CMPT 745
Software Engineering

Basic Formalisms for Software Engineering

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  - High school algebra
  - Classic formal logic
  - Euclidean geometry

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  - Classic formal logic
  - Euclidean geometry
- They serve multiple useful purposes
  - Limit the possibilities that you may consider
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  - Enable automated techniques for finding solutions
- Choosing the *right* tool for the job can be hard

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  (in CS and program analysis)

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How to compare elements of a set

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  - Order Theory
  - Formal Grammars & Automata

Use structure to constrain
the elements of a set

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  - Formal Grammars & Automata
  - Formal Logic (Classical & otherwise)

How and when to infer facts

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  - These techniques are critical for *static program analysis*

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Set: $\mathbb{Z}$ Relation: $\leq$

- $\leq$ is a *total order* on $\mathbb{Z}$.
  - Intuitively, $\forall$ a, b $\in \mathbb{Z}$, either a $\leq$ b or b $\leq$ a

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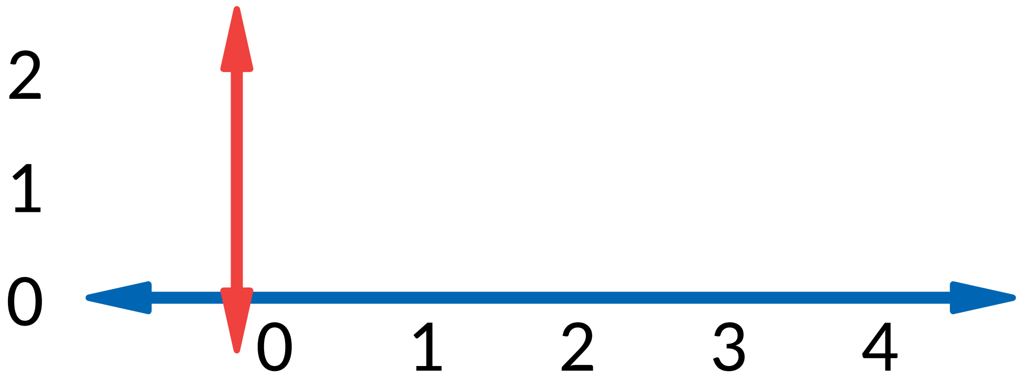

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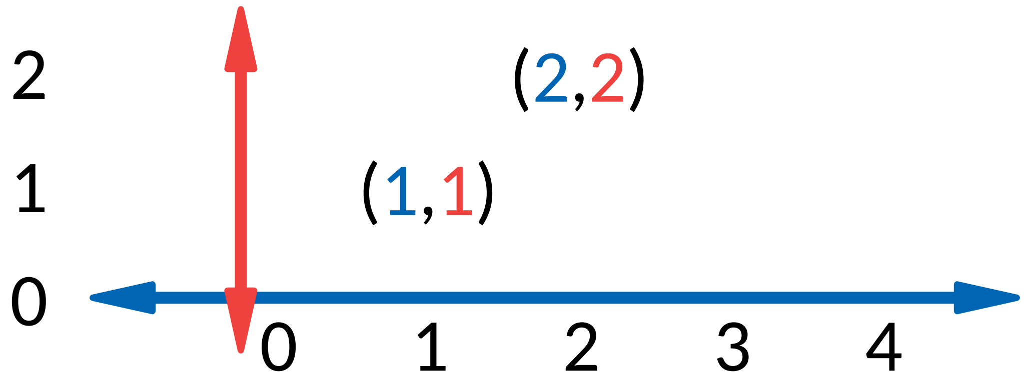

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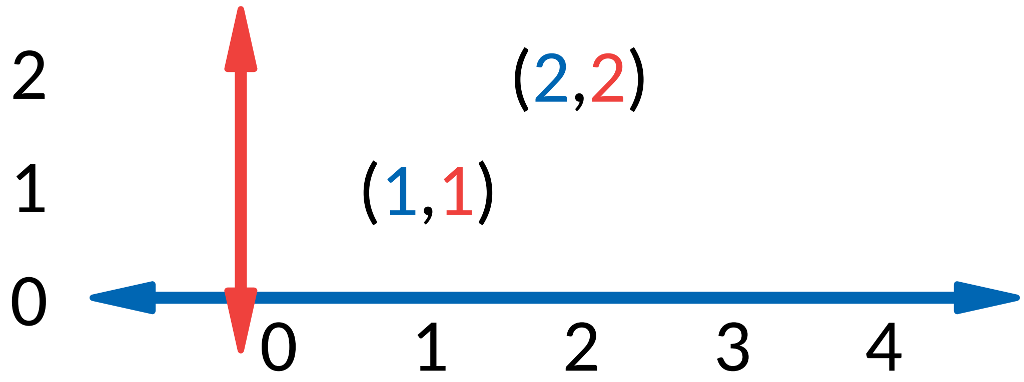

What is the result of
$(1,1) \leq (2,2)$?

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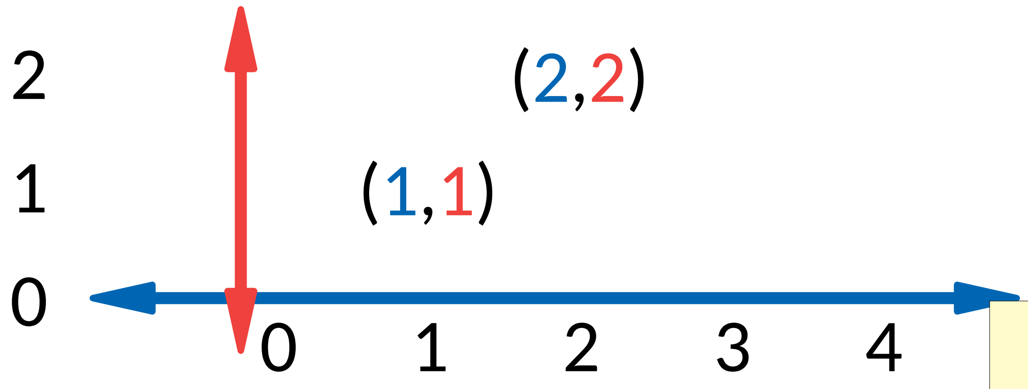

What is the result of (1,1) ≤ (2,2)?

We can take ≤ to be componentwise comparison.

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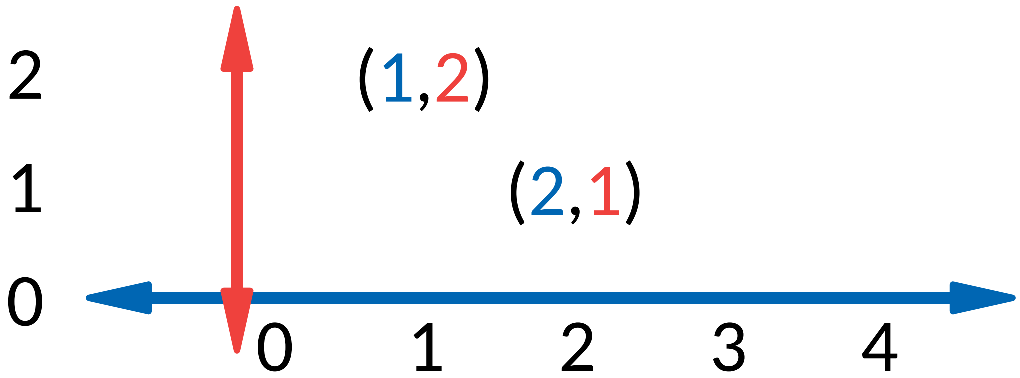

What is the result of
(1,2) ≤ (2,1)?

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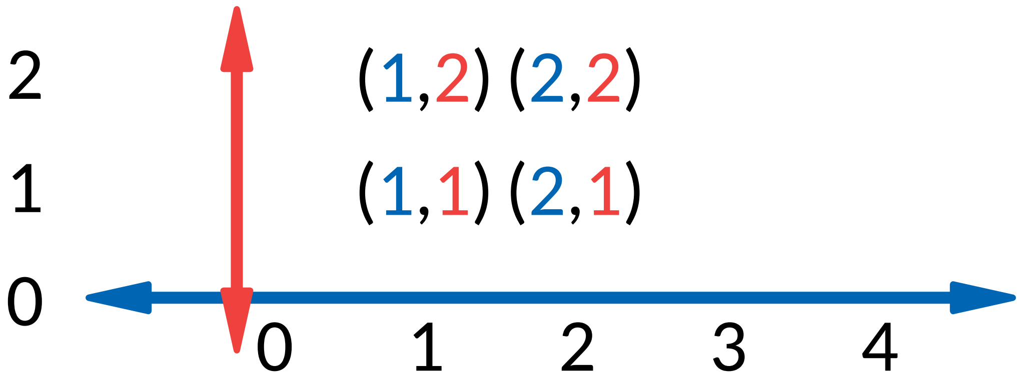

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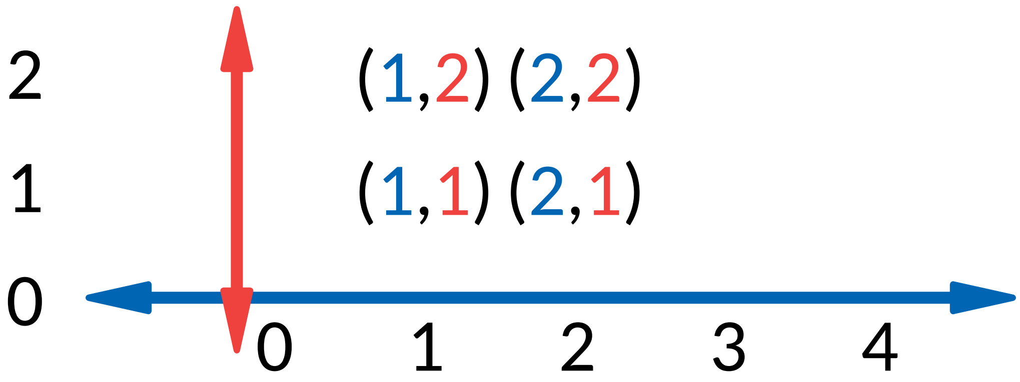

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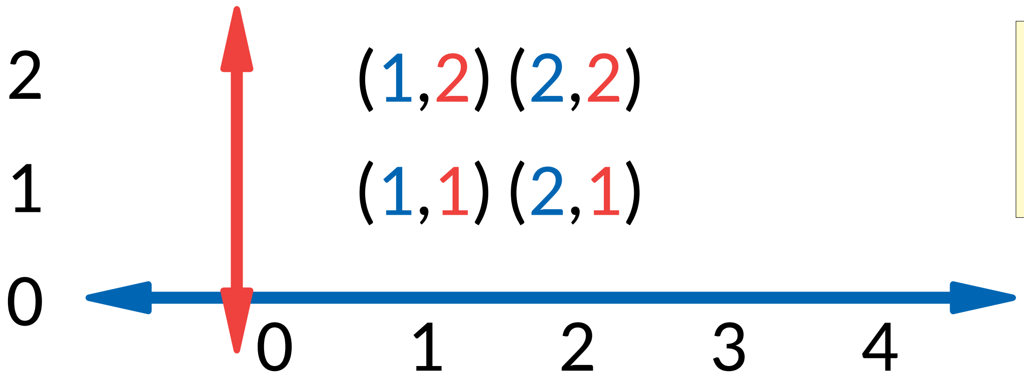

Which of these 4 elements are comparable?

- Componentwise comparison with tuples yields a *partial order*
  - Intuitively, *not all elements are comparable*

Partial Orders

- A relation $\leq$ is a *partial order* on a set S if $\forall$ a,b,c $\in$ S
  - Reflexive: $a \leq a$
  - Antisymmetric: $a \leq b$ & $b \leq a \Rightarrow a = b$
  - Transitive: $a \leq b$ & $b \leq c \Rightarrow a \leq c$

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How does a
total order compare?

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  - substring, subsequence, subset relationships

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$$ab \preceq_{str} xabyz$$

$$ab \preceq_{seq} xaybz$$

$$\{a,b\} \subseteq \{a,b,x,y,z\}$$

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$$(1,1) \sqsubseteq (1,2)$$
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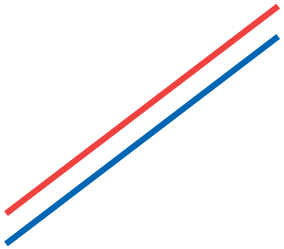

$f(x) = x + 1 \sqsubseteq g(x) = x + 2$

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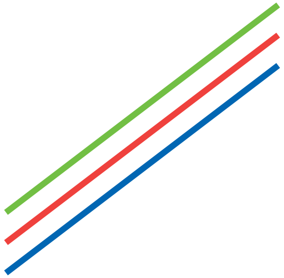

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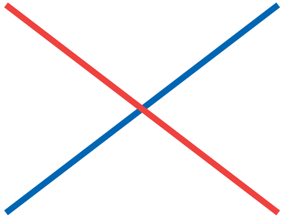

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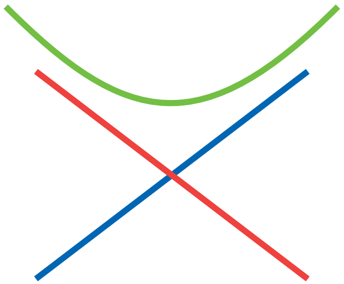

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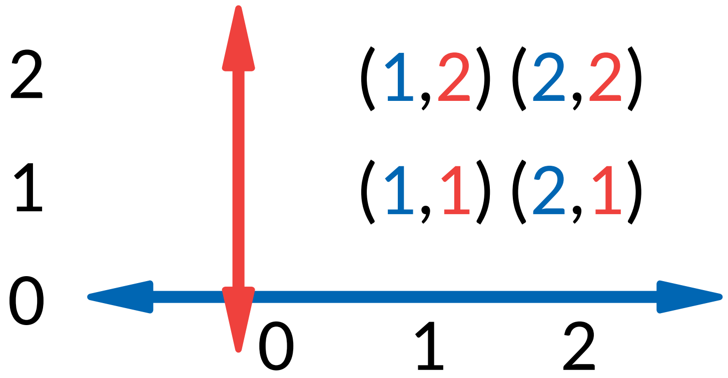
Partial Orders

- We can express the structure of partial orders using *(semi-)lattices*.



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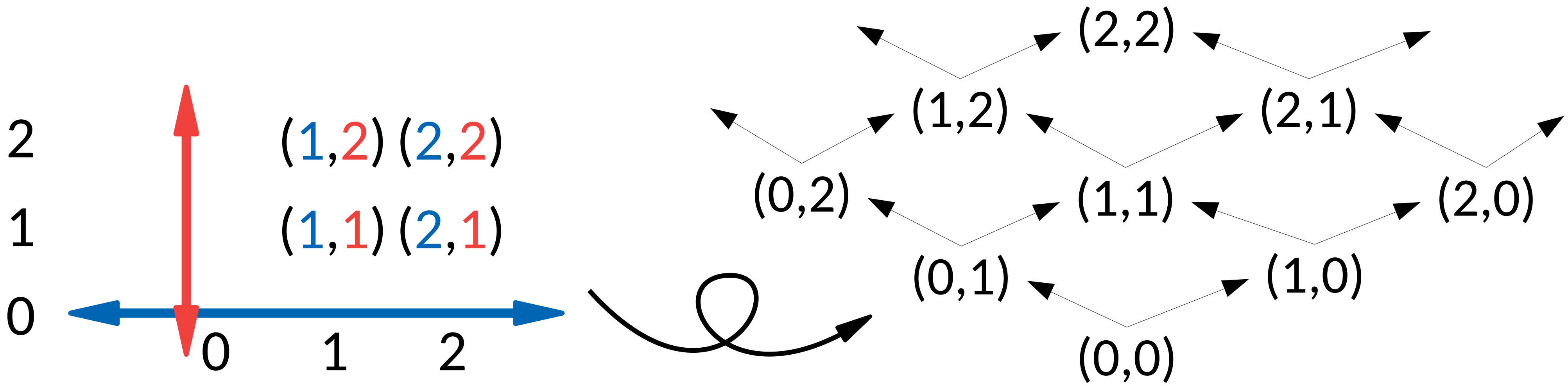

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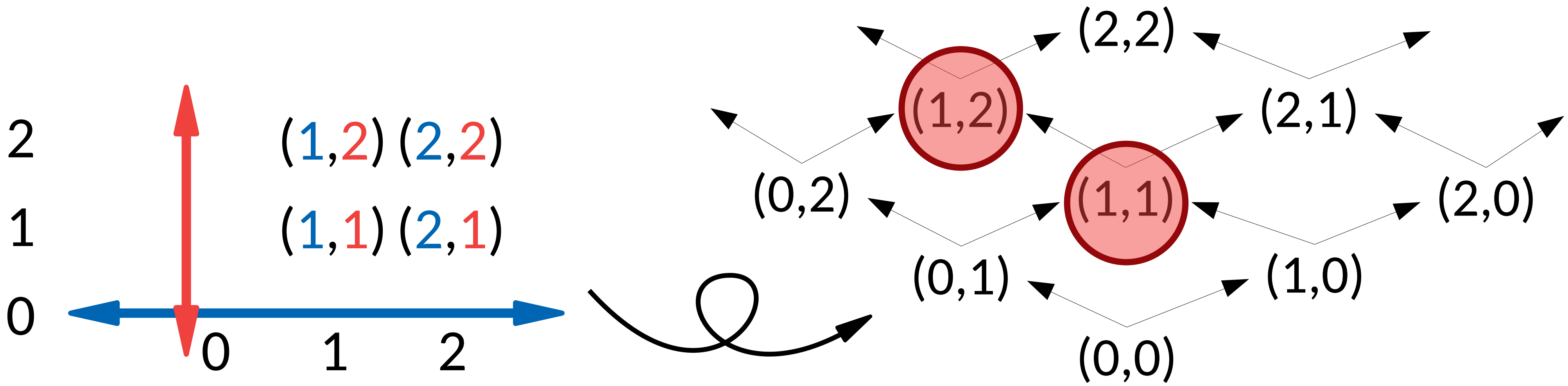

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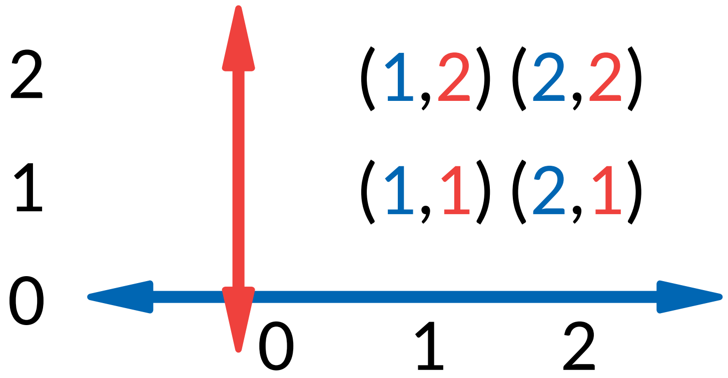

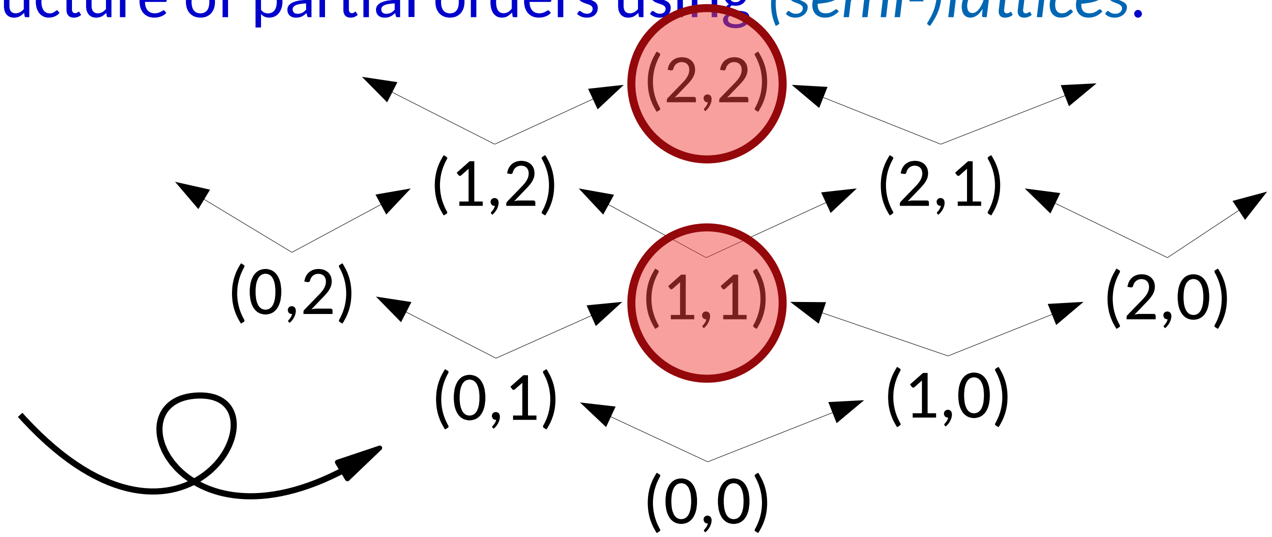

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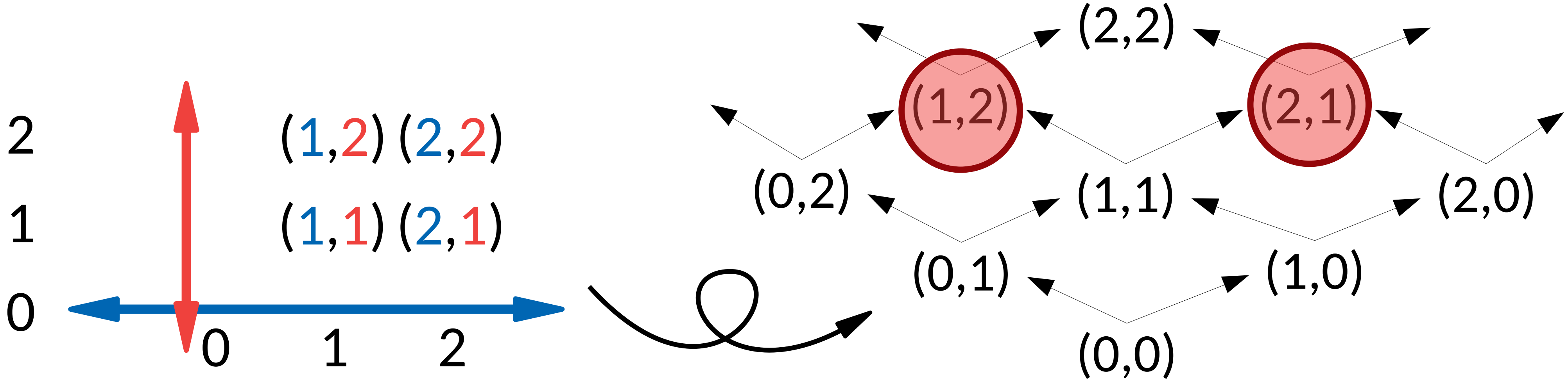

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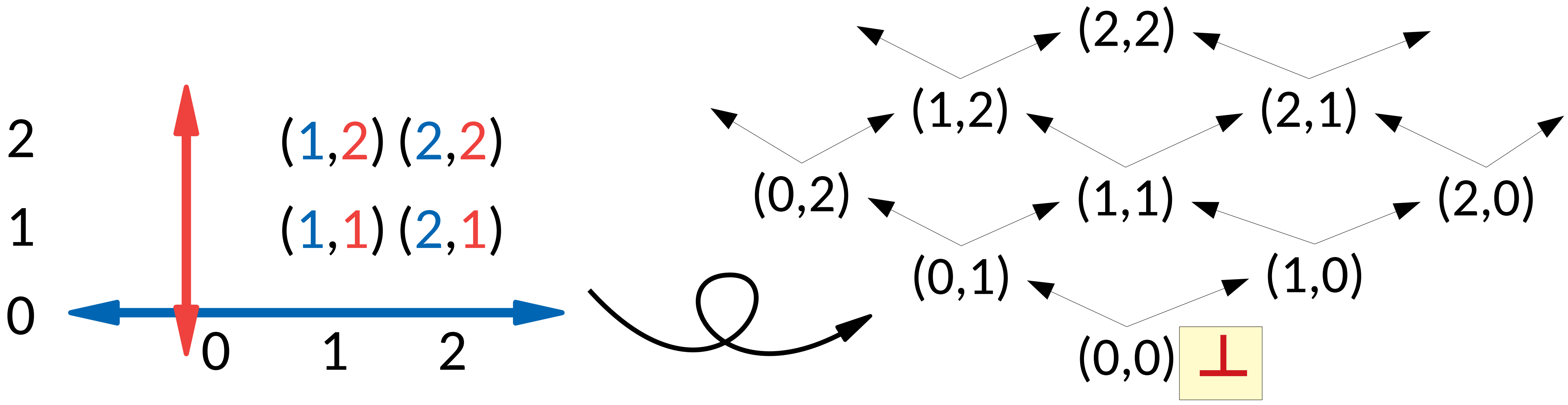

- If unique least/greatest elements exist, we call them ⊥(bottom)/⊤(top)

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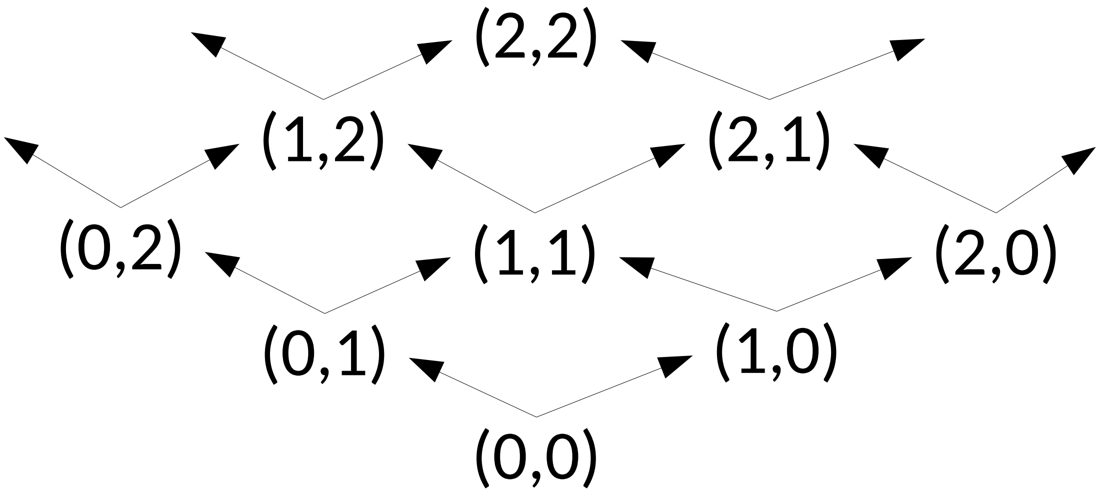

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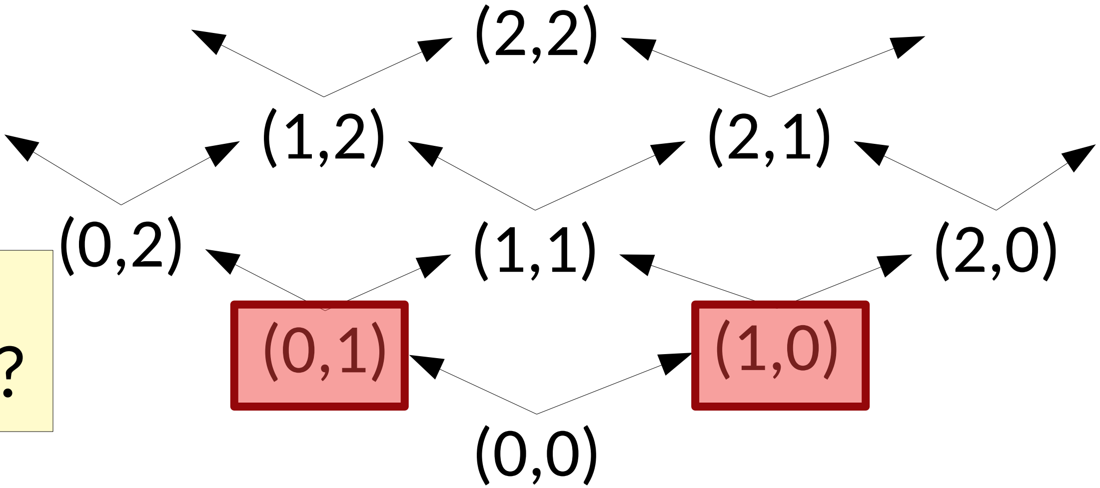

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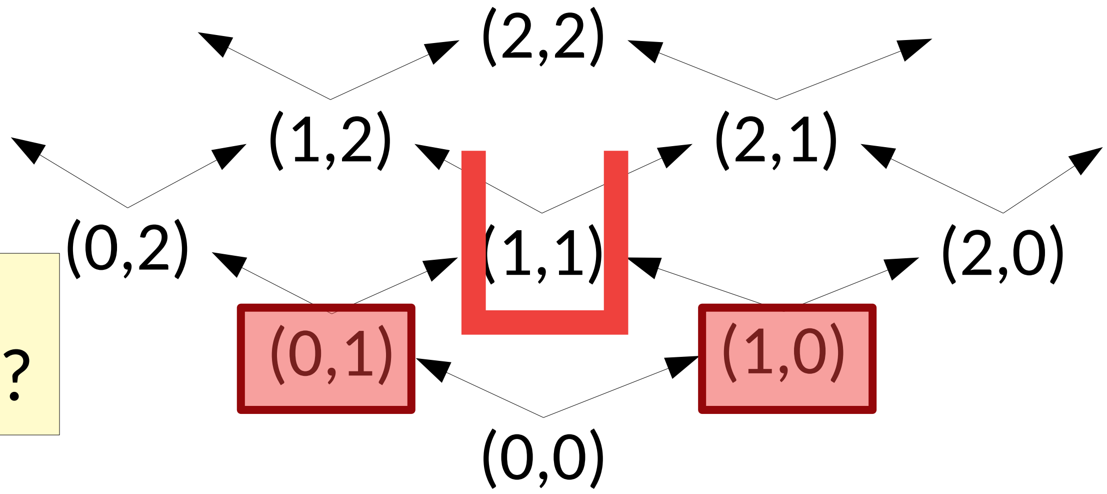

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    a ⊑ (a⊔b) & b ⊑ (a⊔b) & (a ⊑ c & b⊑c → (a⊔b) ⊑ c)

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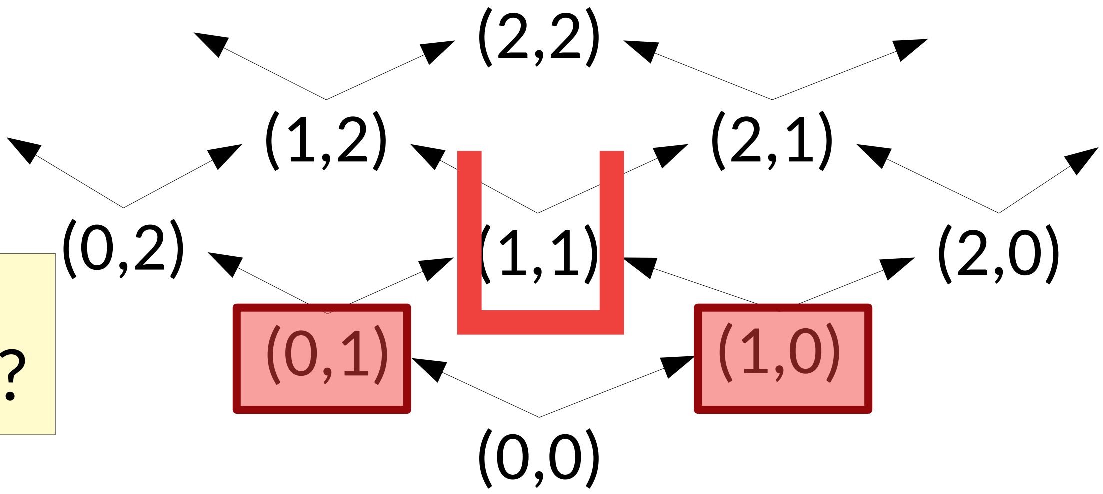

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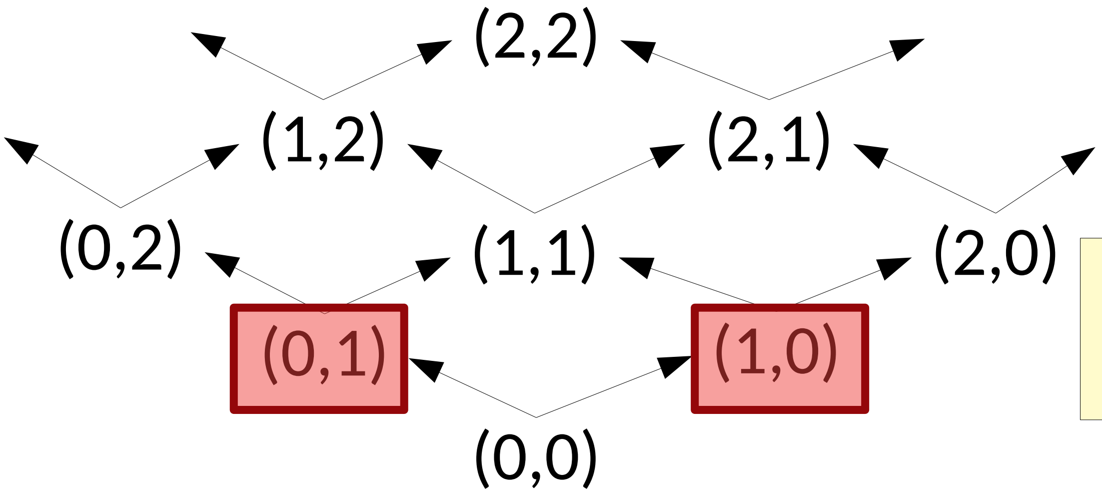

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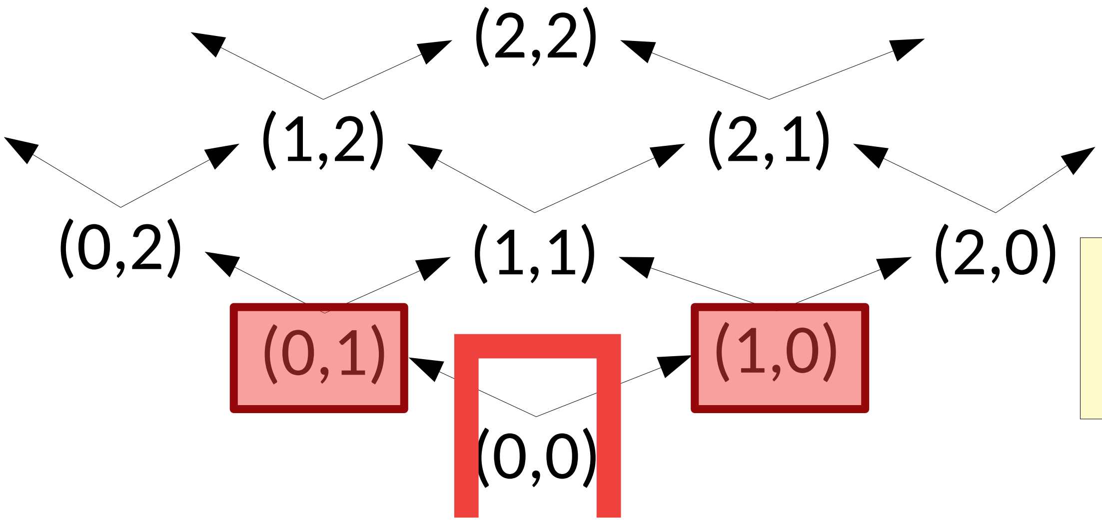

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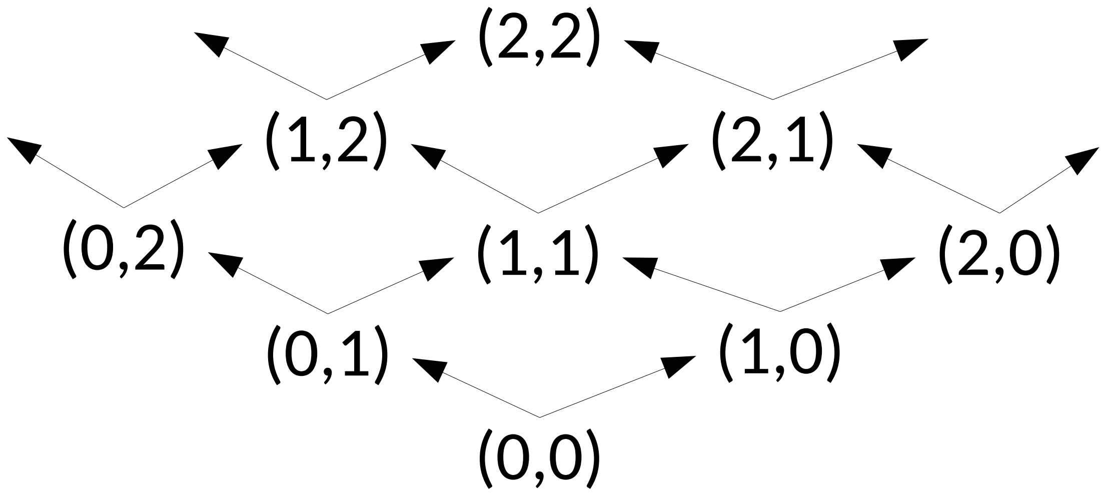

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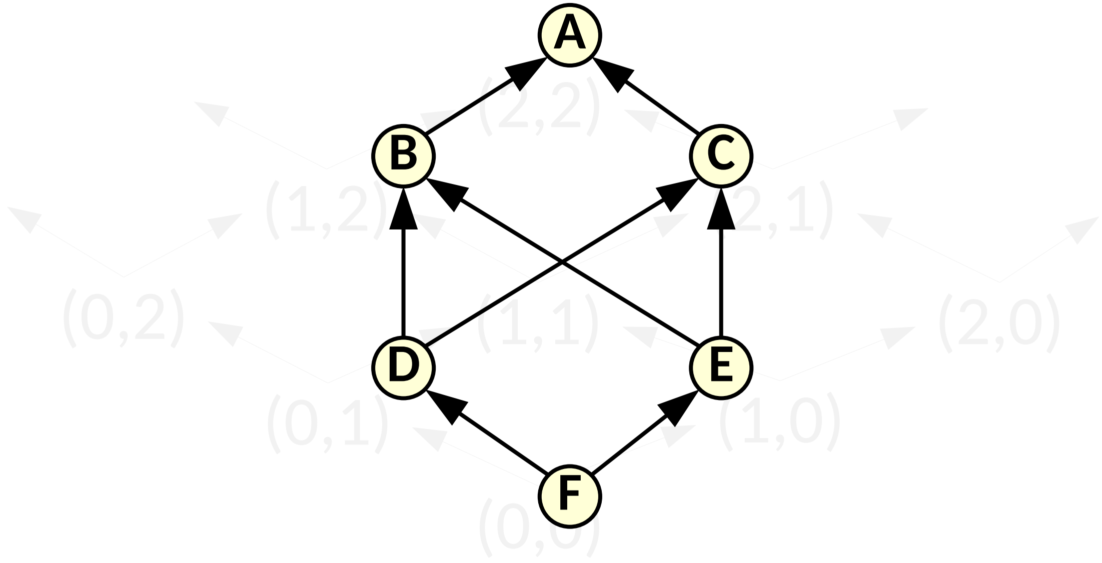

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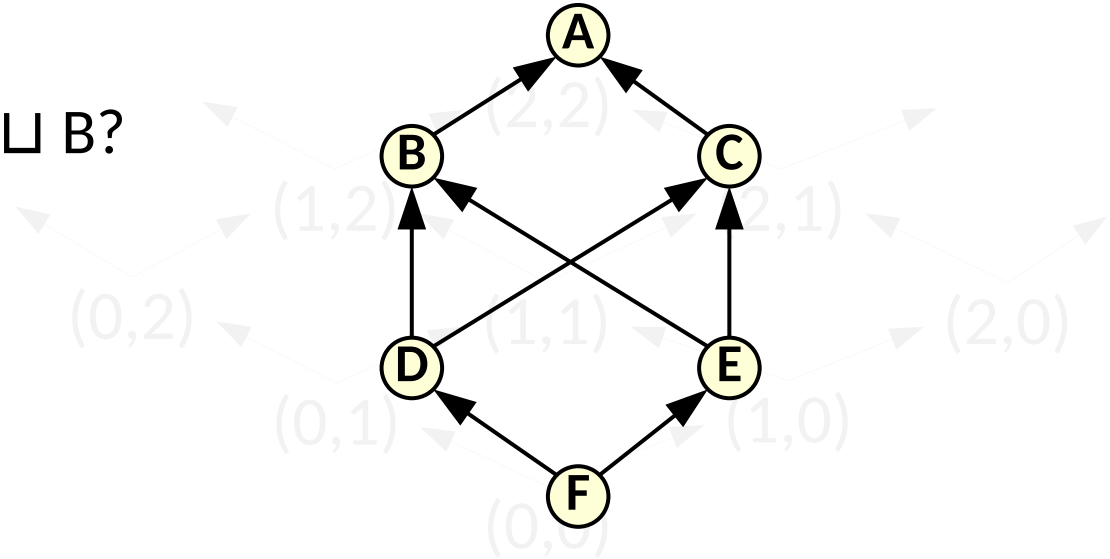

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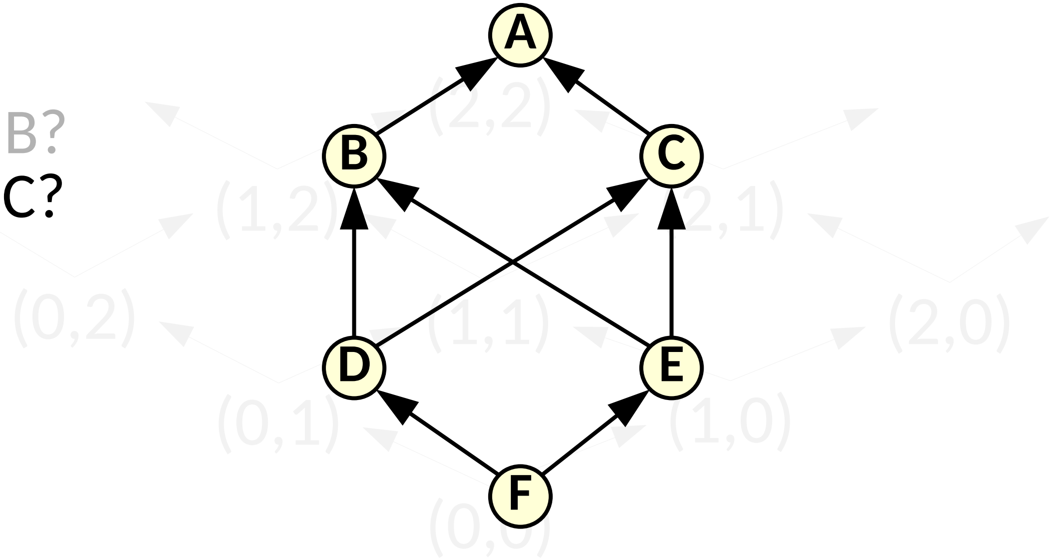
What is B ⊔ C?



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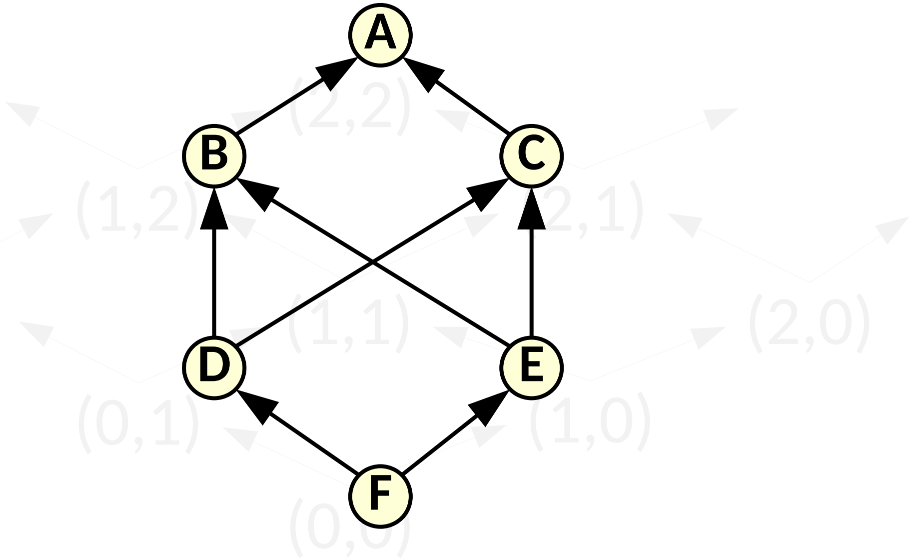
What is D ⊔ E?



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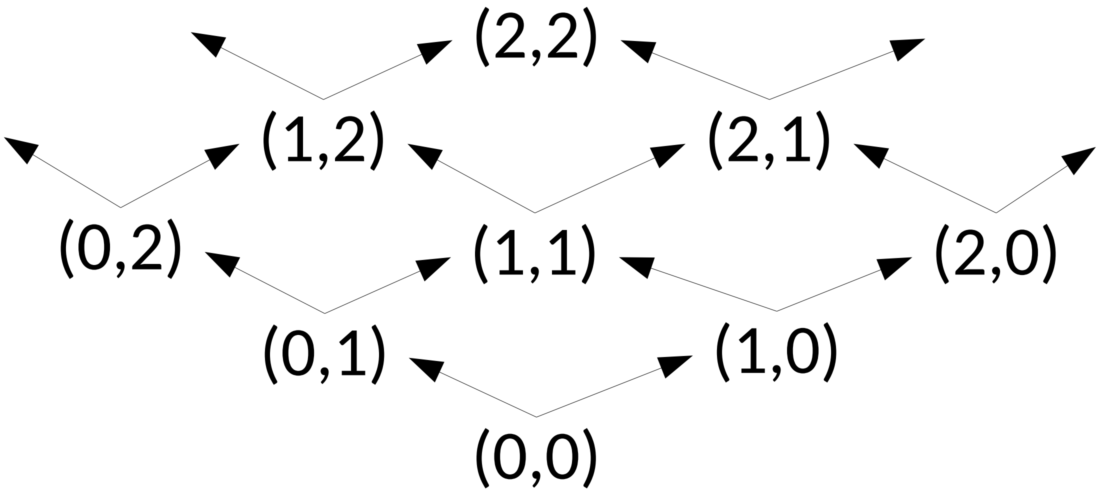
  - ∀S’⊆S,



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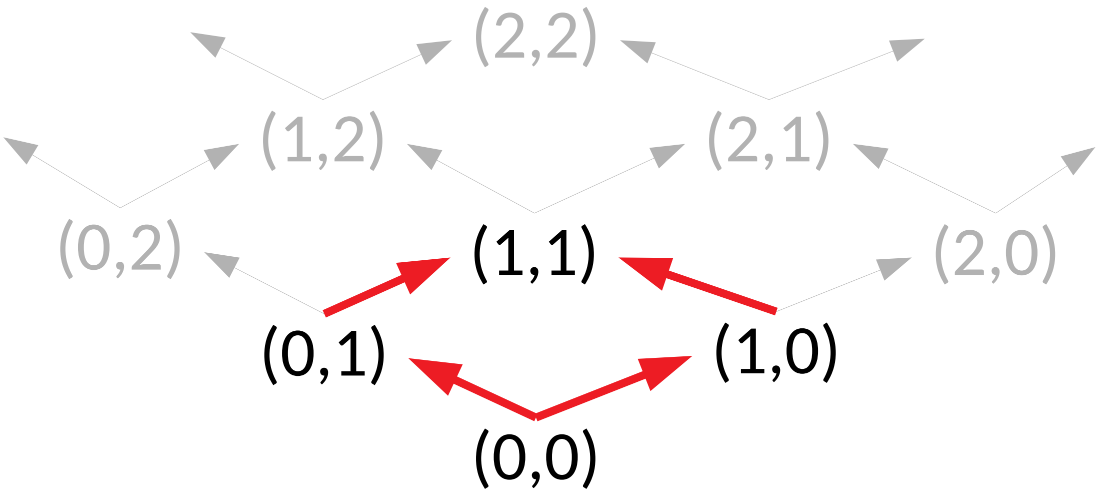
  - ∀S'⊆S, ∃⊔S' & ⊓S'⇒ lattice



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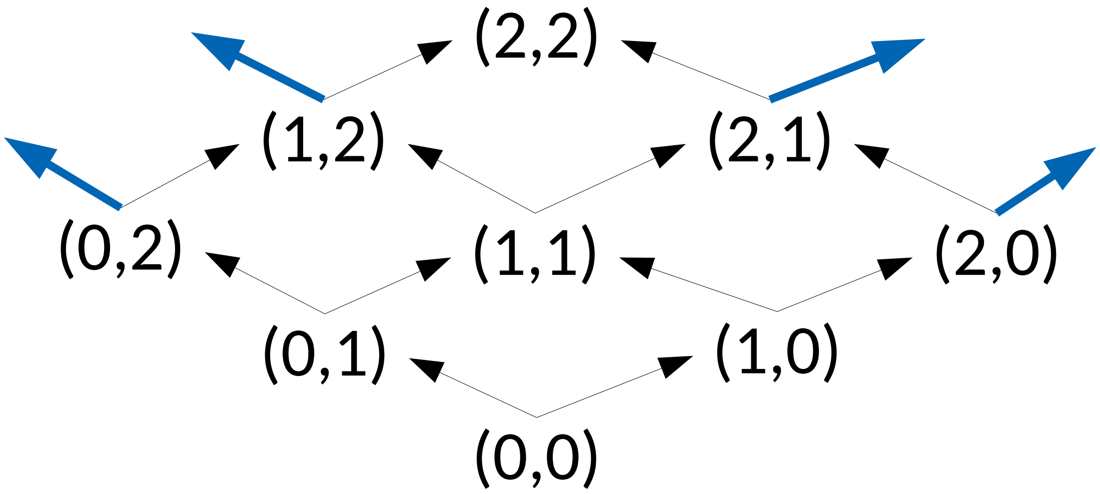
  - ∀S'⊆S, ∃⊔S' & ⊓S'⇒ lattice, ∃⊔S' **or** ∃⊓S'⇒ semilattice



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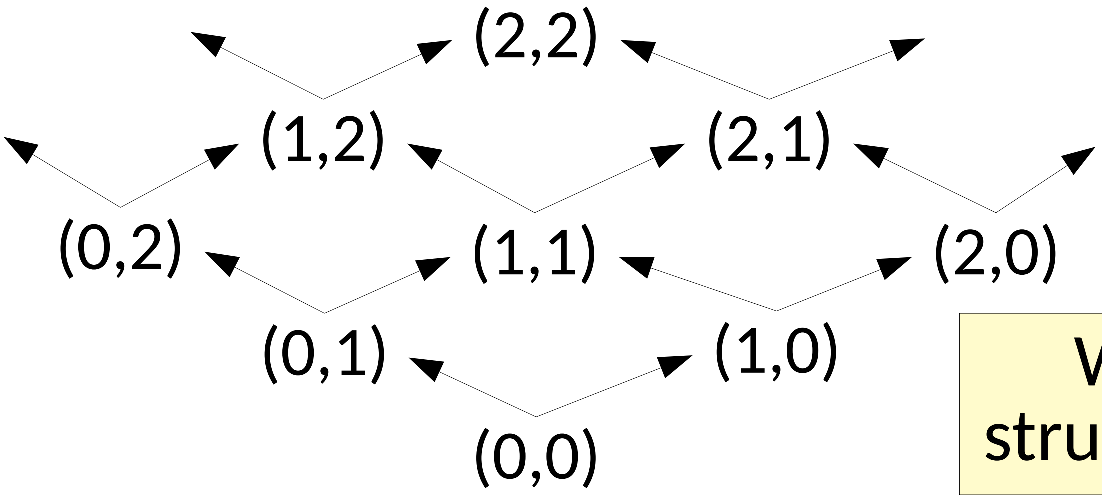

What is the structure shown?

Partial Orders

- A product of lattices (partial orders) yields a lattice (partial order)

$$L_1 \times L_2$$

Partial Orders

- A product of lattices (partial orders) yields a lattice (partial order)
  - We already saw componentwise orderings for tuples.
    This is the same.

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A total order is a partial order.
Products of total orders are partial orders

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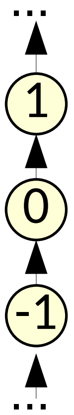

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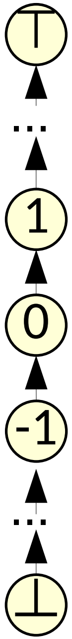

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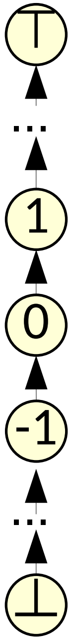

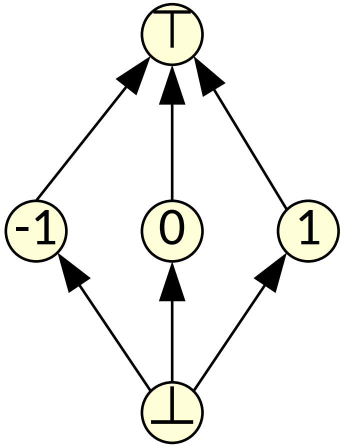

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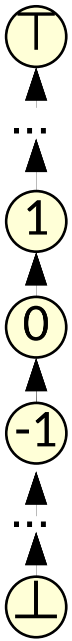

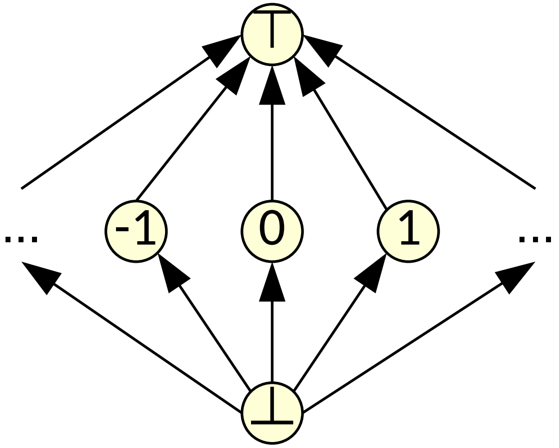

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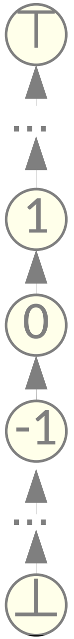

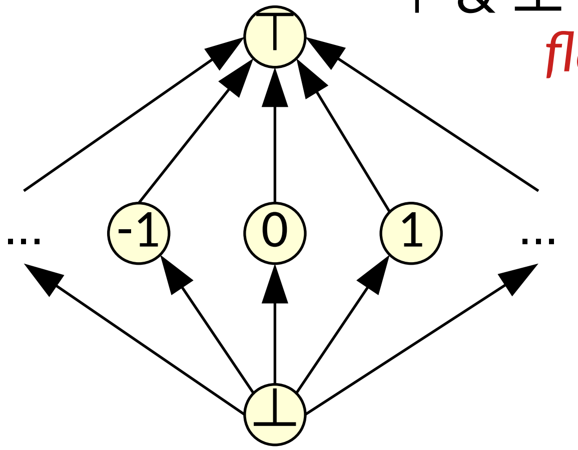

Combining a set with ⊤ & ⊥ like this yields a *flat lattice*.

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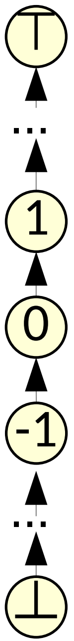
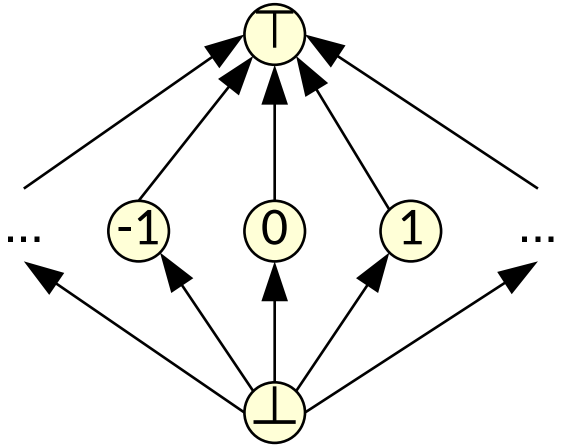
Partial Orders





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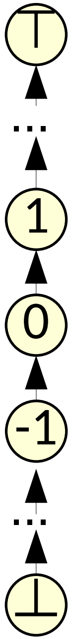

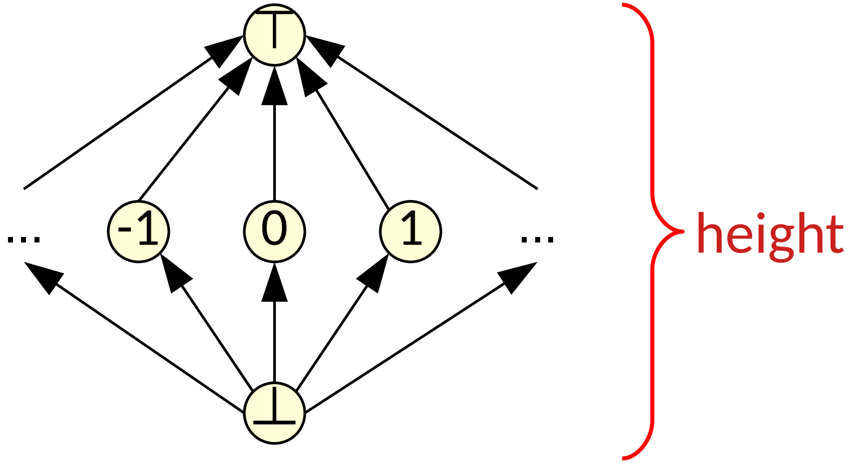

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  - Many use cases can also be affected by the height of a lattice.

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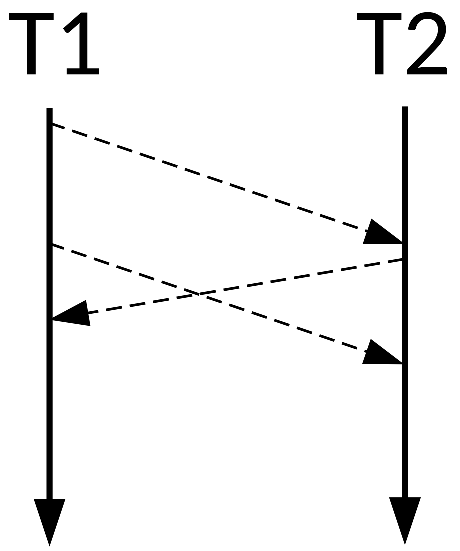

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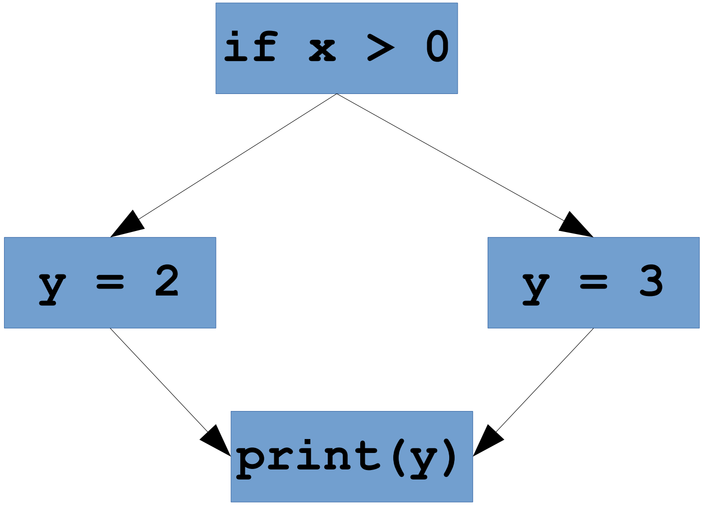

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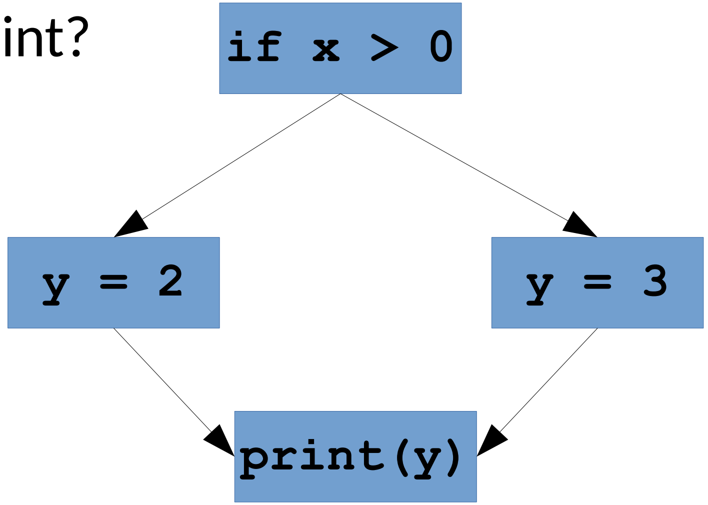

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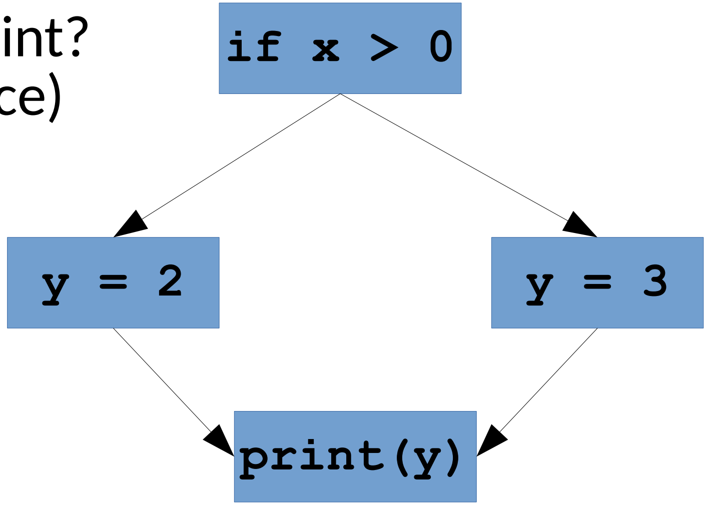
2 or 3? (set lattice)



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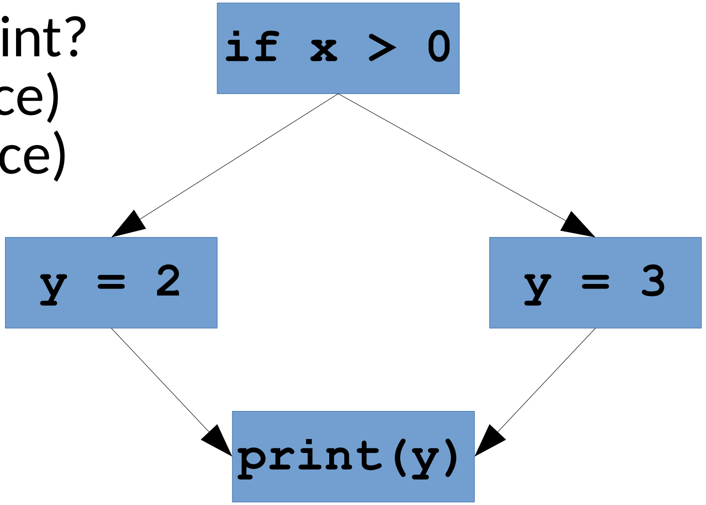
unknown? (flat lattice)



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  - Alternatively, they generate the set via structure
- They commonly define *formal languages*
  - Sets of strings over a defined alphabet
- They are effective at *constraining* sets & search spaces

Regular Languages & Finite Automata

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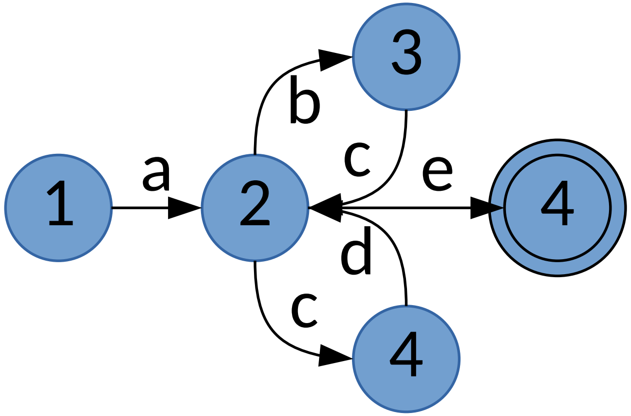

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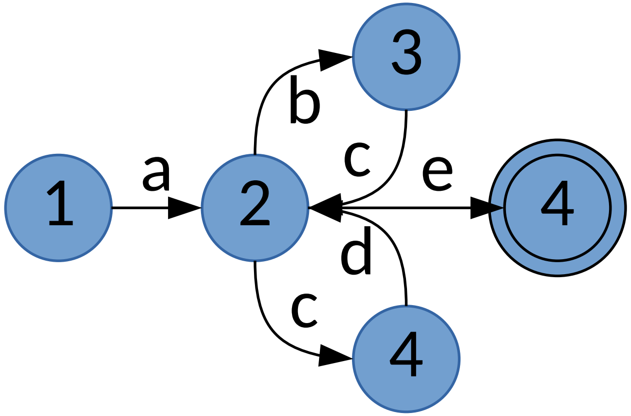

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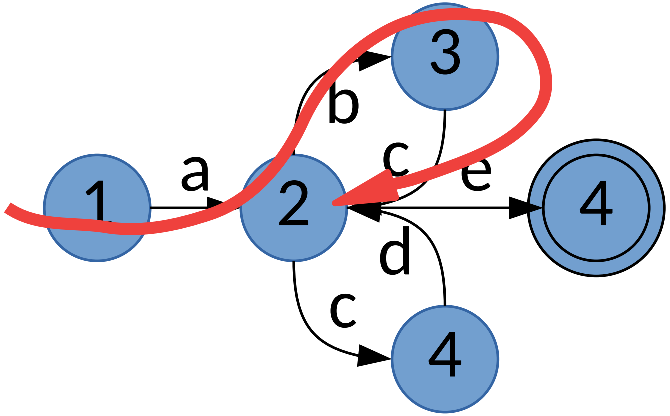

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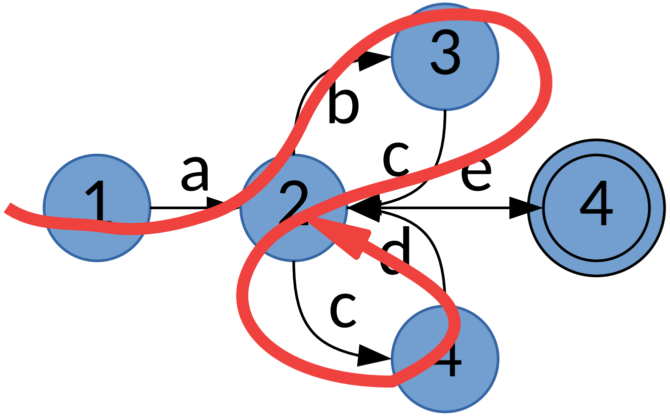

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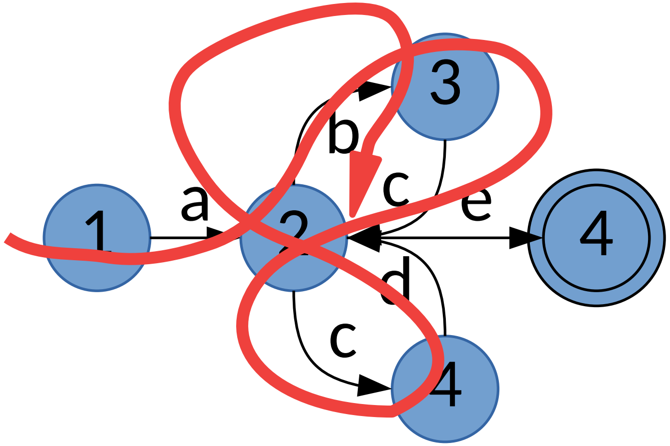

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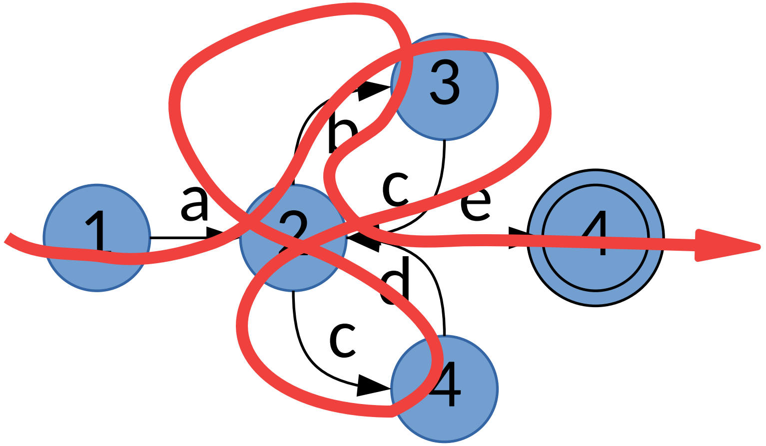

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- Recall, regular languages cannot express matched parentheses (Dyck languages)

$a^n b^n$

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$$\begin{array}{lll} \text{Start} = A & & \\ A & \rightarrow & cBd \\ B & \rightarrow & eBf \\ & | & g \end{array}$$

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This requires some kind of *memory*

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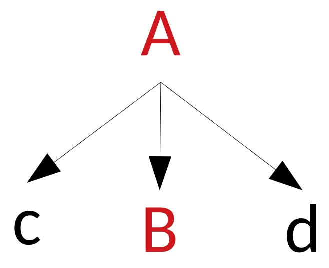

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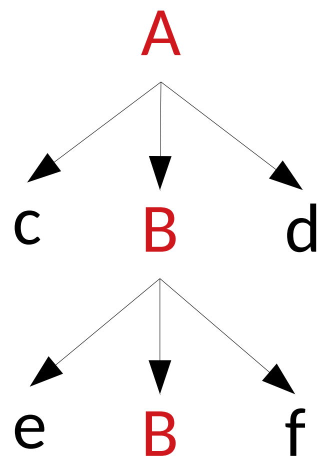

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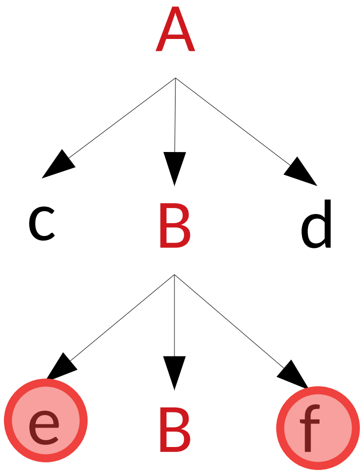

Generating symbols out of order acts as a form of memory.

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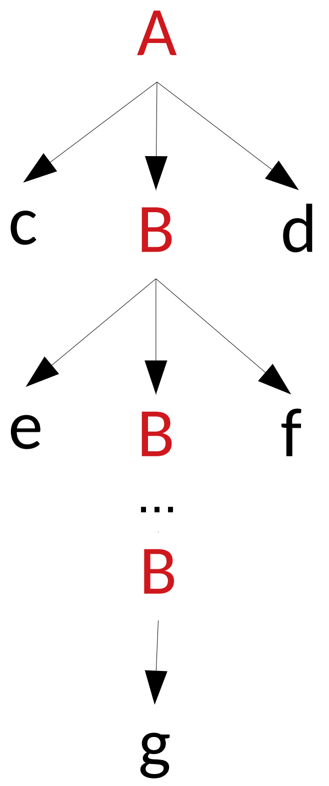

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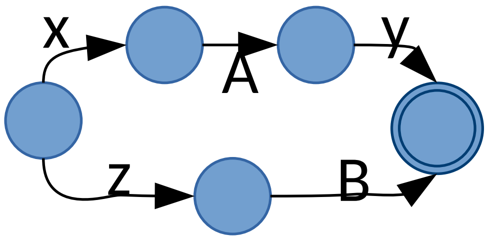

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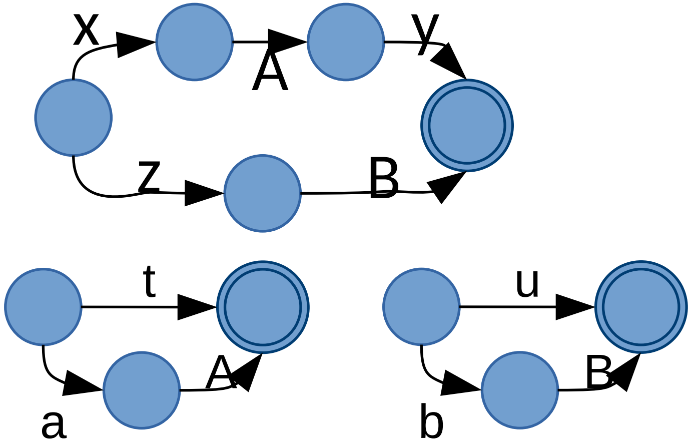

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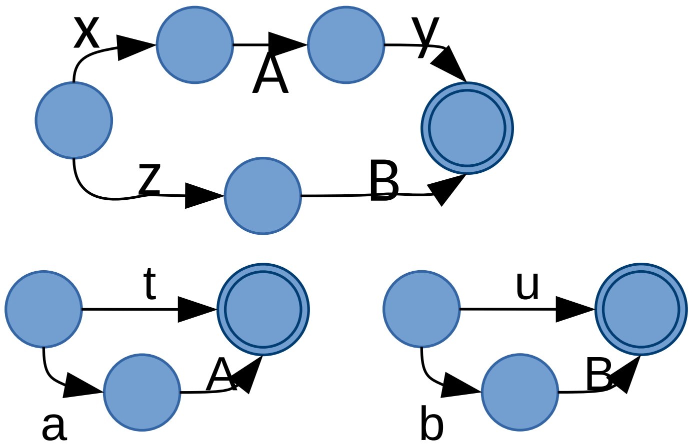

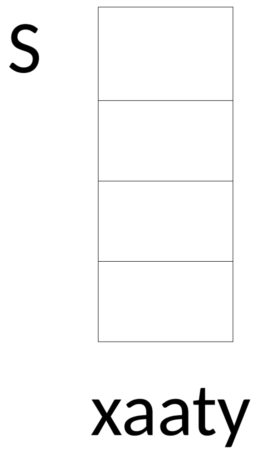

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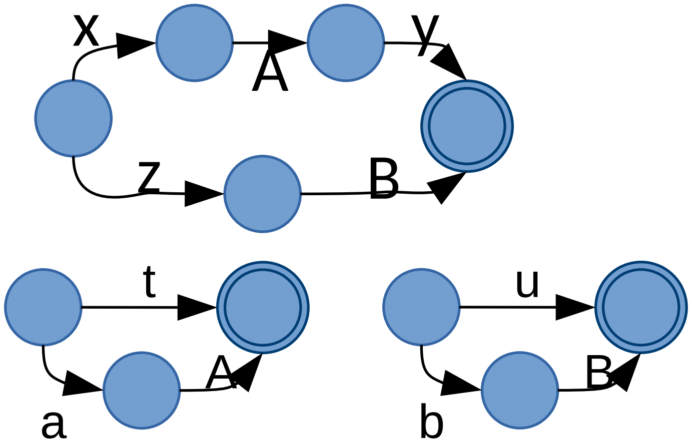

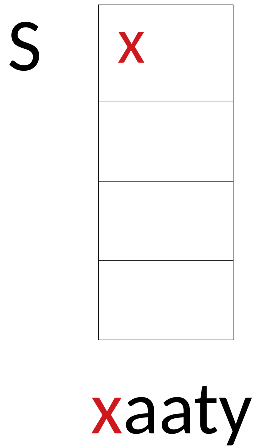

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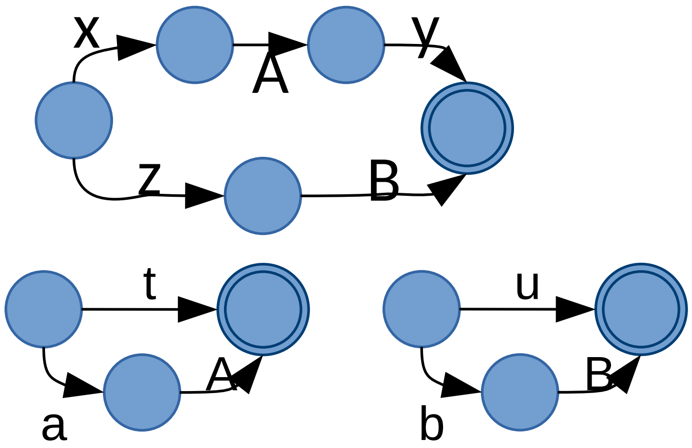

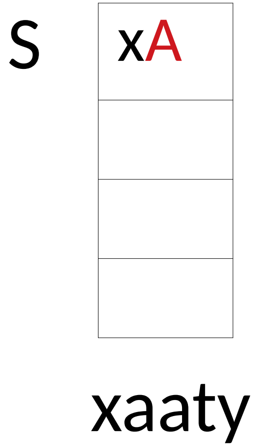

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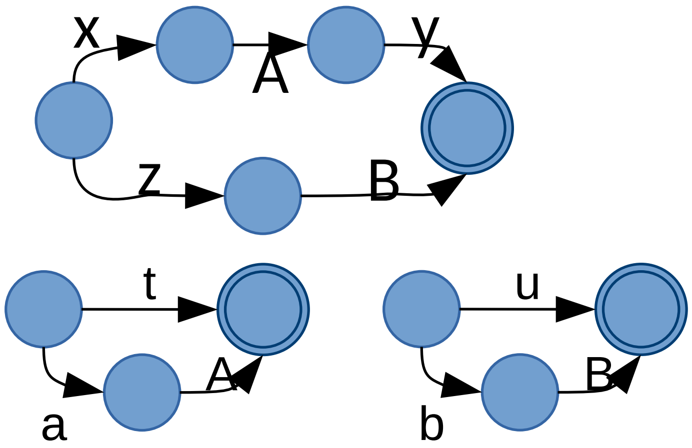

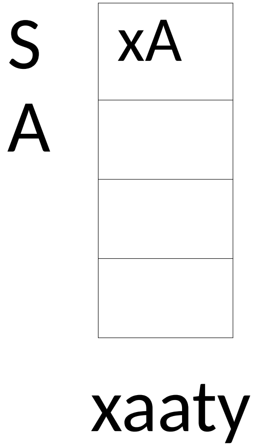

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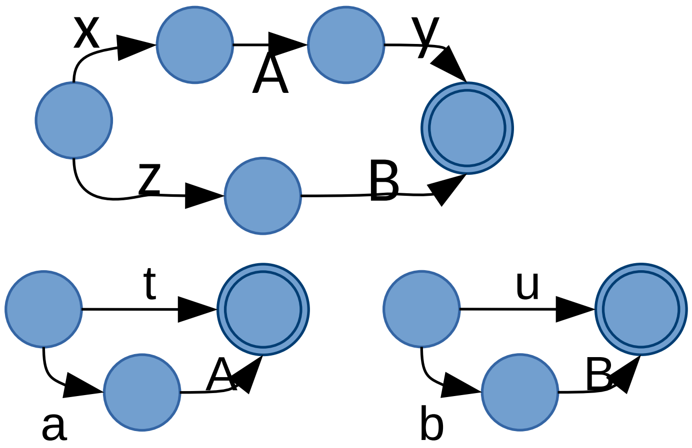

S	xA
A	a

xaaty

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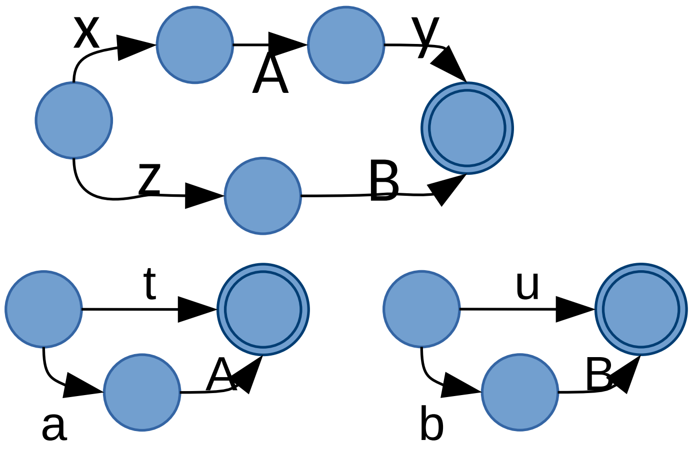

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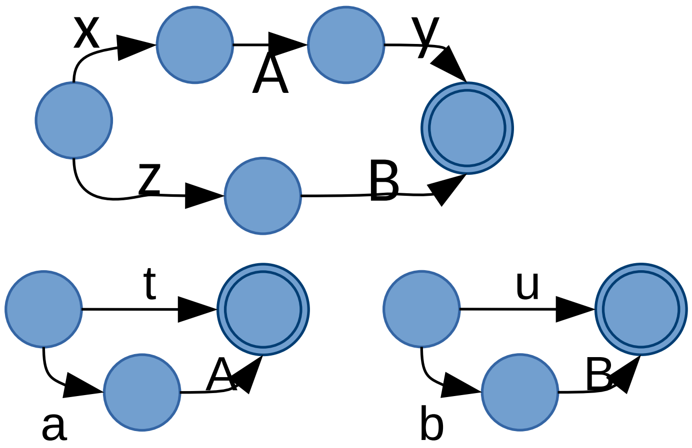

S	xA
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A	aA
A	t

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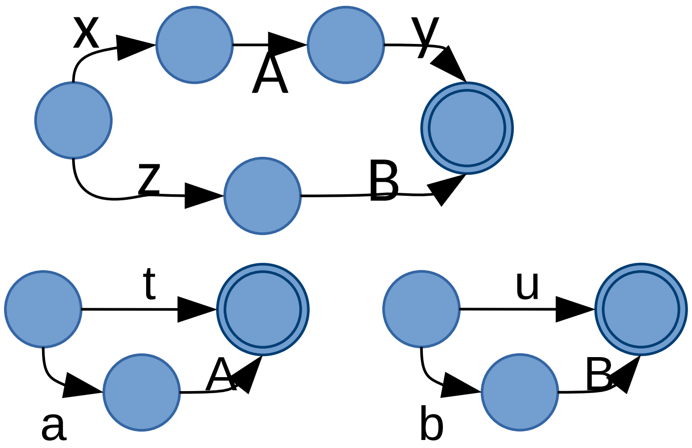

S	xA
A	aA
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xaaty

Context Free Grammars & Pushdown Automata

- *Context free grammars* add recursion and enable Dyck language recognition
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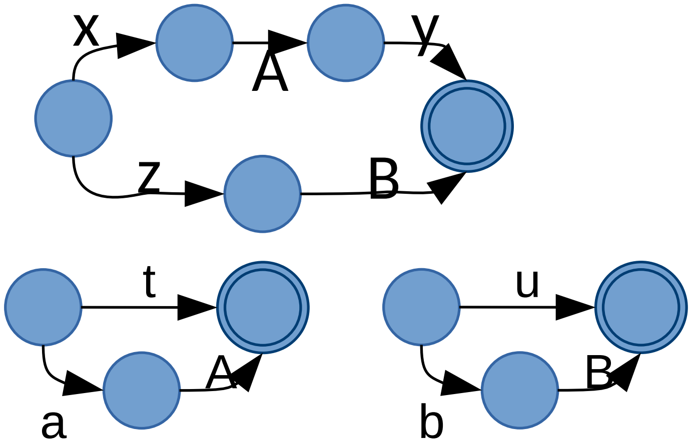

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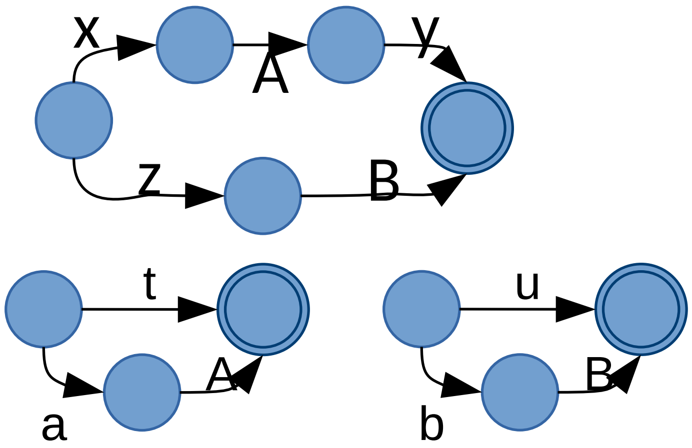

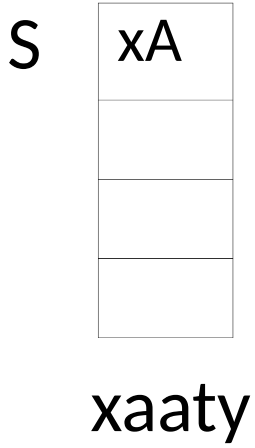

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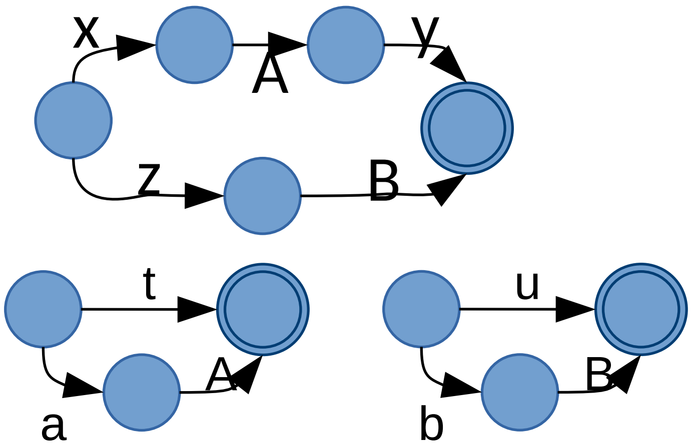

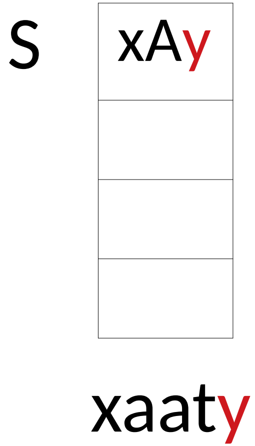

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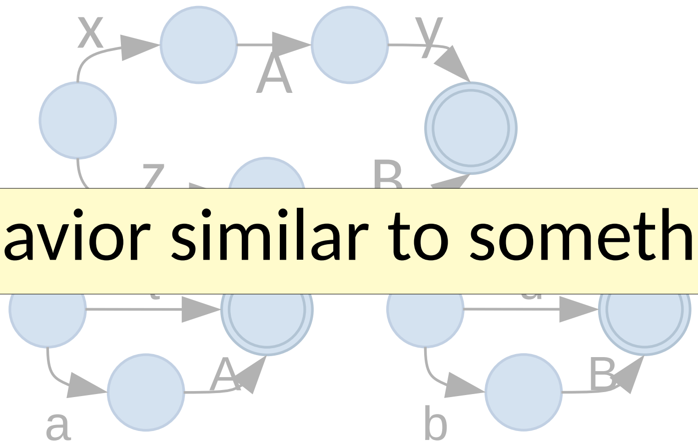

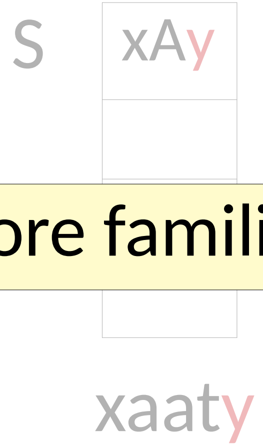

Is this behavior similar to something more familiar?

Context Free Grammars & Pushdown Automata

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- Adding additional rules can extend the expressiveness
  - context sensitive languages
  - tree adjoining grammars
  - …

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- Grammars can constrain far more than strings.
  - graphs
  - semantic objects (furniture layout? sequences of actions? …)

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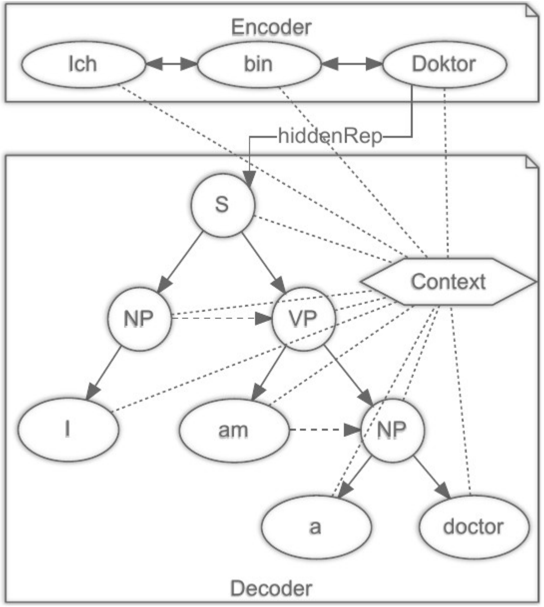

[Gu 2019]

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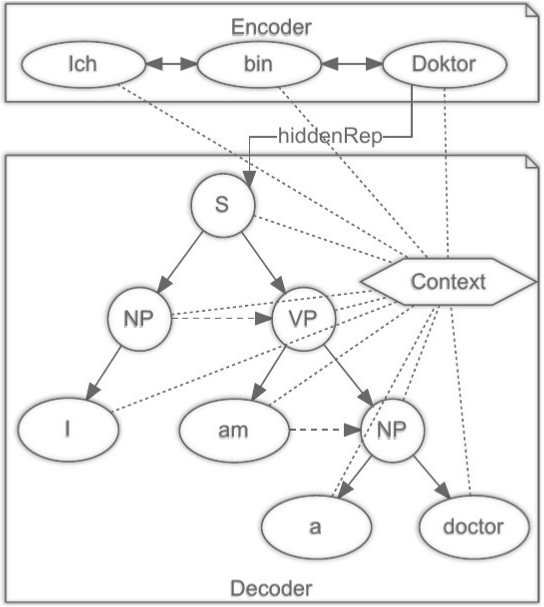
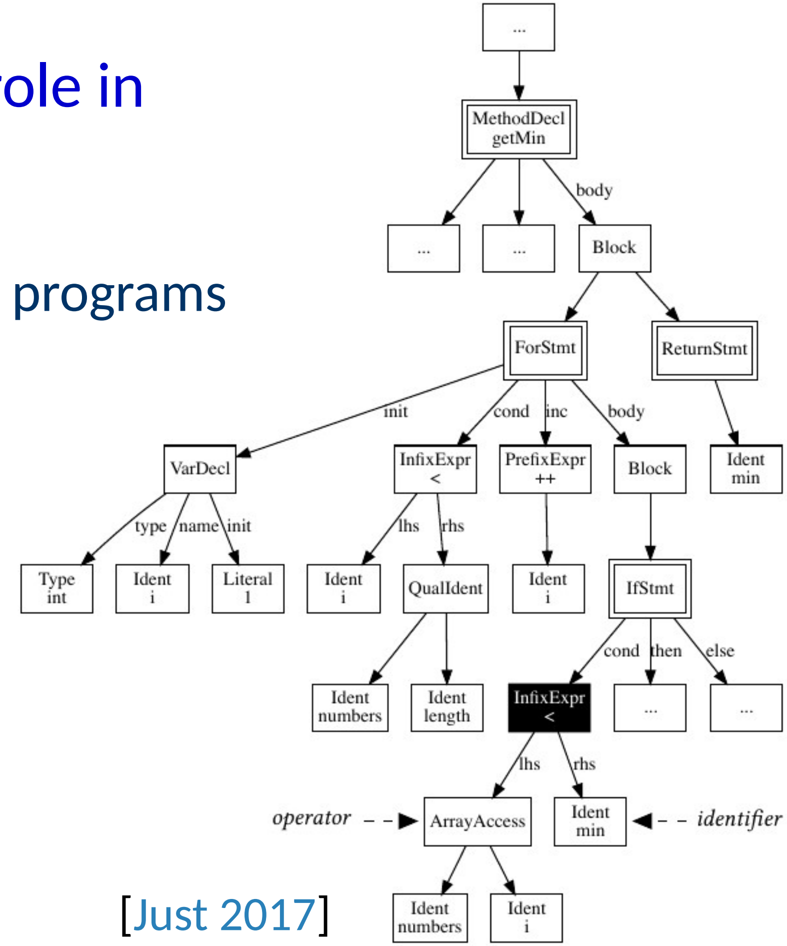


[Gu 2019]



[Just 2017]

Context Free Grammars & Pushdown Automata

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  - Compact encodings of complex sets

Formal Logic

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- Program analysis has actually been one of the driving forces behind satisfiability in recent years.

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$$\mathbf{x} \wedge \neg\mathbf{y} \wedge \mathbf{z}$$

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$$I \vdash x \text{ and } I \vdash y \text{ iff } I \vdash x \wedge y$$

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  - Bug finding
  - Model checking (proving correctness)
  - Explaining defects
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Inference using classical logic

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Id} \qquad \dfrac{\dfrac{}{A \multimap B \vdash A \multimap B}\,\mathrm{Id} \qquad \dfrac{}{A \vdash A}\,\mathrm{Id}}{A \multimap B,\, A \vdash B}\,{\multimap}\text{-E}}{A,\, A \multimap B,\, A \vdash A \times B}\,{\times}\text{-I}}{A \multimap B,\, A,\, A \vdash A \times B}\,\text{Exchange}}{A \multimap B,\, A \vdash A \times B}\,\text{Contraction}$$

Wadler, “A Taste of Linear Logic”. 2014.

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- *Intuitionistic logic* restricts the rules of inference to require direct evidence

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$$\frac{}{\vdash p \lor \neg p}$$

Law of excluded middle

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$$\frac{\Gamma \vdash \neg\neg p}{\Gamma \vdash p}$$

Double negation elimination

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- Note, this is commonly used in *type systems*

Linear & Substructural Logic

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Classical & intuitionistic logic
have trouble expressing consumable facts

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Idea: Some facts (resources) require careful accounting

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Logics that remove additional rules
from intuitionistic logic are *substructural*

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Error ($\vdash$ c:?)

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```
y = w[20]
x = *y + 5
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```

```
y = w[20]
x = *y + 5
```

We want a logic that reasons about changes in state.

Hoare Logic

- *Hoare logic* reasons about the behavior of programs and program fragments

Hoare Logic

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$$\boxed{\{\varphi\}}\, C \{\psi\}$$

Precondition

Hoare Logic

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$$\{\varphi\}\boxed{C}\{\psi\}$$

Precondition

Command

Hoare Logic

- *Hoare logic* reasons about the behavior of programs and program fragments

$$\{\varphi\}\,C\,\boxed{\{\psi\}}$$

Precondition

Command

Postcondition

Hoare Logic

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$$\{\varphi\}C\{\psi\}$$

- If φ holds before C, ψ will hold after

$$\{x=3 \land y=2\}x \leftarrow 5\{x=5\}$$

Hoare Logic

- *Hoare logic* reasons about the behavior of programs and program fragments

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- A ***weakest precondition*** $\mathrm{wp}(C,\psi)$ captures all states leading to ψ after C.

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$$\{\#t\}x\leftarrow5\{x=5\}$$

Hoare Logic

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- If φ holds before C, ψ will hold after

$$\{x=3 \land y=2\}\texttt{x} \leftarrow \texttt{5}\{x=5\}$$

- A ***weakest precondition*** $\mathrm{wp}(C,\psi)$ captures all states leading to ψ after C.

$$\{\#t\}\texttt{x}\leftarrow\texttt{5}\{x=5\}$$

$$\{???\}\texttt{if c then x}\leftarrow\texttt{5}\{x=5\}$$

Hoare Logic

- *Hoare logic* reasons about the behavior of programs and program fragments

$$\{\varphi\}C\{\psi\}$$

- If φ holds before C, ψ will hold after

$\{x=3 \land$

You already have an *intuition* for weakest preconditions

- A ***weakest precondition*** $\mathrm{wp}(C,\psi)$ captures all states leading to ψ after C.

$$\{\#t\}x\leftarrow 5\{x=5\}$$

$$\{???\}\texttt{if c then x}\leftarrow 5\{x=5\}$$

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?

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- A store/state σ is a partial function mapping variables to values
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Command

$x \leftarrow 5$

Hoare Logic – weakest preconditions

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Store	Command
$\sigma = \{x \mapsto 3,\ y \mapsto 1\}$	
	x ← 5

Hoare Logic – weakest preconditions

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$\sigma = \{\mathbf{x \mapsto 5},\ y \mapsto 1\}$	

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	x ← 5	
$\sigma=\{x\mapsto 5,\ y\mapsto 1\}$		$\{x=5\}$

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?
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  - Commands in a program can modify the store

Store	Command	Conditions
$\sigma=\{x\mapsto 3,\ y\mapsto 1\}$		$\{x=3 \wedge y=\overset{?}{2}\}$
	x ← 5	
$\sigma=\{x\mapsto 5,\ y\mapsto 1\}$		$\{x=5\}$

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?

- A store/state σ is a partial function mappi
  - Commands in a program can modify the sto

This was *technically* true,
but not so useful
(…or even compatible with our states)

Store	Command	Conditions
$\sigma=\{x\mapsto 3,\ y\mapsto 1\}$		$\{x=3 \wedge y=2\}$
	x ← 5	
$\sigma=\{x\mapsto 5,\ y\mapsto 1\}$		$\{x=5\}$

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?
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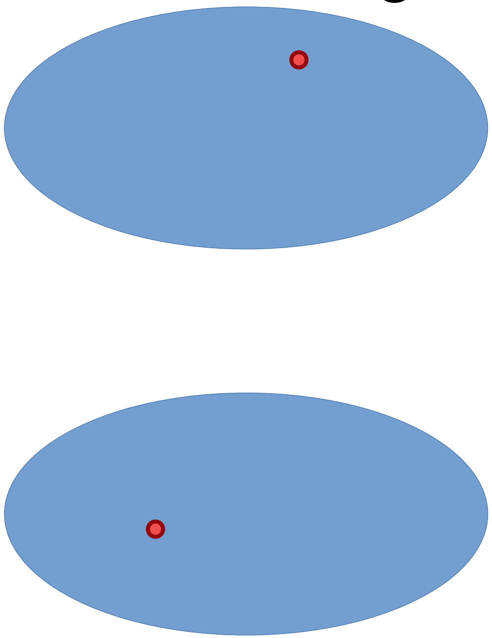
Store	Command	Conditions
$\sigma=\{x\mapsto 3,\ y\mapsto 1\}$		$\{x=3 \wedge y=2\}$
	$x \leftarrow 5$	
$\sigma=\{x\mapsto 5,\ y\mapsto 1\}$		$\{x=5\}$

  - **$\sigma \in \Sigma$**(all possible states), and we can reason about *subsets* of Σ

Hoare Logic – weakest preconditions

- What do we really mean by *captures all states*?

$\sigma = \{x \mapsto 3,\ y \mapsto 1\}$



x ← 5

$\sigma = \{x \mapsto 5,\ y \mapsto 1\}$

Hoare Logic – weakest preconditions

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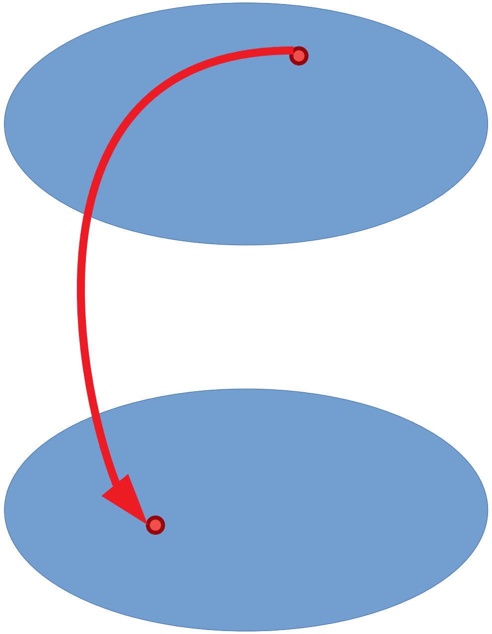
$\sigma = \{x \mapsto 3,\ y \mapsto 1\}$



$x \leftarrow 5$

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Hoare Logic – weakest preconditions

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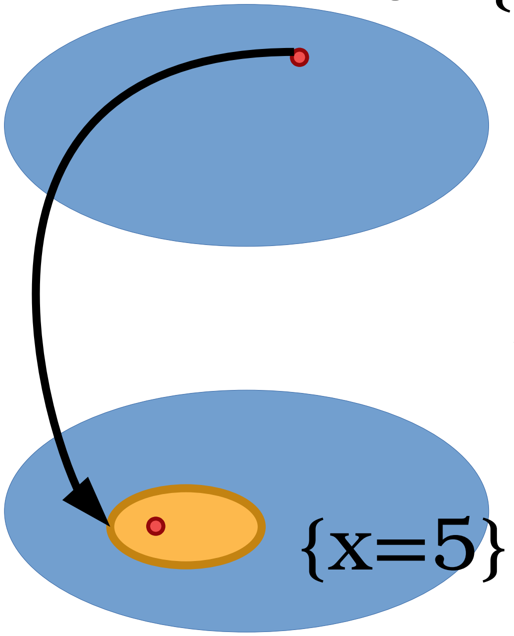

Hoare Logic – weakest preconditions

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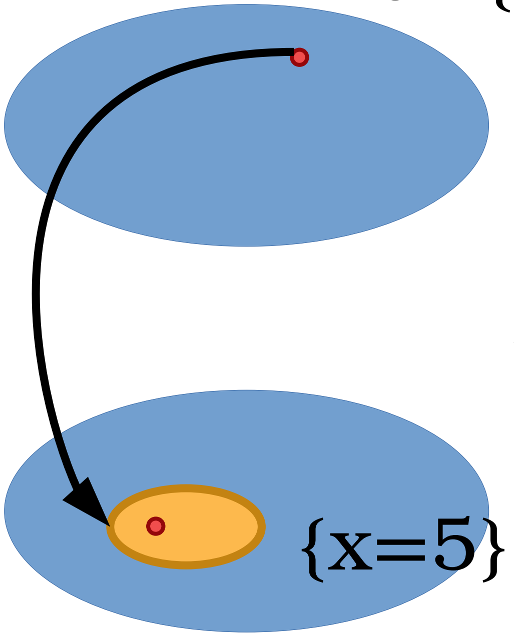

x ← 5

Each set of states corresponds to a condition defining the set

$\sigma = \{x \mapsto 5,\ y \mapsto 1\}$

Hoare Logic – weakest preconditions

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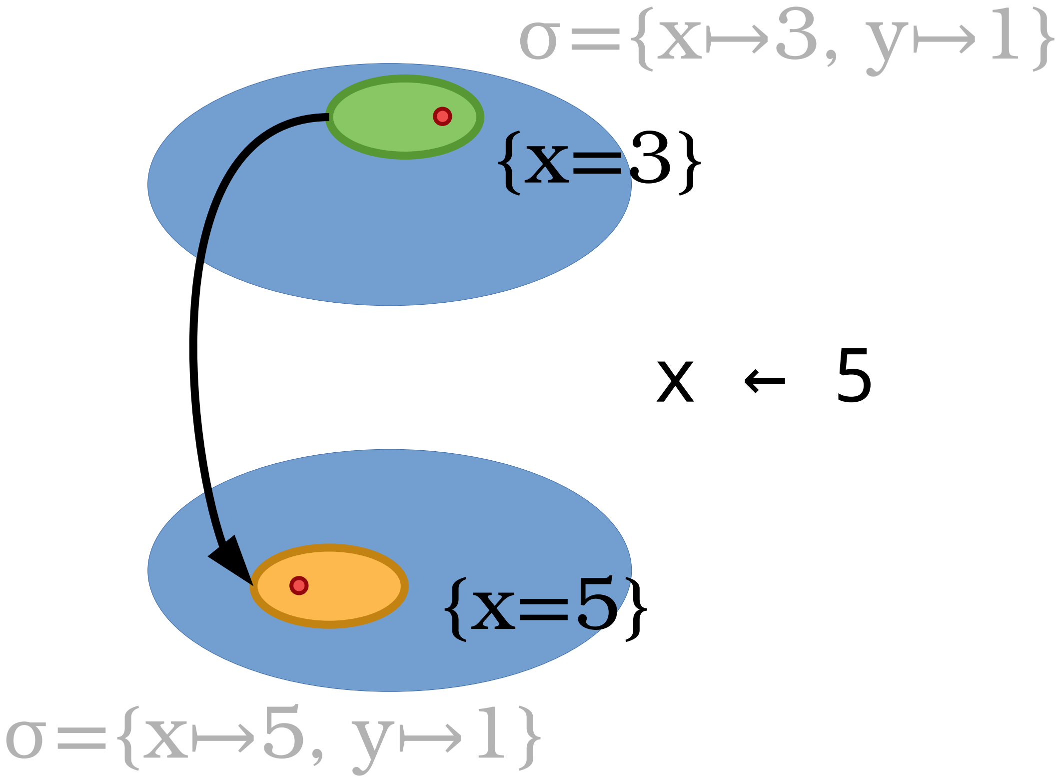

Hoare Logic – weakest preconditions

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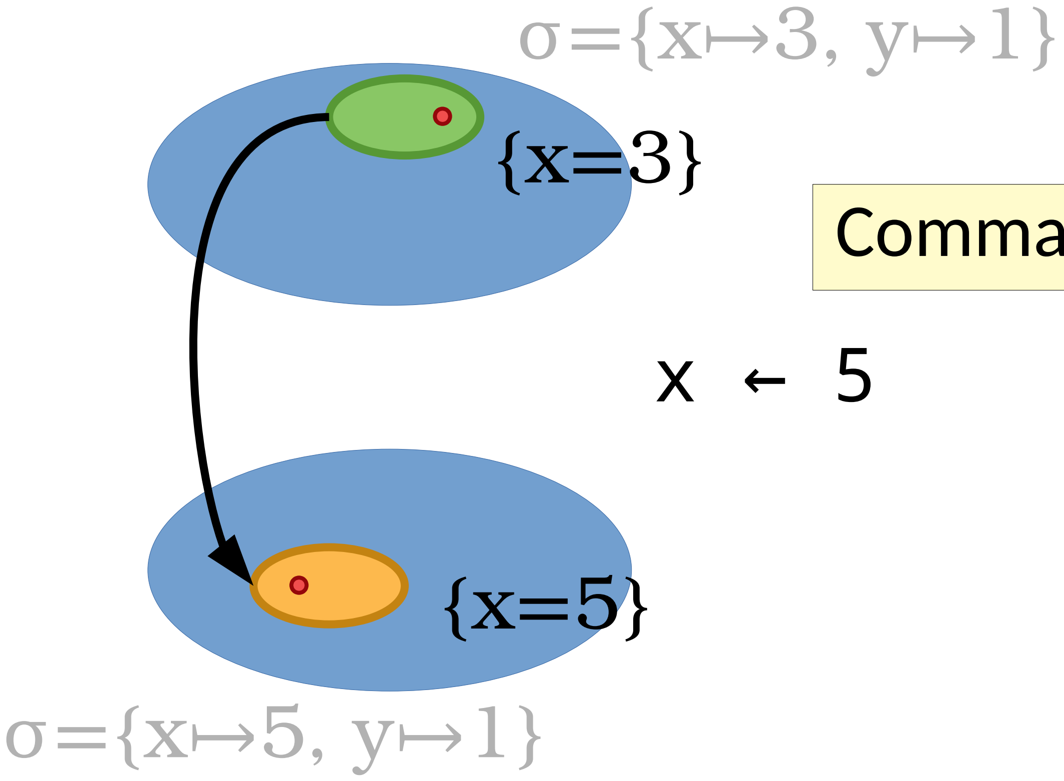

Commands map sets to sets

x ← 5

Hoare Logic – weakest preconditions

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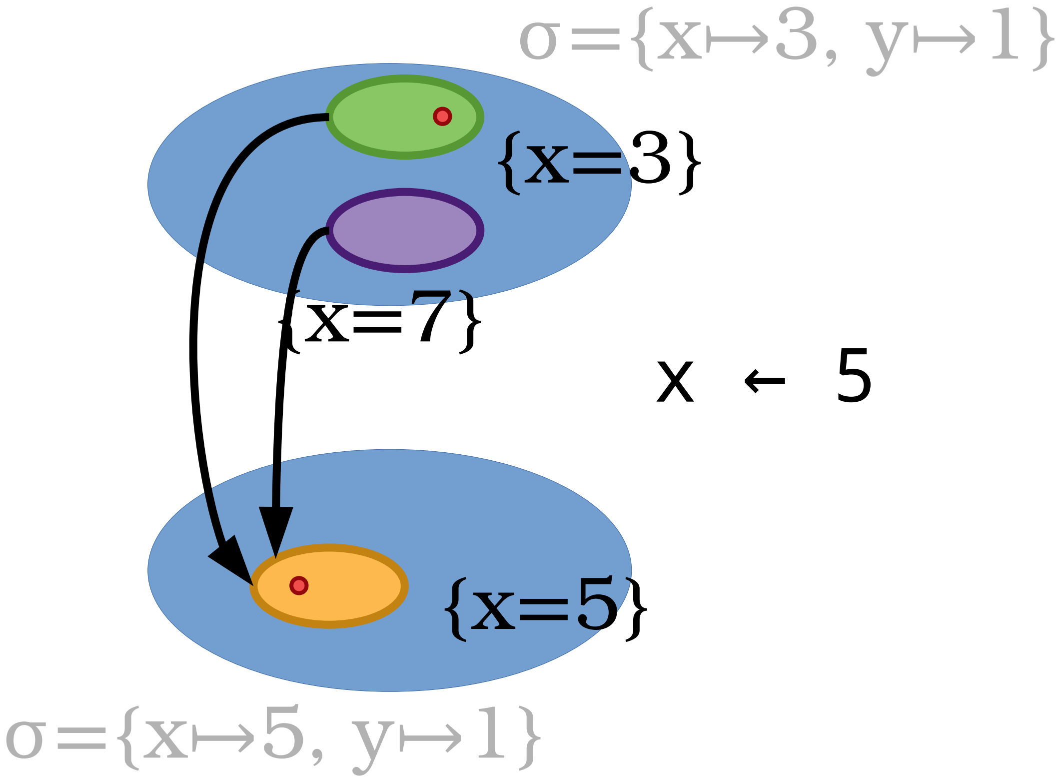

Hoare Logic – weakest preconditions

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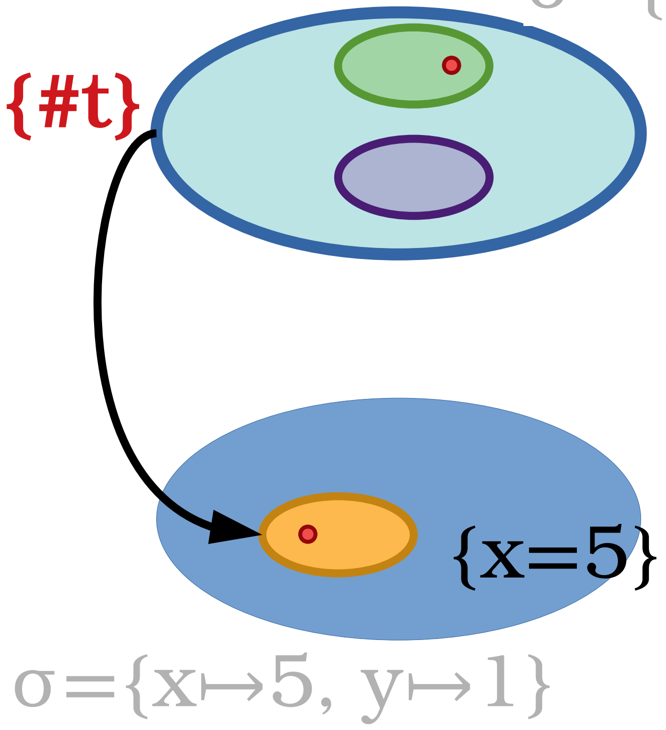

All states lead
to the postcondition!

x ← 5

Hoare Logic – weakest preconditions

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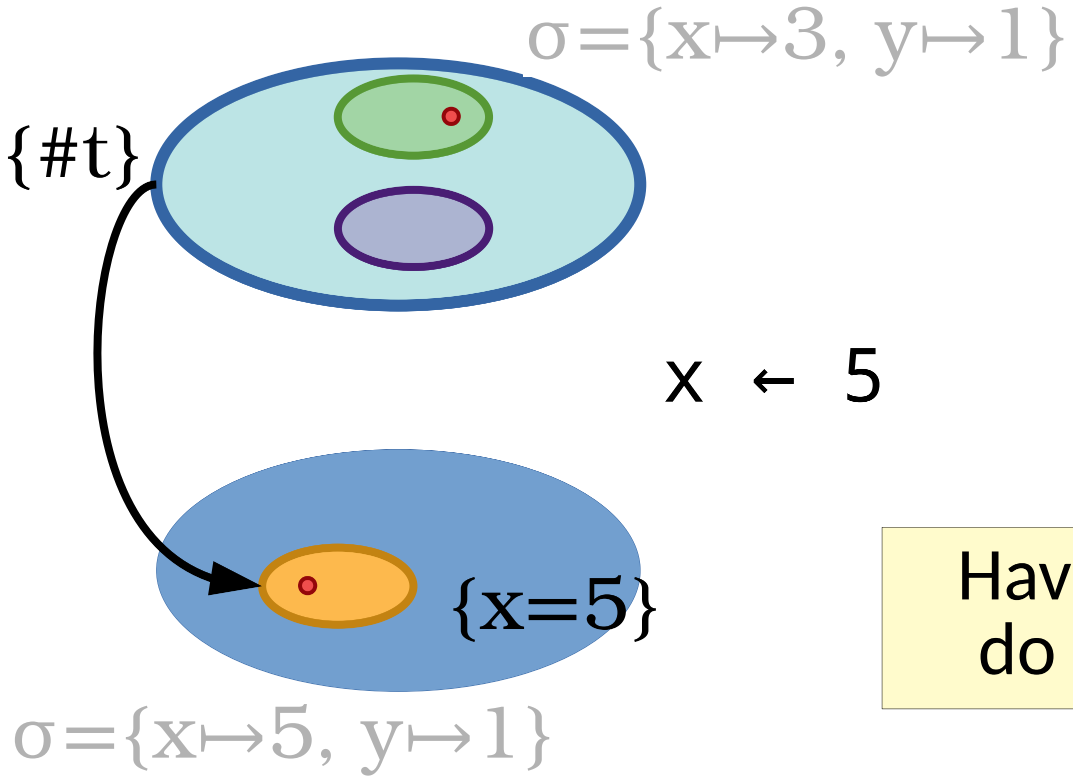

Have we already seen a way do describe this structure?

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?
- $wp(C, \psi) = \bigsqcup \{x \mid \{x\}\ C\ \{\psi\}\}$
  - Where $(A \rightarrow B) \vdash (A < B)$

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?
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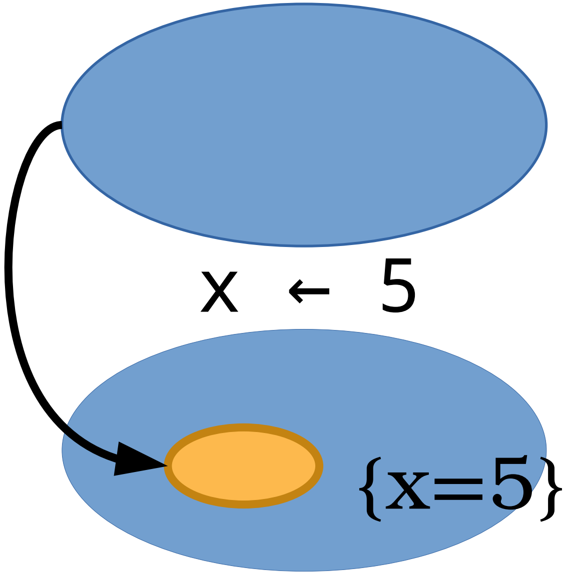

$\psi = \{x=5\}$

Hoare Logic – weakest preconditions

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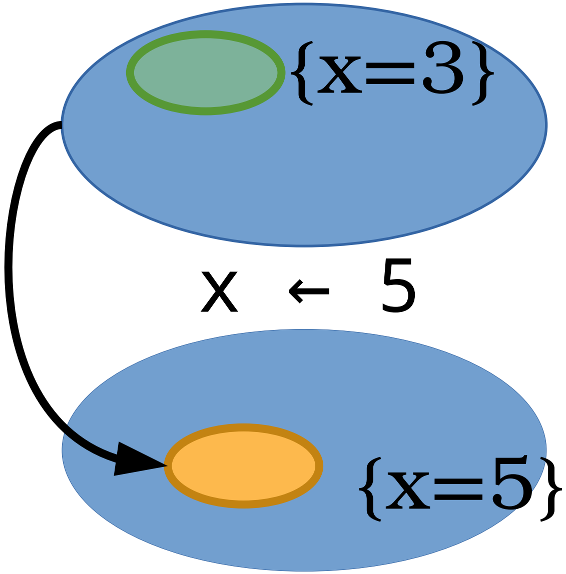

$\{x=3\}$

$\psi = \{x=5\}$

Hoare Logic – weakest preconditions

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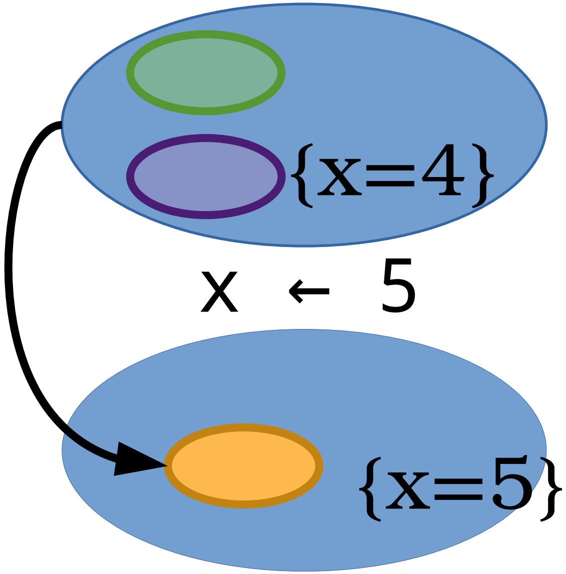

$\{x=3\} \bigsqcup \{x=4\}$
$= ?$

$\psi = \{x=5\}$

Hoare Logic – weakest preconditions

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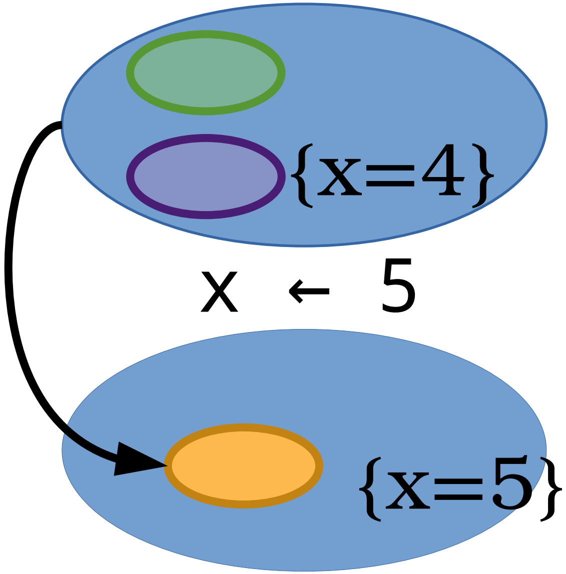

$\{x=3\} \bigsqcup \{x=4\}$

$= ?$

$\{x=3\} \rightarrow \{x=3 \lor x=4\}$

$\{x=4\} \rightarrow \{x=3 \lor x=4\}$

$\psi = \{x=5\}$

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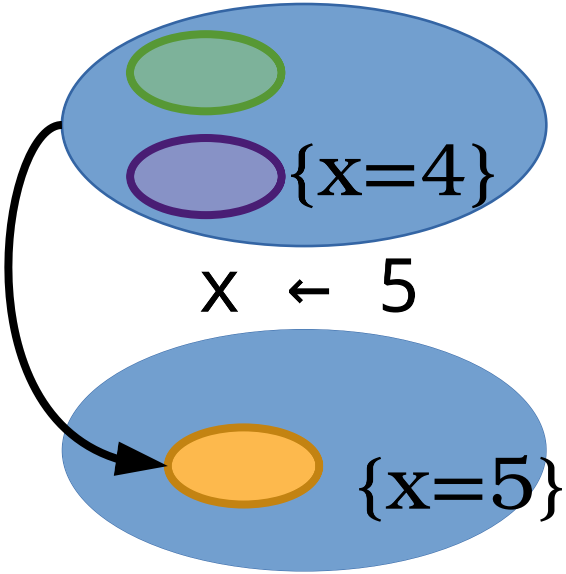

$$\{x=3\} \sqcup \{x=4\}$$
$$= \{x=3 \lor x=4\}$$

$$\{x=3\} \rightarrow \{x=3 \lor x=4\}$$
$$\{x=4\} \rightarrow \{x=3 \lor x=4\}$$

$$\psi = \{x=5\}$$

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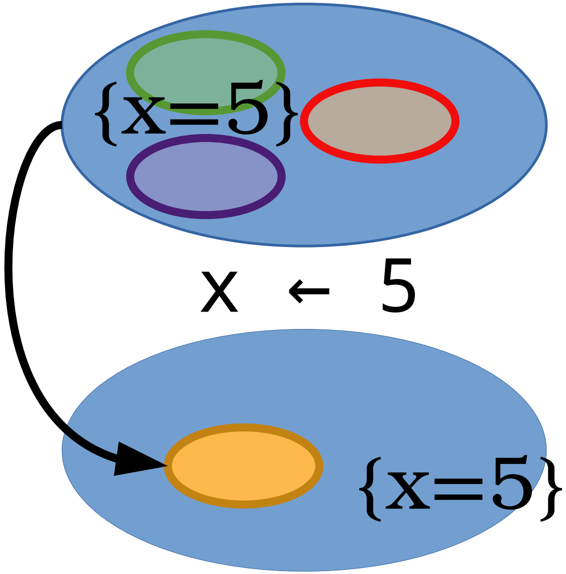

$$\{x=3\} \bigsqcup \{x=4\} \bigsqcup \{x=5\}$$
$$= \{3 \leq x \wedge x \leq 5\}$$

$$\{x=3\} \rightarrow \{3 \leq x \wedge x \leq 5\}$$
$$\{x=4\} \rightarrow \{3 \leq x \wedge x \leq 5\}$$
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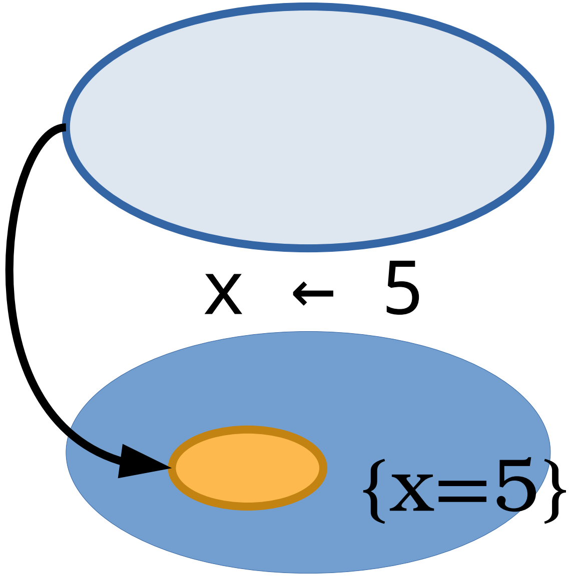

$$\{x=3\} \bigsqcup \{x=4\} \bigsqcup \{x=5\} \bigsqcup \ldots$$
$$= \{\#T\}$$

$$\{x=3\} \rightarrow \{\#T\}$$
$$\{x=4\} \rightarrow \{\#T\}$$
$$\ldots$$

$$\psi = \{x=5\}$$

Hoare Logic – weakest preconditions

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Intuitively, B is at least as general as A
(it holds in at least as many states)

Hoare Logic – weakest preconditions

- What do we really mean by captures all states?
- $wp(C, \psi) = \bigsqcup \{x \mid \{x\}\ C\ \{\psi\}\}$
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- Technically, these are ***Weakest Sufficient Preconditions***

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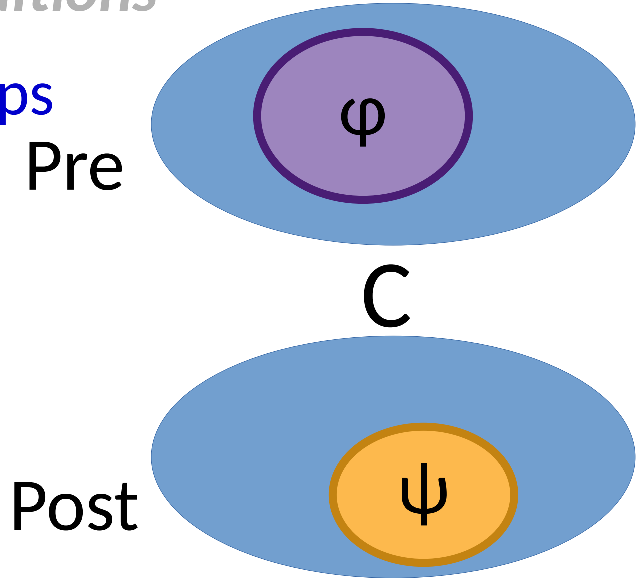

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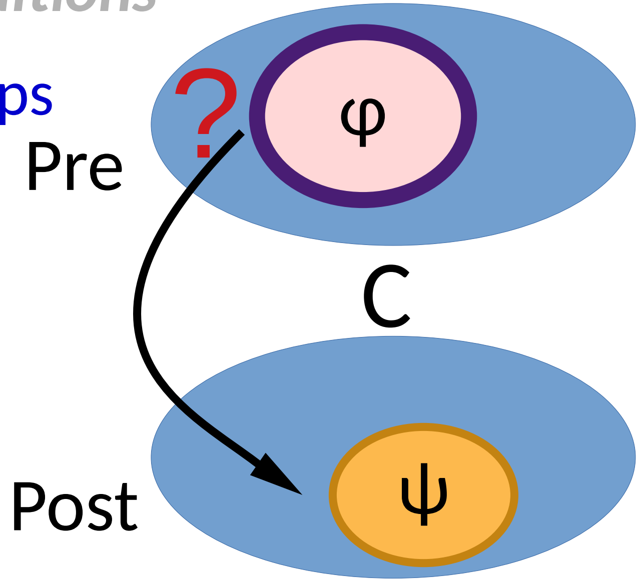
- We may also consider/compute other relationships
  - Weakest Sufficient Preconditions (wsp)

What states φ lead to ψ?

"Given ψ, what must be true for it to hold?"



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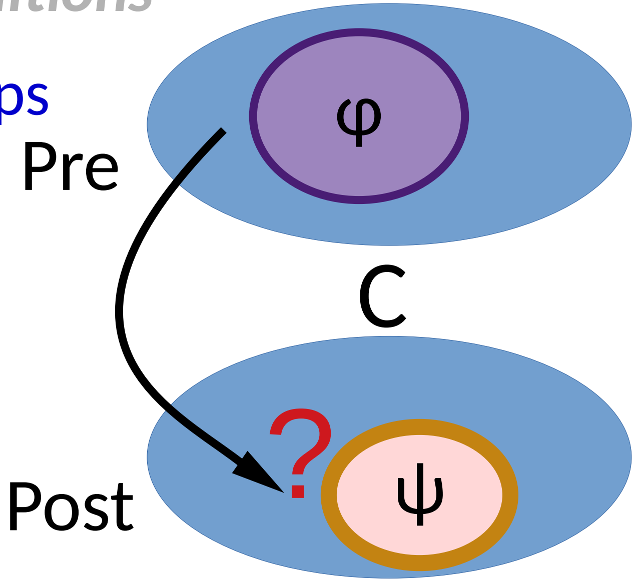
  - Weakest Sufficient Preconditions
  - Strongest Necessary Postconditions (snp)

What states ψ must φ lead to?

"Given φ, what is guaranteed when it holds?"



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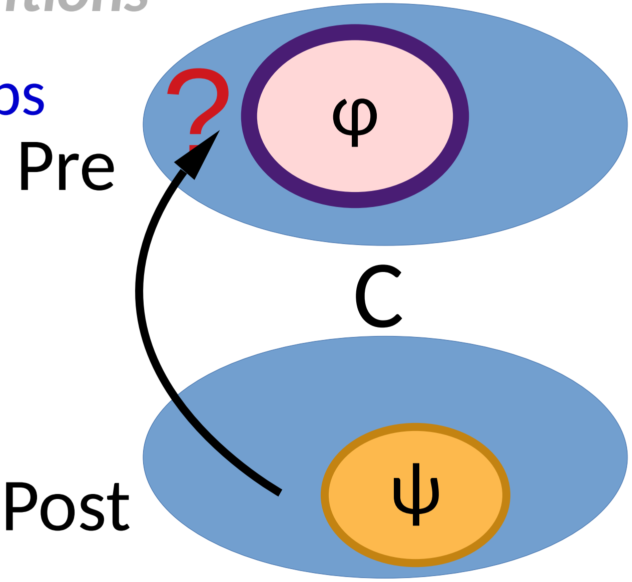

"Given ψ, what if false at φ would exclude it?"

Hoare Logic

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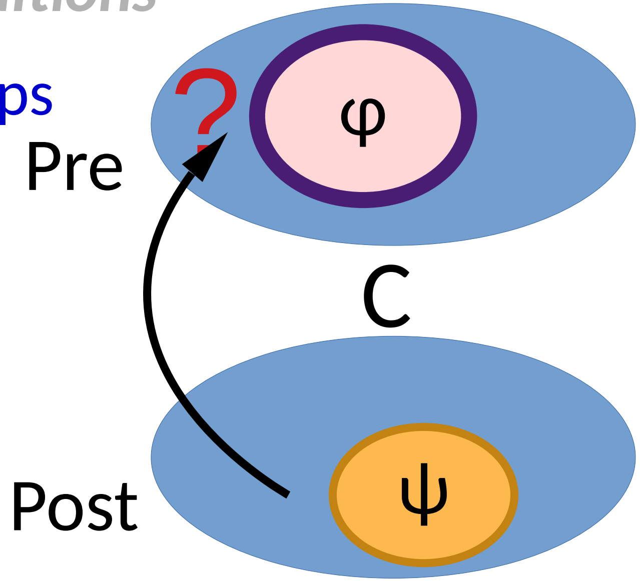
What states φ lead to ψ?

Then how does this differ from *wsp*?



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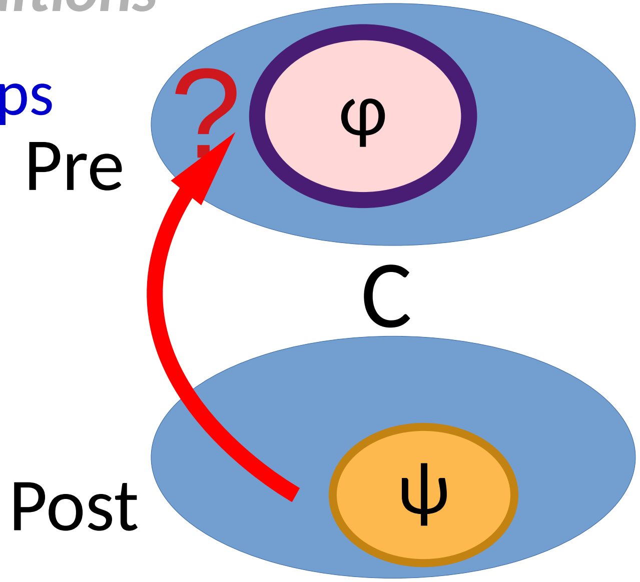

WSP

φ@pre $\rightarrow$ ψ@post

SNPre

φ@pre $\leftarrow$ ψ@post

Hoare Logic

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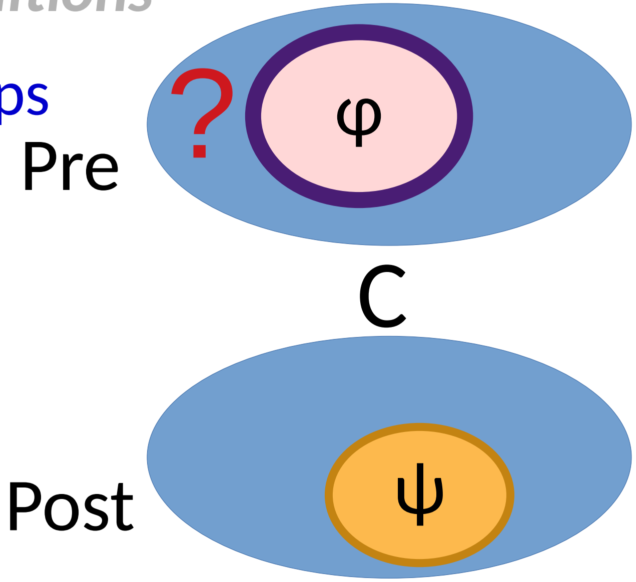
WSP SNPre

Since solving them is technically impossible, these differ in practice!



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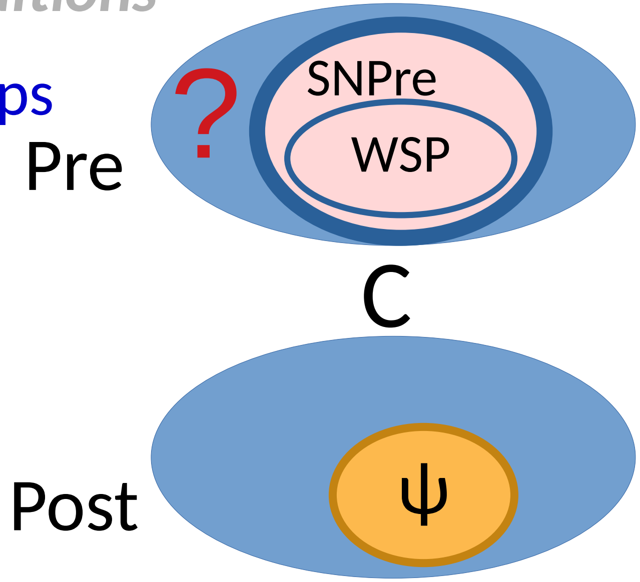

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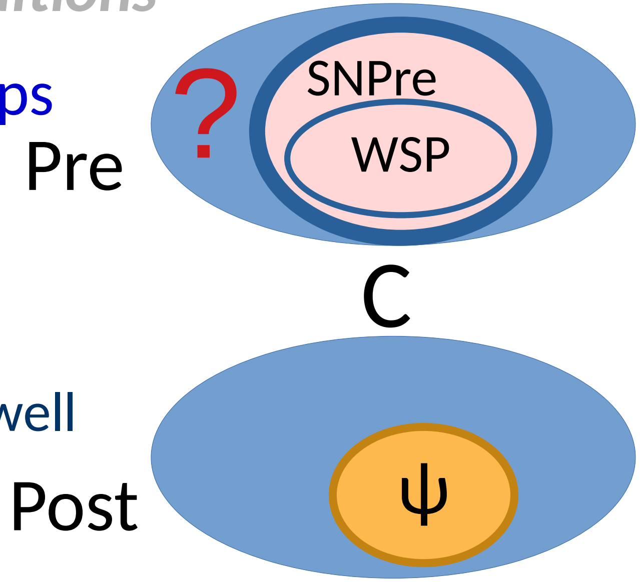
    In practice, SNPre captures *precondition assertions* well
    [Cousot 2013]



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- We may also consider/compute other relationships
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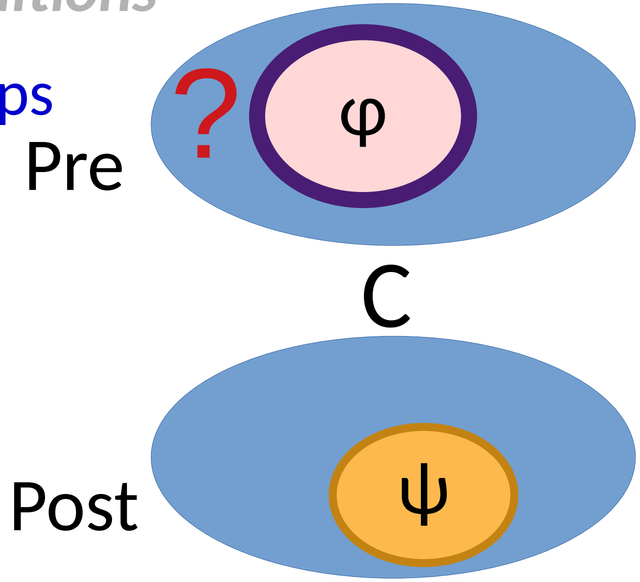
  - Strongest Necessary Postconditions
  - Strongest Necessary Preconditions
  - Weakest *Liberal* Preconditions

What states φ lead to ψ
or *do not terminate*?



Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$\mathrm{wp}(x \leftarrow E, \psi) = [E/x]\psi$$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$\mathrm{wp}(x \leftarrow E, \psi) = [E/x]\psi$$

$$\{ \quad ??? \quad \}$$

$$x \leftarrow a + b$$

$$\{x<5\}$$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$wp(x \leftarrow E, \psi) = [E/x]\psi$$

$$\{a + b < 5\}$$

x ← a + b

$$\{x<5\}$$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$wp(x \leftarrow E, \psi) = [E/x]\psi$$

$$wp(S;\ T, \psi) = wp(S,\ wp(T, \psi))$$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$\mathrm{wp}(x \leftarrow E, \psi) = [E/x]\psi$$

$$\mathrm{wp}(S;\, T, \psi) = \mathrm{wp}(S,\, \mathrm{wp}(T, \psi))$$

$\{\quad ???\quad \}$

```
b ← 7;

x ← a + b
```

$\{x<5\}$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$wp(x \leftarrow E, \psi) = [E/x]\psi$$

$$wp(S;\, T, \psi) = wp(S,\, wp(T, \psi))$$

$\{ \quad ??? \quad \}$

```
b ← 7;
```

$\{a + b < 5\}$

```
x ← a + b
```

$\{x<5\}$

Hoare Logic – weakest preconditions

- Inference rules for weakest preconditions

$$wp(x \leftarrow E, \psi) = [E/x]\psi$$

$$wp(S;\, T, \psi) = wp(S,\, wp(T, \psi))$$

$\{a + 7 < 5\}$

```
b ← 7;
```

$\{a + b < 5\}$

```
x ← a + b
```

$\{x<5\}$

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$$\mathrm{wp}(\texttt{if } B \texttt{ then } S \texttt{ else } T, \psi) = \boxed{B \to \mathrm{wp}(S, \psi)} \wedge \boxed{\neg B \to \mathrm{wp}(T, \psi)}$$

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```
if c then
  d = y + 2
else
  d = y + 5
x/d
```

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if c then
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$\{d \neq 0\}$

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	{ ??? }
`if c then`	
`  d = y + 2`	$\{y+2 \neq 0\}$
`else`	
`  d = y + 5`	$\{y+5 \neq 0\}$
`x/d`	$\{d \neq 0\}$

Hoare Logic – weakest preconditions

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```
if c then
  d = y + 2
else
  d = y + 5
x/d
```

$\{c \rightarrow y+2 \neq 0\ \land\ \neg c \rightarrow y+5 \neq 0\}$

$\{y+2 \neq 0\}$

$\{y+5 \neq 0\}$

$\{d \neq 0\}$

Hoare Logic

- Careful points
  - Redefinition of variables

Pre: {a < 5, c < 2}

```
b = a + 2
a = 3*c
```

Post: {??}

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It can be necessary to rename variables that are redefined.

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Pre: {??}

```
*a = *a + 5
```

Post: {*a + *b < 10}

Hoare Logic

- Careful points
  - Redefinition of variables
  - Pointers

Pre: {??}

```
*a = *a + 5
```

Post: {*a + *b < 10}

Efficiently modeling memory is challenging!
Newer logics target this directly.
(points-to analysis allows for ***weak*** and ***strong*** updates)

Hoare Logic

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  - Loops

Hoare Logic

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  - Redefinition of variables
  - Pointers
  - Loops

Loops run head first into undecidability!
They require deriving an *inductive invariant*.

Hoare Logic

- Careful points
  - Redefinition of variables
  - Pointers
  - Loops

$\{\varphi\}C\{\psi\}$

```
while B do S done
```

Hoare Logic

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$Inv \land \neg B \rightarrow \psi$ exit

Hoare Logic

- Careful points
  - Redefinition of variables
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$\{\varphi\}C\{\psi\}$

```
while B do S done
```

$\{Inv \land \neg B \rightarrow \psi\}$ exit

$\{Inv \land B\}\ S\ \{Inv\}$ continue

Hoare Logic

- Careful points
  - Redefinition of variables
  - Pointers
  - Loops

$\{\varphi\}C\{\psi\}$ `while B do S done`

$\{Inv \wedge \neg B \rightarrow \psi\}$ exit

$\{Inv \wedge B\}\ S\ \{Inv\}$ continue

$\{\varphi \rightarrow Inv\}$ enter

Hoare Logic

- Careful points
  - Redefinition of va
  - Pointers
  - Loops

Automatically inferring such invariants
is used for verifying safe:
avionics
machine learning
…

$\{\varphi\}C\{\psi\}$

```
while B do S done
```

$\{Inv \land \neg B \rightarrow \psi\}$ exit

$\{Inv \land B\}\ S\ \{Inv\}$ continue

$\{\varphi \rightarrow Inv\}$ enter

Separation Logic

- Linear logic allows facts to be used exactly once <> or arbitrarily many times [].

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  or arbitrarily many times [].
- *Separation logic* (informally) distinguishes separate facts (counting), allowing them to be used separately
  - This helps to solve reasoning about pointers as we saw earlier

Separation Logic

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- Hoare logic is extended with a separating conjunction *

Separation Logic

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$$\{x \mapsto y * y \mapsto x\}\texttt{x = z}\{x \mapsto z * y \mapsto x\}$$

Facts separated by * do not “mix” (overlap)

Separation Logic

- Linear logic allows facts to be used exactly once <>
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$$\{x \mapsto y * y \mapsto x\}\ x = z\ \{x \mapsto z * y \mapsto x\}$$

Suppose we used $\wedge$ instead, what problem exists?

Separation Logic

- Linear logic allows facts to be used exactly once <>
  or arbitrarily many times [].
- *Separation logic* (informally) distinguishes separate facts (counting), allowing them to be used separately
- Hoare logic is extended with a separating conjunction $*$

$$\{x \mapsto y * y \mapsto x\}\, x \;=\; z \{x \mapsto z * y \mapsto x\}$$

- Separation logic enables efficient compositional reasoning
  - It is the backbone of Facebook's Infer engine!
  - It combines Hoare logic with a substructural logic

Separation Logic

- The ***frame rule*** enables reasoning about the logical footprint of a command

$$\frac{\{\varphi\}C\{\psi\}}{\{\varphi * r\}C\{\psi * r\}}$$

Separation Logic

- The ***frame rule*** enables reasoning about the logical footprint of a command

$$\frac{\{\varphi\}C\{\psi\}}{\{\varphi * r\}C\{\psi * r\}}$$

- Part of the power is that frames can be inferred via ***bi-abduction***

Solving Problems Using Logic

Solving problems using logic

- We will look at a few ways logic can attack real problems

Solving problems using logic

- We will look at a few ways logic can attack real problems
- The exact techniques may have flaws,
  but how they attack problems with logic is interesting

Discovering & Disproving Bugs

```
foo(a,b,c) {
  if (a != null) {
    b = c;
    t = new...;
    c.f = t;
  }
  d = a;
  if (d != null) {
    b.f.g = 10;
  }
}
```

[Margoor & Komondoor, 2015]

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[Margoor & Komondoor, 2015]

Can accessing the field g cause a null pointer exception?

Discovering & Disproving Bugs

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{b.f=null}

[Margoor & Komondoor, 2015]

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$\{b.f = null \wedge d \neq null\}$

$\{b.f = null\}$

[Margoor & Komondoor, 2015]

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{b.f=null ∧ a≠null}

{b.f=null ∧ d≠null}

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{$(\boxed{b \neq c} \wedge b.f = null \wedge a \neq null) \vee (\boxed{b = c} \wedge t = null \wedge a \neq null)$}

{$b.f = null \wedge a \neq null$}

{$b.f = null \wedge d \neq null$}

{$b.f = null$}

[Margoor & Komondoor, 2015]

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```

{b≠c ∧ b.f=null ∧ a≠null}

{(b≠c ∧ b.f=null ∧ a≠null) v (b=c∧t=null ∧ a≠null)}

{b.f=null ∧ a≠null}

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{#f}

$\{b \neq c \land b.f = null \land a \neq null\}$

$\{(b \neq c \land b.f = null \land a \neq null) \lor (b = c \land t = null \land a \neq null)\}$

$\{b.f = null \land a \neq null\}$

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$\{(a \neq null \rightarrow \#f) \lor (a = null \rightarrow b.f = null \land a \neq null)\}$

$\{\#f\}$

$\{b \neq c \land b.f = null \land a \neq null\}$

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Discovering & Disproving Bugs

Safe!

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}
```

$\{(\cancel{a \neq null} \rightarrow \#f) \vee (a = null \rightarrow \cancel{b.f = null \wedge a \neq null})\}$ **= #f**

$\{\#f\}$

$\{b \neq c \wedge b.f = null \wedge a \neq null\}$

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```

[Margoor & Komondoor, 2015]

{(~~a≠null~~ → #f) v (~~a=null → b.f=null ∧~~ a≠null)} **= #f**

{#f}

{b≠c ∧ b.f=null ∧ a≠null}

{(b≠c ∧ b.f=null ∧ a≠null) v (b=c∧t=null ∧ a≠null)}

{b.f=null ∧ a≠null}

{b.f=null ∧ d≠null}

{b.f=null}

Note: this can be automated within a tool!

Localizing Bugs

[Jose & Majumdar, 2011]

```
int arr[3];
...
if (index != 1) {
  index = 2;
} else {
  index = index + 2;
}
i = index;
print(arr[i]);
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assert(0 ≤ `i` < 3) *should* hold

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When the starting `index` is 1, `i` is out of bounds

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We will generate constraints in the forward direction

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$\texttt{index}_1 = 1$

$\wedge\ (0 \leq \texttt{i} < 3)$

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Localizing Bugs

$$
\begin{aligned}
& index_1 = 1 \\
\wedge\ & guard_1 = (index_1 \neq 1) \\
\wedge\ & index_2 = 2 \\
\wedge\ & index_3 = (index_1 + 2) \\
\wedge\ & (guard_1 \rightarrow i = index_2) \\
\wedge\ & (\neg guard_1 \rightarrow i = index_3) \\
\wedge\ & (0 \leq i < 3)
\end{aligned}
$$

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\end{aligned}
$$

P

```
int arr[3];
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```

assert($0 \leq$ i < 3) *should* hold

When the starting index is 1,
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Localizing Bugs

[Jose & Majumdar, 2011]

$$
\left.\begin{array}{l}
\texttt{index}_1 = 1 \\
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\end{array}\right\} \mathrm{snp}(P,\#t)
$$

$\wedge\ (0 \leq \texttt{i} < 3)$

P

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} else {
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}
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assert(0 ≤ `i` < 3) *should* hold

When the starting `index` is 1, `i` is out of bounds

Localizing Bugs

[Jose & Majumdar, 2011]

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$\wedge\ (guard_1 \rightarrow i = index_2)$
$\wedge\ (\neg guard_1 \rightarrow i = index_3)$
$\wedge\ (0 \leq i < 3)$

This is *always false*,
but we can *use* that!

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} else {
  index = index + 2;
}
i = index;
print(arr[i]);
```

assert($0 \leq i < 3$) *should* hold

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These constraints define our goal, so they are essential

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These constraints define our goal, so they are essential

Some of these constraints *conflict* with our goal

```
if (index != 1) {
  index = 2;
[illegible]
  index = index + 2;
}
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print(arr[i]);
```

assert(0 ≤ i < 3) *should* hold

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Localizing Bugs

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```
if (index != 1) {
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  index = index + 2;
print(arr[i]);
```

Minimum unsat cores & ***partial MAX-SAT*** can discover the conflicts

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$\land\ (\neg\text{guard}_1 \rightarrow i = \text{index}_3)$
$\land\ (0 \leq i < 3)$

These constraints define our goal, so they are essential

Some of these constraints ***conflict*** with our goal

```
if (index != 1) {
    index = 2;
    index = index + 2;
print(arr[i]);
```

Minimum unsat cores & ***partial MAX-SAT*** can discover the conflicts

Must SAT

assert(0 ≤ i < 3) *should* hold

When the starting index is 1, i is out of bounds

Localizing Bugs

[Jose & Majumdar, 2011]

$$
\begin{aligned}
&\boxed{index_1 = 1}\\
&\wedge\ guard_1 = (index_1 \neq 1)\\
&\wedge\ index_2 = 2\\
&\wedge\ index_3 = (index_1 + 2)\\
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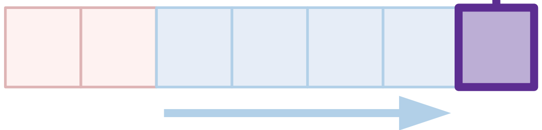

Could not SAT;
Blame for inconsistency

assert(0 ≤ i < 3) *should* hold

Must SAT   Max # satisfiable

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Further notes

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- Interpolants can help synthesize information as if “out of thin air”

Recap

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- Many solutions involve a careful combination of
  - order theory (for comparison)
  - formal grammars (for structure)
  - formal logic (for inference)