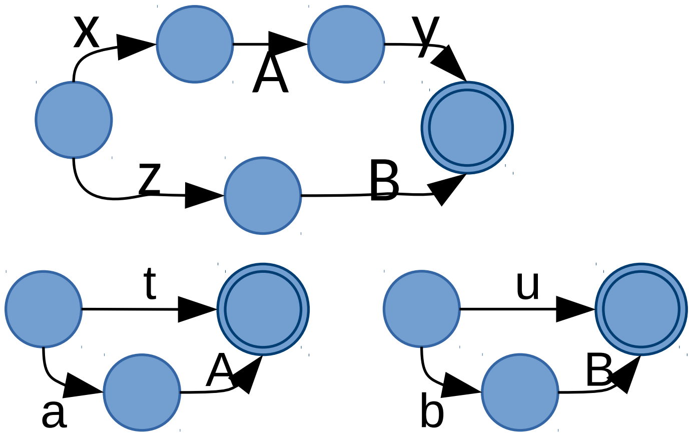

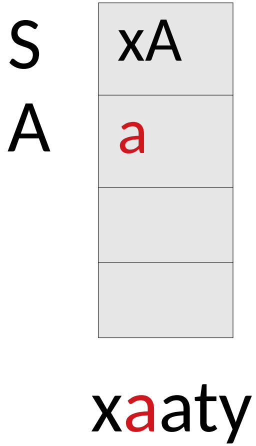

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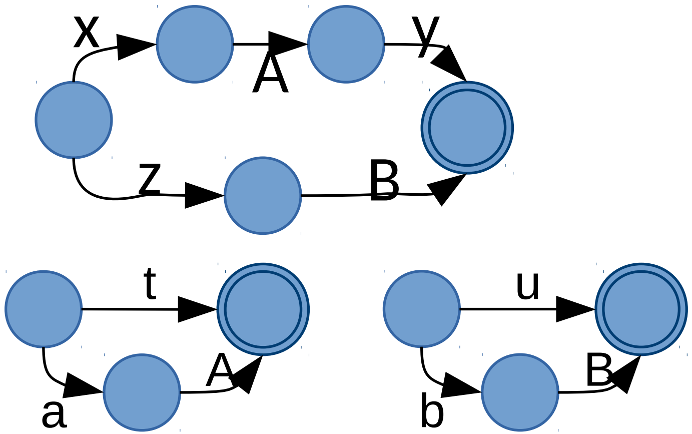

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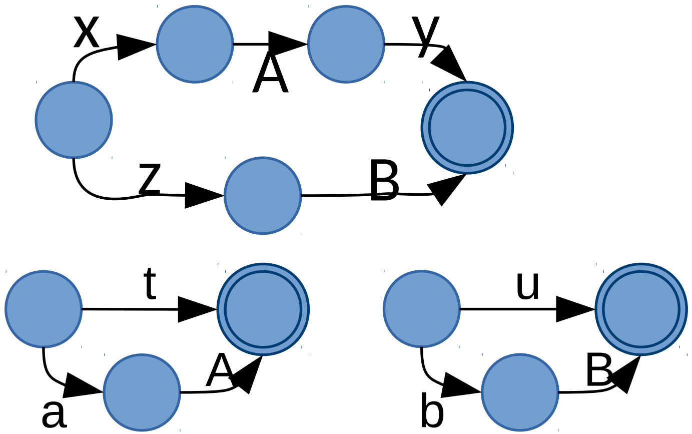

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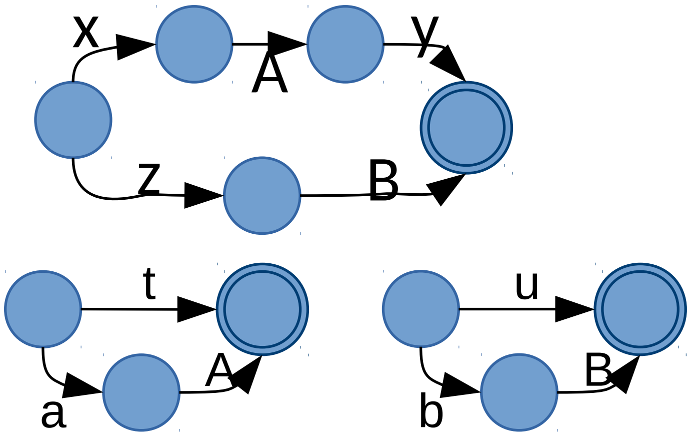

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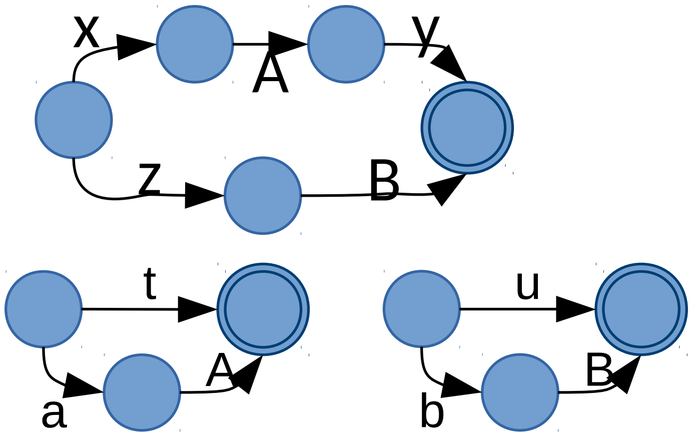

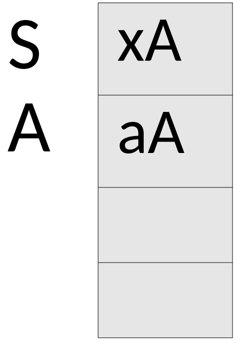

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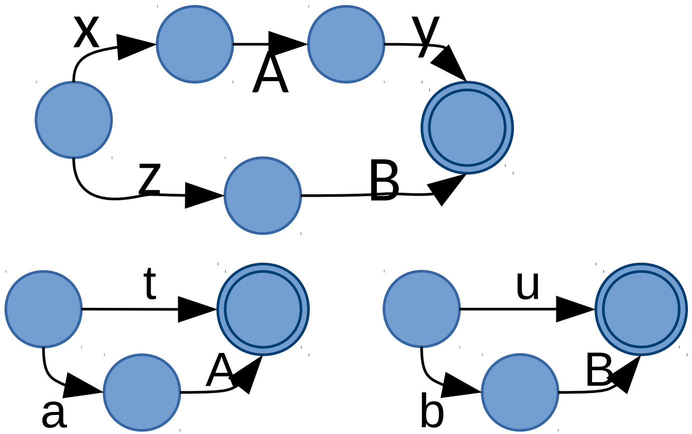

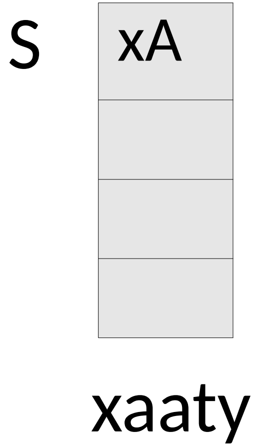

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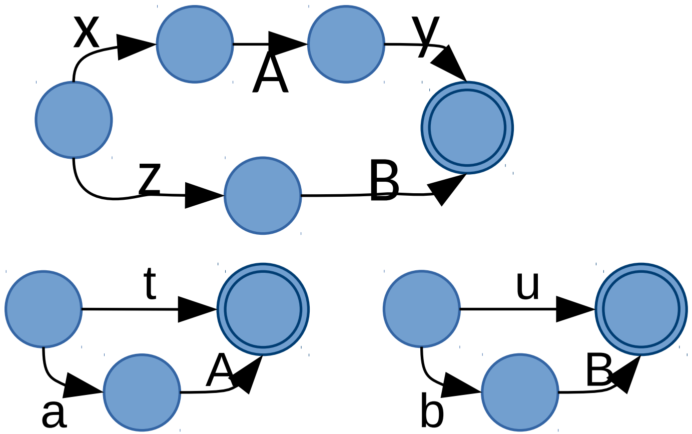

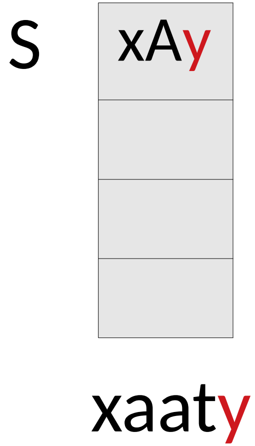

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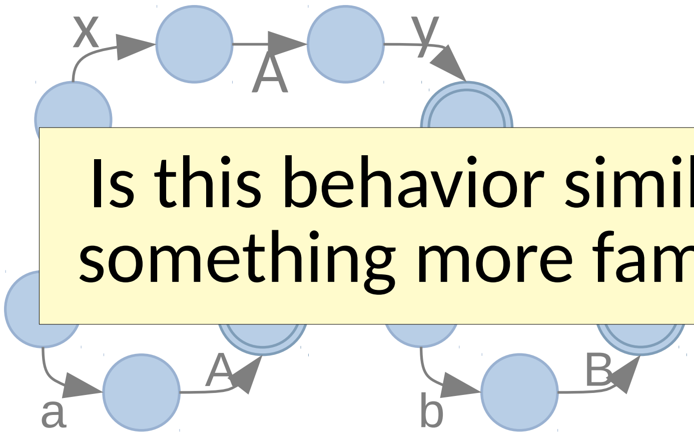
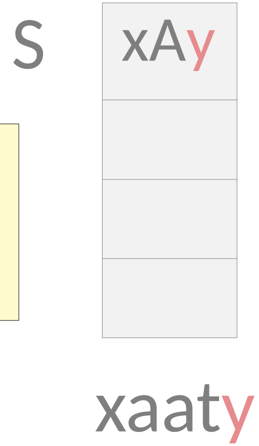


Is this behavior similar to something more familiar?



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- Program analysis has actually been one of the driving forces behind satisfiability in recent years.

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$\mathrm{I} \vdash \mathrm{x}$ and $\mathrm{I} \vdash \mathrm{y}$ iff $\mathrm{I} \vdash \mathrm{x} \wedge \mathrm{y}$

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Wadler, “A Taste of Linear Logic”. 2014.

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Law of excluded middle

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Double negation elimination

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- Note, this is commonly used in type systems

Linear & Substructural Logic

$$\begin{array}{l}\text{sellsBurritos(store)}\\ \text{has10Dollars(me)}\end{array} \vdash \text{buyBurrito(me,store)}$$

Linear & Substructural Logic

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Classical & intuitionistic logic have trouble expressing consumable facts

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  - [Intuitionistic] as before
  - <Linear> cannot be used with contraction or weakening

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$$\frac{\Gamma, A, A, \Delta \vdash p}{\Gamma, A, \Delta \vdash p} \qquad \frac{\Gamma, \Delta \vdash p}{\Gamma, A, \Delta \vdash p}$$

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Logics that remove additional rules from intuitionistic logic are *substructural*

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  - [Intuitionistic] as before
  - <Linear> cannot be used with contraction or weakening
  - In essence, linear facts must be consumed exactly once in a proof.
- This forms the backbone of *ownership types* in languages like Rust!

Separation Logic

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Suppose we used ∧ instead, what problem exists?

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- Separation logic enables efficient compositional reasoning
  - It is the backbone of Facebook's Infer engine!

Recap

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- Formalism is a tool that can simplify reasoning about tasks
- Many solutions involve a careful combination of
  - order theory (for comparison)
  - formal grammars (for structure)
  - formal logic (for inference)