

A Language-Independent Framework for Reasoning about Preferences for Declarative Problem Solving

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Abstract. Automated decision making often requires solving difficult and primarily NP-hard problems. In many AI applications (e.g., planning, robotics, recommender systems, etc.), users can assist decision making by specifying their preferences over some domain of interest. To take preferences into account, we take a model-theoretic approach to computationally hard problems with preferences. Computational problems are characterized as Model Expansion, that is, the logical task of expanding an input structure to satisfy a set of specifications. The uniformity of the model-theoretic approach allows us to combine computational problems with preferences regardless of the syntax of their specifications. We introduce a formalism to represent preferences of users associated with computationally hard problems. We introduce Prioritized Model Expansion, which is Model Expansion based on preferences. We investigate properties of Prioritized Model Expansion and conduct a thorough study of the impact of introducing preferences on the computational complexity of Σ_k^P -complete Model Expansion problems. We also discuss how Prioritized Model Expansion is related to other preference-based declarative approaches, such as SAT with preferences and preference-based Logic Programming.

Keywords: Preference · Model Expansion · Computational Problems

1 Introduction

Solving computationally hard problems (e.g., NP-hard) is in the core of many AI tasks. Due to the significant progress in the performance of modern solvers, finding solutions to such problems (e.g., planning, travelling salesman, graph colouring, etc.) has become feasible in many applications. We view such hard problems as Model Expansion [25], which is the logical task of expanding a structure (a problem instance) to a solution structure that satisfies a formula (the problem specification) that is written in a certain language. The main declarative approaches in AI, including Satisfiability problems (SAT), Constraint Satisfaction Problems (CSP), and Answer Set Programming (ASP), can be encoded as Model Expansion [25]. By distinguishing between problem instances and problem specifications, Model Expansion provides a robust modelling framework and establishes a connection to Descriptive Complexity [23].

It is common that a decision maker prioritizes the solutions. Since preferences play a key role in AI, a large number of frameworks for handling preferences have been

proposed during the last two decades, e.g., [1,12,15,5,29]. As it was discussed in [19], these proposals are often language-dependent because they are added to a host formal language such as ASP [12,10,18] or default logic [17,11].

In real-world applications, search and decision problems often consist of a number of sub-problems that interact with each other. For example, suppose a vacation planner includes a component that generates a vacation package based on the constraints of a travel agency by solving answer set programs and a component that solves integer linear programs to find the best plan based on the needs and priorities of a traveler. The modular nature of many AI tasks necessitates the integrating of preference-based problems regardless of the language of their specifications. We tackle this issue by proposing a preference framework for Model Expansion. Given that Model Expansion underlies all predominant declarative frameworks such as ASP, SAT, and CSP, we show how our language-independent preference-based framework corresponds to these approaches with preferences, such as Answer Set Optimization [12], Logic Programming with Preferences [26], CSP with CP-nets [6], etc. The main motivation for our work is to connect model theory, descriptive complexity, and preference modelling to study computationally hard problems with preferences. To the best of our knowledge, this is the first proposal of this kind in the literature. We propose that preferences are expressed as an ordering relation on ground atoms. The relation among atoms is lifted to a preference ordering relation among structures using a number of different preference semantics (lifting methods). We define the *Dominant Structure* problem, which is the problem of deciding whether a structure is preferred to another structure. We prove that solving the *Dominant Structure* problem is polynomial in the size of the domain of structures. Our framework allows other methods of lifting as long as the *Dominant Structure* problem is polynomial, which makes the following complexity results applicable.

A model-theoretic view on modular systems with preferences was presented in [21]. The authors showed the connection of their proposal to other formalisms such as CP-nets [5]. They used Codd's relational algebra [16] to combine modules with preferences. The combination was static – the authors did not focus on Model Expansion.

Our main contributions are as follows: First, we introduce the notion of Prioritized Model Expansion, a declarative framework for specifying computational problems with preferences. Prioritized Model Expansion extends Model Expansion by modelling preferences of a decision maker. Second, we show that adding preferences even in the simplest formulations leads to a rise of the computational complexity of Σ_k^P -complete Model Expansion problems to Σ_{k+1}^P -complete Model Expansion problems. Third, we study the relations of some preference-based frameworks with Prioritized Model Expansion. We apply the computational complexity result to some associated reasoning tasks to obtain similar results for those frameworks.

2 Background

A τ -structure $\mathcal{A} = (A, R_1^{\mathcal{A}}, \dots, R_n^{\mathcal{A}}, C_1^{\mathcal{A}}, \dots, C_m^{\mathcal{A}})$ is a tuple where τ is a vocabulary, which is a set of non-logical relation symbols R_i with associated arity k_i where $i \in [1, n]$ and constant symbols C_j where $j \in [1, m]$ and A is a domain. For any $R_i \in \tau$, $R_i^{\mathcal{A}}$ is called the interpretation of R_i and $R_i^{\mathcal{A}} \subseteq A^{k_i}$. Also, for every $C_j \in \tau$, $C_j^{\mathcal{A}} \in Dom$.

A structure is called finite if its domain is finite. Throughout this paper, we assume that all structures are finite. For a formula ψ in a logic $\mathcal{L}$, $\text{vocab}(\psi)$ denotes the set of vocabulary symbols appearing in ψ . The definition of expansion is standard in logic literature and is defined inductively as follows: Let $\mathcal{A}$ be a σ -structure and $\mathcal{B}$ be a τ -structure where $\sigma \subseteq \tau$. $\mathcal{B}$ expands $\mathcal{A}$ if the domain of $\mathcal{B}$ is the same as the domain of $\mathcal{A}$ and for all $R \in \sigma$, $R^{\mathcal{A}} = R^{\mathcal{B}}$. Model Expansion (MX) is the task of expanding a structure to satisfy a formula in logic $\mathcal{L}$ [25]. The Model Expansion problem is defined as follows:

Definition 1. (Model Expansion Problem $\text{MX}_{\sigma,\psi}$)

Given: an arbitrary σ -structure $\mathcal{I}$ over a finite domain Dom , formula ψ in a logic $\mathcal{L}$ where vocabulary $\sigma \subseteq \text{vocab}(\psi)$,

Find a τ -structure $\mathcal{A}$ where $\mathcal{A} \models \psi$ and expands $\mathcal{I}$. (The decision version: is there a τ -structure $\mathcal{A}$ such that $\mathcal{A} \models \psi$ and $\mathcal{A}$ expands $\mathcal{I}$?)

We call $\mathcal{A}$ an *expansion structure* of $\text{MX}_{\sigma,\psi}$. Each expansion structure $\mathcal{A}$ is a τ -structure with domain Dom . In this paper, we are interested in data complexity (ψ is fixed and the domain of input is variable). For logic $\mathcal{L}$, the data complexity of Model Expansion (MX) is always in-between model checking (MC) and satisfiability (SAT). For example, for first-order logic, MC is AC_0 , MX is in NP, and SAT is undecidable. Graph colouring can be characterized as a first-order Model Expansion problem (i.e., the problem specification is in first-order logic) as follows:

Example 1. Let E be a binary relation. Let unary relation symbols R, G , and B denote red, green, and blue, respectively. The following formula specifies three-colouring:

$$\begin{aligned} \psi = & \forall x [(R(x) \vee B(x) \vee G(x)) \\ & \wedge \neg((R(x) \wedge B(x)) \vee (R(x) \wedge G(x)) \vee (B(x) \wedge G(x))) \\ & \wedge \forall x \forall y [E(x, y) \supset (\neg(R(x) \wedge R(y)) \\ & \wedge \neg(B(x) \wedge B(y)) \wedge \neg(G(x) \wedge G(y)))]]. \end{aligned}$$

A graph $\mathcal{G} = (V, E)$ is an instance structure with vocabulary $\sigma = \{E\}$ and domain V , which is the set of vertices. Model Expansion problem $\text{MX}_{\{E\},\psi}$ finds expansion structure $\mathcal{A}$ (i.e., three-colouring of $\mathcal{G}$) that interprets symbols R, B , and G satisfying ψ as follows:

$$\underbrace{(V; E^{\mathcal{G}}, R^{\mathcal{A}}, B^{\mathcal{A}}, G^{\mathcal{A}})}_{\mathcal{A}} \models \psi.$$

Computational complexity, a Review: Let X be a complexity class.

Notation 1. P^X is the class of languages (complexity class) that can be computed in polynomial time by a deterministic Turing machine with an oracle in X . Also, NP^X is the class of languages that can be computed in polynomial time by a nondeterministic Turing machine using an oracle in X .

Notation 2. $\text{co-}X$ is the complexity class of decision problems whose complements are in X .

The Polynomial Hierarchy (PH) is defined as $\Sigma_0^P = \Pi_0^P = \Delta_0^P = P$, $\Sigma_{k+1}^P = \text{NP}^{\Sigma_k^P}$, $\Delta_{k+1}^P = P^{\Delta_k^P}$, and $\Pi_{k+1}^P = \text{coNP}^{\Sigma_k^P}$ for $k > 0$.

3 Model Expansion with Preferences

In this section, we introduce the notion of Prioritized Model Expansion (PMX). We study the computational complexity of solving problems related to the PMX including *Dominant Structure* (i.e., given two structures, whether one is preferred to another), *Optimal Expansion* (i.e., given a structure, whether it is an optimal expansion of a Model Expansion problem), and *Goal-Oriented Optimal Expansion* (i.e., deciding whether there is an optimal expansion satisfying a certain goal).

3.1 Preference Expression

Let Dom be a domain of elements. Consider first-order variables $X = \{x_1, \dots, x_r\}$ over Dom , vocabulary τ , and k -ary $R \in \tau$. Let $\nu : X \rightarrow Dom$ be an assignment function that assigns a domain element to each variable. For an ordered set of variables $\bar{x} = (x_1, \dots, x_k)$, we call $\bar{a} = (a_1, \dots, a_k)$ a k -ary tuple when there is an assignment ν such that for $1 \leq i \leq k$, $\nu(x_i) = a_i$. We use the symbol $\bar{a}[x_i]$ to denote value a_i . For k -ary predicate symbol $R \in \tau$, we call $R(\bar{a})$ a ground atom of τ over Dom . We say a structure $\mathcal{A}$ satisfies a ground atom $R(\bar{a})$ (notation $\mathcal{A} \models R(\bar{a})$) if $\bar{a} \in R^{\mathcal{A}}$.

Definition 2. (Preference Expression) A preference expression P over Dom is defined as a pair $P = (\mathcal{S}_\tau, \sqsubseteq_P)$ where $\mathcal{S}_\tau$ is the set of all ground atoms of vocabulary τ over Dom and $\sqsubseteq_P$ is a preorder on $\mathcal{S}_\tau$.

Let $R(\bar{a})$ and $T(\bar{b})$ be ground atoms where k -ary predicate R and k' -ary predicate T are in τ , k -ary tuple $\bar{a} \in Dom^k$, and k' -ary tuple $\bar{b} \in Dom^{k'}$. The expression $R(\bar{a}) \sqsubseteq_P T(\bar{b})$ is read as $R(\bar{a})$ is preferred to $T(\bar{b})$. Also, $R(\bar{a})$ is called strictly preferred to $T(\bar{b})$ with notation $R(\bar{a}) \sqsubset_P T(\bar{b})$ if $R(\bar{a}) \sqsubseteq_P T(\bar{b})$ is true and $T(\bar{b}) \sqsubseteq_P R(\bar{a})$ does not hold.

We shall introduce a preference ordering $\geq_P^s$ on structures based on a preference expression P and a preference semantics (lifting method) s . Lifting method s specifies how $\geq_P^s$ is constructed from $\sqsubseteq_P$. Comparing two sets with members that are prioritized has been widely studied in different areas, such as in database systems [27], or even beyond the realm of theoretical computer science, such as in economics and decision theory [14]. Inspired by [13, 27, 1, 9], here we introduce three different methods to lift a preference ordering on ground atoms to a preference ordering on structures. In each of the following methods, relation $\geq_P^s$ among τ -structures with the same domain is constructed from $\sqsubseteq_P$.

Definition 3. (Preference Relations on Structures)

Given a preference expression $P = (\mathcal{S}_\tau, \sqsubseteq_P)$ over domain Dom , let $\mathcal{A}$ and $\mathcal{B}$ be two τ -structures with domain Dom ,

- **Weak Pareto (WP).** $\mathcal{A} \geq_P^{wp} \mathcal{B}$ iff for all $R, S \in \tau$ and for all $\bar{a} \in R^{\mathcal{A}}$ and all $\bar{b} \in S^{\mathcal{B}}$, $R(\bar{a}) \sqsubseteq_P S(\bar{b})$.
- **Upper Bound Dominance (UBD).** $\mathcal{A} \geq_P^{ubd} \mathcal{B}$ iff for all $S \in \tau$ and for all $\bar{b} \in S^{\mathcal{B}}$, for some $R \in \tau$, there is $\bar{a}$ such that $\bar{a} \in R^{\mathcal{A}}$ and $R(\bar{a}) \sqsubseteq_P S(\bar{b})$.

- **Element Dominance (ED).** $\mathcal{A} \geq_P^{ed} \mathcal{B}$ iff for some $R, S \in \tau$, there is $\bar{b} \in S^{\mathcal{B}}$ and there is $\bar{a} \in R^{\mathcal{A}}$ such that $R(\bar{a}) \sqsupseteq_P S(\bar{b})$ and there is no $\bar{c}$ for some $T \in \tau$ such that $\bar{c} \in T^{\mathcal{B}}$ and $T(\bar{c}) \sqsupseteq_P R(\bar{a})$.

The Weak Pareto semantic uses the idea of Pareto dominance [28] such that $\mathcal{A}$ is preferred to $\mathcal{B}$ if every ground atom that is satisfied by $\mathcal{A}$ is at least as preferred as any ground atom which $\mathcal{B}$ satisfies. In a stronger version, all ground atoms that are satisfied by $\mathcal{A}$ must be at least as preferred as ground atoms satisfied by $\mathcal{B}$ except one ground atom satisfied by $\mathcal{A}$ that is strictly preferred to a ground atom that $\mathcal{B}$ satisfies. The Upper Bound Dominance approach is a weaker version of the Weak Pareto such that if $\mathcal{A}$ satisfies any ground atom that is at least as preferred as the maximal element (based on preorder $\sqsupseteq_P$) of atoms satisfied by $\mathcal{B}$, then $\mathcal{A}$ is preferred to $\mathcal{B}$. Finally, based on the Element Dominance semantics, there is an adequate reason to drive that $\mathcal{A}$ is preferred to $\mathcal{B}$ if there is a ground atom, say $R(\bar{a})$, that $\mathcal{A}$ satisfies and is preferred to some ground atoms satisfied by $\mathcal{B}$ and no ground atom satisfied by $\mathcal{B}$ is strictly preferred to $R(\bar{a})$.

These semantics (lifting methods) may illustrate similar, or in some cases, different behavior. For example, the computational complexity of reasoning tasks associated with preferred models using each of these semantics is the same. Also, relation $>_P^y$ is transitive when y is the Upper Bound Dominance or the Weak Pareto semantics. However, if y is the Element Dominance semantics, $>_P^y$ is not necessarily transitive. For example, consider the case where the relational vocabulary is $\tau = \{R\}$ and the domain is $Dom = \{1, 2, 3, 4\}$. Assume the preorder over ground atoms is defined as $R(1) \sqsupseteq R(2)$ and $R(3) \sqsupseteq R(4)$. Assume $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are τ -structures such that $R^{\mathcal{A}} = \{1\}$, $R^{\mathcal{B}} = \{2, 3\}$, and $R^{\mathcal{C}} = \{4\}$. One can check that for the Element Dominance semantics, $\mathcal{A} >^{ed} \mathcal{B}$, $\mathcal{B} >^{ed} \mathcal{C}$, but $\mathcal{A} >^{ed} \mathcal{C}$ does not hold. However, the transitivity is not a requirement for relation $>_P^y$ in solving *Optimal Expansion* and *Goal-oriented Optimal Expansion* problems. We note that the lifting methods are not limited to what was proposed in Definition 3. Other lifting methods are allowed if the *Dominant structure* problem remains in polynomial time. Defining different preference semantics on the condition that the *Dominant Structure* problem is in polynomial time gives us the alternatives to pick stronger or weaker versions of preference semantics based on a particular application while the complexity of associated reasoning tasks does not vary.

The strict version of $\geq_P^s$ where $s \in \{wp, ubd, ed\}$ is defined as $\mathcal{A} >_P^s \mathcal{B}$ if $\mathcal{A} \geq_P^s \mathcal{B}$ and $\mathcal{B} \not\geq_P^s \mathcal{A}$ does not hold. Also, we say $\mathcal{A}$ is *dominant* to $\mathcal{B}$ based on a preference semantics $s \in \{wp, ubd, ed\}$ whenever $\mathcal{A} >_P^s \mathcal{B}$.

Definition 4. (Dominant Structure Problem)

Input: a preference expression $P = (S_\tau, \sqsupseteq_P)$ over Dom , τ -structures $\mathcal{A}$ and $\mathcal{B}$ with domain Dom , and a preference semantics $s \in \{wp, ubd, ed\}$,

Question: is $\mathcal{A} >_P^s \mathcal{B}$?

The following result indicates that the problem of deciding whether a structure is dominant to another structure using one of the preference semantics in Definition 3 is solvable in polynomial time in the size of the domain of the structures.

Proposition 1. *The Dominant Structure problem is solvable in polynomial time in the size of Dom .*

Proof: As stated by Definition 3, at most, we compare all tuples in $R^{\mathcal{A}}$ and $R^{\mathcal{B}}$ for all $R \in \tau$. The total possible number of k -ary tuples is $|Dom|^k$ where k is the maximum arity of predicate symbols in τ . Therefore, $\mathcal{O}(|Dom|^{2k})$ comparisons are required for each $R \in \tau$. Thus, deciding whether $\mathcal{A} >_P^s \mathcal{B}$ is in $\mathcal{O}(m \cdot |Dom|^{2k})$ (polynomial in the size of Dom) where m is the number of elements in τ . ■

We note that vocabulary τ is considered to be fixed and our discussion of computational complexity is focused on the size of the domain of $\mathcal{A}$ and $\mathcal{B}$ in the *Dominant Structure* problem.

3.2 Prioritized Model Expansion

We characterize search and decision problems with preferences as Prioritized Model Expansion (PMX), which is the task of expanding an input structure to the most preferred expansion structures with respect to a preference expression.

Definition 5. (Prioritized Model Expansion Problem)

Input: an arbitrary input σ -structure $\mathcal{I}$, formula ψ , input vocabulary $\sigma \subseteq \text{vocab}(\psi)$, a preference expression $P = (\mathcal{S}_\tau, \sqsupseteq_P)$ over the domain of $\mathcal{I}$, and a preference semantics s

Find: structure $\mathcal{A}$ such that $\mathcal{A}$ is an expansion structure of $MX_{\sigma,\psi}$ and there is no expansion structure $\mathcal{B}$ such that $\mathcal{B} >_P^s \mathcal{A}$.

Notation 3. $\Pi_{\sigma,\psi} = (MX_{\sigma,\psi}, P)$ is a Prioritized Model Expansion problem (based on a preference semantics s) where $MX_{\sigma,\psi}$ is a Model Expansion problem and $P = (\mathcal{S}_\tau, \sqsupseteq_P)$ is a preference expression over the domain of the input structure.

Each solution of a Prioritized Model Expansion problem $\Pi_{\sigma,\psi}$ is called an optimal expansion of $\Pi_{\sigma,\psi}$. It is worth noting that all expansion structures retain the domain of input structure $\mathcal{I}$. Also, ψ and $\text{vocab}(\psi)$ are assumed to be fixed. Hereafter, unless it is mentioned otherwise, we only study the data complexity of associated reasoning tasks. The data complexity points to the fact that the domain of input $\mathcal{I}$ (and hence the size of $\mathcal{I}$) varies while the vocabulary of input (i.e., σ) and specifications (and therefore $\text{vocab}(\psi) = \tau$) are considered to be fixed.

Example 2. Consider the problem of graph colouring that was described as Model Expansion in Example 1. Let $\mathcal{G} = (V, E)$ be the input graph where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E^{\mathcal{G}} = \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_1, v_3), (v_3, v_1), (v_2, v_4), (v_4, v_2), (v_4, v_5), (v_5, v_4), (v_3, v_5), (v_5, v_3)\}$. Assume that we prefer red for v_1 . Also, a red v_4 is favoured over a red v_5 and a blue v_2 is preferred to a green v_2 . These preference statements can be encoded by a preference expression P such that $R(v_1) \sqsupseteq_P B(v_1)$ and $R(v_1) \sqsupseteq_P G(v_1)$. Also, $R(v_4) \sqsupseteq_P R(v_5)$ and $B(v_2) \sqsupseteq_P G(v_2)$. Prioritized Model Expansion problem $\Pi_{\{E\},\psi} = (MX_{\{E\},\psi}, P)$ where $MX_{\{E\},\psi}$ is the characterization of three-colouring for input graph $\mathcal{G}$ and P is the preference expression. The input graph $\mathcal{G}$ has 18 possible three-colourings. Among these solutions, $\mathcal{A}$ is an optimal expansion of $\mathcal{G}$ (based on the Element Dominance semantics) where $R^{\mathcal{A}} = \{v_1, v_4\}$, $B^{\mathcal{A}} = \{v_2, v_5\}$, and $G^{\mathcal{A}} = \{v_3\}$.

In the rest of this subsection, we discuss some decision problems that are associated with Prioritized Model Expansion.

Definition 6. (Optimal Expansion Problem)

Input: a τ -structure $\mathcal{A}$ and a Prioritized Model Expansion problem $\Pi_{\sigma,\psi} = (MX_{\sigma,\psi}, P)$ based on a preference semantics s where $MX_{\sigma,\psi}$ is a Model Expansion problem with an arbitrary input σ -structure $\mathcal{I}$ such that $\sigma \subseteq \tau = \text{vocab}(\psi)$ and $P = (\mathcal{S}_\tau, \sqsupseteq_P)$ is a preference expression over the domain of $\mathcal{I}$,
Question: Is $\mathcal{A}$ an optimal expansion of $\Pi_{\sigma,\psi}$?

Proposition 2. For a Model Expansion problem $MX_{\sigma,\psi}$, let model checking of ψ (given a structure $\mathcal{A}$, decide if $\mathcal{A} \models \psi$) be in a complexity class Y . The problem of Optimal Expansion is in co-NP^Y .

Proof: The complementary problem is deciding whether there is an expansion structure $\mathcal{B}$ such that $\mathcal{B} >_P^s \mathcal{A}$. The complementary problem can be solved by a non-deterministic polynomial Turing machine guessing $\mathcal{B}$ with access to an oracle in Y that decides whether $\mathcal{B}$ is an expansion of $MX_{\sigma,\psi}$ (this includes checking if $\mathcal{B}$ expands $\mathcal{I}$ in polynomial time and whether $\mathcal{B} \models \psi$ in complexity Y) and, based on Proposition 1, in polynomial time checks whether $\mathcal{B} >_P^s \mathcal{A}$. Thus, the complementary problem is in NP^Y and the original problem is in co-NP^Y . ■

To put the impact of defining preferences in perspective, consider that deciding whether a given structure $\mathcal{A}$ is an expansion of $MX_{\sigma,\psi}$ is in complexity Y because it is required to first check whether $\mathcal{A}$ is an expansion of $\mathcal{I}$ that can be decided in polynomial time and then determine whether $\mathcal{A} \models \psi$ in complexity Y . On the other hand, deciding whether $\mathcal{A}$ is an optimal expansion is in co-NP^Y . In fact, to decide whether a structure is an optimal expansion, not only must it be verified as an expansion, but it must also be compared to all possible expansion structures.

One of the common tasks in many AI applications is to determine if a certain goal is achieved by solutions to a problem, e.g., in automated AI planning [2]. In the context of Prioritized Model Expansion, we ask whether there is an optimal expansion that satisfies a certain formula (goal). The problem is formulated as follows:

Definition 7. (Goal-Oriented Optimal Expansion Problem)

Input: $\Pi_{\sigma,\psi} = (MX_{\sigma,\psi}, P)$ where $MX_{\sigma,\psi}$ is a Model Expansion problem with an input σ -structure $\mathcal{I}$ with a domain Dom , $\tau = \text{vocab}(\psi)$, $P = (\mathcal{S}_\tau, \sqsupseteq_P)$ is a preference expression over the domain of $\mathcal{I}$, and ϕ is a formula of the form $R_i(\bar{a}_j) \wedge \dots \wedge R_l(\bar{a}_k)$ where $R_i, \dots, R_l \in \tau$ and $R_i(\bar{a}_j), \dots, R_l(\bar{a}_k)$ are ground atoms over Dom where every element of Dom is also a constant in vocabulary τ that is interpreted as itself by every expansion structure of $MX_{\sigma,\psi}$,
Question: Is there an optimal expansion $\mathcal{A}$ of $\Pi_{\sigma,\psi}$ such that $\mathcal{A} \models \phi$?

Proposition 3. Let solving Optimal Expansion problem $\Pi_{\sigma,\psi} = (MX_{\sigma,\psi}, P)$ be in the complexity class X . The problem of Goal-Oriented Optimal Expansion is in NP^X .

Proof: First, we non-deterministically guess a τ -structure $\mathcal{A}$ and in polynomial time check if $\bar{a} \in R^{\mathcal{A}}$, for all ground atoms $R(\bar{a})$ appearing in ϕ , that can be done by

means of a non-deterministic polynomial Turing machine. Second, we check whether our guess is an optimal expansion that is in complexity class X by the assumption. Thus, the problem can be solved by a non-deterministic polynomial Turing machine using an oracle in X . Hence, the problem is in NP^X . ■

A generalization of the *Goal-Oriented Optimal Expansion* problem is to find an optimal expansion satisfying a formula ϕ in a certain logic $\mathcal{L}^*$. In this case, the complexity of model checking in logic $\mathcal{L}^*$ (i.e., given a structure $\mathcal{A}$ if $\mathcal{A} \models \phi$?) is taken into account. However, for the sake of simplicity, in this chapter we consider goal ϕ as a conjunction of ground atoms. Hence, deciding whether a structure $\mathcal{A}$ satisfies ϕ can be verified in polynomial time.

Prioritized Σ_k^P -complete Model Expansion problems In this subsection, we discuss the computational complexity impact of introducing preferences on Σ_k^P -complete Model Expansion problems. As was discussed in [25], any boolean query computable in NP can be expressed as a first-order Model Expansion $MX_{\sigma, \psi}$ where ψ is a first-order formula. Based on Fagin's theorem [22], NP is the class of boolean queries expressible in existential second-order logic ($\exists SO$). This shows that a first-order MX and existential second-order logic have the same expressive power. Similarly, the Polynomial Hierarchy is the set of boolean queries expressible in second-order logic and any query computable in Σ_k^P ($k > 1$) can be encoded as $MX_{\sigma, \psi}$ where ψ is a formula of the form $Q_1, \dots, Q_{k-1} \psi^*$ such that for $1 \leq i \leq k$, Q_i s are alternating second-order quantifiers (alternation between $\exists$ and $\forall$), $Q_1 = \forall$, Q_{k-1} is $\forall$ if k is even and $\exists$ otherwise, and ψ^* is a first-order formula.

If the decision version of a Model Expansion problem $MX_{\sigma, \psi}$ is in Σ_k^P , then the problem has second-order specification ψ of the form $Q_1, \dots, Q_{k-1} \psi^*$. Thus, the complexity of model checking of ψ is in Π_{k-1}^P and, hence, based on Proposition 2, solving the Optimal Expansion problem $(MX_{\sigma, \psi}, P)$ is in Σ_k^P . For Σ_1^P -complete Model Expansion problems, we show that the problem of deciding the existence of minimal solutions to an abductive logic program [20] satisfying a goal can be reduced to Goal-Oriented Optimal Expansion similar to [26]. An abductive logic program is defined as $ALP = \langle H, M, \mathcal{P} \rangle$ over a set A of propositional atoms where $\mathcal{P}$ is a logic program, $H \subseteq A$ is called hypothesis and $M \subseteq A \cup \{\neg a \mid a \in A\}$ is manifestation. A solution of ALP is a set $N \subseteq H$ such that there is a stable model S of $\mathcal{P} \cup N$ and $M \subseteq S$. A solution N is called (H) minimal if there is no solution N' such that $N' \subset N$. For a given hypothesis $h \in H$, deciding whether there is a minimal solution N such that $h \in N$ is Σ_2^P -complete.

For problems in the higher levels of the Polynomial Hierarchy, we consider the following: Σ_k^P -complete problems can be encoded as a combined logic program [4]. $\Pi = (\mathcal{P}_g, \mathcal{P}_t)$ is called a combined logic program where $\mathcal{P}_g$ and $\mathcal{P}_t$ are logic programs over a set of propositional variables G and T respectively. M is a model of Π if it is a stable model of $\mathcal{P}_g$ and there is not a stable model N of $\mathcal{P}_t$ such that $M \cap G = N \cap T$. The decision version of this problem is Σ_2^P -complete. Recursively, the existence of a model of a combined program in depth 2 defined as $\Pi_2 = (\mathcal{P}_{g_2}, (\mathcal{P}_{g_1}, \mathcal{P}_t))$ is Σ_3^P -complete and, similarly, in depth k , the existence of a model of $(\mathcal{P}_{g_{k-1}}, \Pi_{k-2})$ is Σ_k^P -complete. We introduce abductive combined program as $\mathcal{C} = \langle H, M, \Pi \rangle$ where $\Pi = (\mathcal{P}_g, \mathcal{P}_t)$ is a combined logic program. W is a solution of $\mathcal{C}$ if there is a model S of

$(\mathcal{P}_g \cup W, \mathcal{P}_t)$ such that $M \subseteq S$. W is minimal if there is not a solution W' such that $W' \subset W$.

Lemma 1. *The problem of deciding whether $\mathcal{C} = \langle H, M, \Pi_k \rangle$ for a given $h \in H$ has a minimal solution containing h is Σ_{k+1}^P -complete.*

Proof: The proof includes a translation from a quantified boolean formula (QBF) to $\mathcal{C}$ for $k = 2$ and then, by induction on k for $k > 2$, the result follows. Let φ be a boolean formula in CNF and $X = \{x_1, \dots, x_m\}$, $W = \{w_1, \dots, w_m\}$, $X' = \{x'_1, \dots, x'_m\}$, $Y = \{y_1, \dots, y_n\}$, and $Z = \{z_1, \dots, z_l\}$ be a set of boolean variables in φ . Let t , h , and f also be boolean variables. Consider $\mathcal{P}_g$ to be a set of rules of the form $\{t \leftarrow x_i, x'_i\}$, $\{w_i \leftarrow x_i\}$, $\{w_i \leftarrow x'_i\}$, $\{t \leftarrow y_1, \dots, y_n, h\}$, and $\{f \leftarrow l_1, \dots, l_r\}$ where $\neg(l_1 \wedge \dots \wedge l_r) \in \varphi$ similar to [20]. For $X \cup X' \subseteq H$, an H -minimal solution of $\langle H, \{t\} \cup W, \mathcal{P}_g \rangle$ does not contain f and it has either x_i or x'_i . On the other hand, similar to [4], assume $\mathcal{P}_t$ determines the truth value of a set of boolean variables Z . Also, for each clause $C \in \varphi$, suppose that $\mathcal{P}_t$ includes a set of rules of the form $t \leftarrow \neg C$ and $f \leftarrow \neg f, t$ that means t must not be in any stable model of $\mathcal{P}_t$. This implies that the validity of $\exists X \forall Y \exists Z \varphi$ is equivalent to the existence of an H -minimal solution of $\mathcal{C}$ that contains h . So, for $k = 2$, the existence of a minimal solution to an abductive combined logic program containing an atom h is Σ_3^P -complete. ■

The following result shows the impact of introducing preferences to Σ_k^P -complete Model Expansion problems.

Theorem 1. *Let the decision version of a Model Expansion problem $MX_{\sigma, \psi}$ be Σ_k^P -complete. The problem of Goal-Oriented Optimal Expansion for $MX_{\sigma, \psi}$ is Σ_{k+1}^P -complete.*

Proof: The membership to Σ_{k+1}^P follows from the results of Proposition 2, Proposition 3, and properties of Model Expansion. Since the Model Expansion problem is in Σ_k^P , it has a second-order specification with $k - 1$ number of alternations between second-order quantifiers. Therefore, the complexity of model checking of ψ is in Π_{k-1}^P . Thus, based on proposition 2, the complexity of the Optimal Expansion problem is in $\text{co-NP}^{\Sigma_{k-1}^P}$ that is equal to Π_k^P . Also, based on Proposition 3, the Goal-Oriented Optimal Expansion problem is in $\text{NP}^{\Pi_k^P}$ or $\text{NP}^{\Sigma_k^P}$ that is equal to Σ_{k+1}^P .

For the proof of hardness, we consider an abductive logic program $ALP = \langle H, M, \mathcal{P} \rangle$. Let us define a logic program $\mathcal{P}'$ as a set of rules of the form $r : R(\bar{a}) \leftarrow \neg S(\bar{b})$ for any $R(\bar{a}) \sqsupseteq_P S(\bar{b})$ such that $S(\bar{b}) \in H$. Rule r says that a better conclusion is drawn from not making a less preferred assumption. Define $\mathcal{P}^* = \mathcal{P} \cup \mathcal{P}'$. The problem of deciding the existence of a stable model of a logic program is NP-complete and it can be translated into the decision version of a Model Expansion problem $MX_{\sigma, \psi}$. The program can be represented by an instance structure and the stable model semantics is characterized by ψ (e.g., a first-order Model Expansion characterization of ASP was shown in [25]). The problem of finding out whether there is an H -minimal solution of ALP can be reduced to deciding whether there is an optimal expansion in $(MX_{\sigma, \psi}, P)$ where $\mathcal{P}^*$ is translated into $MX_{\sigma, \psi}$. Assume X_1 and X_2 are two stable models of $\mathcal{P}^*$. If X_1 is preferred to X_2 with respect to one of the preference semantics in Definition 3, then there is $R(\bar{a}) \in X_1$ and $S(\bar{b}) \in X_2$ such that $R(\bar{a}) \sqsupseteq_P S(\bar{b})$. So, we have $X_1 \cap H \subseteq X_2 \cap H$ and therefore, each preferred answer set is H -minimal. Hence, finding a minimal solution

for $\langle H, M, \mathcal{P} \rangle$ is reduced to finding an optimal expansion of $\Pi_{\sigma, \psi}$ that satisfies a goal M . Thus, *Goal-Oriented Optimal Expansion* for an NP-complete MX is Σ_2^P -complete. By using the same argument and according to Lemma 1, finding a minimal solution for an abductive combined logic program in level k can be translated into a *Goal-Oriented Optimal Expansion* where the Model Expansion problem is Σ_k^P -complete and hence the result follows. ■

Theorem 1 presents an important consequence of adding preferences to a Σ_k^P -complete Model Expansion problem. For the problem of deciding whether there is an expansion that satisfies a goal ϕ , adding preferences leads to a jump in the Polynomial Hierarchy. So, the preference relation between expansion structures derived from a preference expression can not be translated into axiomatization ψ with polynomial time model checking unless $P=NP$ or the Polynomial Hierarchy collapses.

Example 3. *Deciding whether a graph is Hamiltonian (i.e., whether a graph has a Hamiltonian cycle) is a well-known NP-complete problem. We first characterize the Hamiltonian graph problem as first-order Model Expansion and then examine the Goal-Oriented Prioritized Model Expansion problem for an input graph with some preferences. Consider vocabulary $\tau = \{E, H\}$. For an arbitrary graph $\mathcal{G} = (V, E^{\mathcal{G}})$ (represented as a structure) such that V is the set of vertices of the graph and $E^{\mathcal{G}}$ specifies edges of $\mathcal{G}$, the Hamiltonian graph problem is defined as a Model Expansion problem $MX_{\{E\}, \Psi}$ where $\{E\}$ is the vocabulary of input and $\Psi = \psi_1 \wedge \psi_2 \wedge \psi_3$ such that*

$$\begin{aligned}\psi_1 &= \forall x \forall y (H(x, y) \vee H(y, x)) \\ \psi_2 &= \forall x \forall y \forall z (H(x, y) \wedge H(y, x) \supset H(x, z)) \\ \psi_3 &= \forall x \forall y ((H(x, y) \wedge \neg \exists z [H(x, z) \wedge H(z, y)]) \supset E(x, y))\end{aligned}$$

where ψ_1 indicates there is a Hamiltonian path between any arbitrary pair of vertices x and y , ψ_2 stipulates the transitivity property of the path, and, based on ψ_3 , if there are two adjacent vertices in the path, they must be connected through an edge of the graph. A τ -structure $\mathcal{A}$ is an expansion structure of $MX_{E, \Psi}$ if $\mathcal{A}$ expands $\mathcal{G}$ and satisfies ψ as follows:

$$\underbrace{(V; E^{\mathcal{G}}, H^{\mathcal{A}})}_{\mathcal{A}} \models \psi$$

Model Expansion problem $MX_{E, \psi}$ asks whether there is a Hamiltonian path (and therefore a Hamiltonian cycle) in the input graph. If the answer is yes, then there is a τ -structure $\mathcal{B}$ such that $H^{\mathcal{B}}$ constitutes a Hamiltonian path. Assume $V = \{v_1, v_2, v_3, v_4\}$ is a set of vertices and $\mathcal{G} = (V, E^{\mathcal{G}})$ is defined $E^{\mathcal{G}} = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_4), (v_3, v_1), (v_4, v_2)\}$. A preference expression P is defined as $H(v_4, v_2) \sqsupseteq_P H(v_3, v_4)$ which means it is preferred that a Hamiltonian path includes $H(v_4, v_2)$ (i.e., edge $E(v_4, v_2)$ due to specification ψ_3) rather than $H(v_3, v_4)$ (i.e., edge $E(v_3, v_4)$ because of specification ψ_3).

The Prioritized Model Expansion problem $\Pi_{\{E\}, \psi} = (MX_{\{E\}, \psi}, P)$ (in the search version) finds the preferred Hamiltonian paths of $\mathcal{G}$ if there are such paths. Let us define

a goal formula $\phi = H(v_3, v_1)$. The Goal-Oriented Model Expansion Problem asks whether there is a preferred expansion structure (i.e., Hamiltonian path) that includes $H(v_3, v_1)$ and hence edge $E(v_3, v_1)$. There are two possible Hamiltonian paths $H^A = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ and $H^B = \{(v_3, v_1), (v_1, v_4), (v_4, v_2), (v_2, v_3)\}$. Based on Definition 3, $\mathcal{A}$ characterizes a path that is preferred and it is comprised of $H(v_3, v_4)$.

3.3 Conditional Preferences

Preferences of users are often expressed in conditional statements. For example, A is preferred to B if C is true. Contextual preferences are widely studied in the literature. Examples include [5], [7], [24], and [8]. Here, we show how the Prioritized Model Expansion framework handles conditional preferences.

Let p be a conditional preference of the form $p : R_q(\bar{a}_u), \dots, R_s(\bar{a}_v) \supset S_i(\bar{b}_l) \sqsupset_p S_j(\bar{b}_m)$. Conditional preference p is read as $S_i(\bar{b}_l)$ is preferred to $S_j(\bar{b}_m)$ if $R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v)$ is true. Intuitively, we aim to construct a partial order $\geq_p$ from p such that for structures $\mathcal{A}$ and $\mathcal{B}$ with the same domain and vocabulary, $\mathcal{A}$ is preferred to $\mathcal{B}$ with respect to p if $\mathcal{A} \models R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v) \wedge S_i(\bar{b}_l)$ and $\mathcal{B} \models R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v) \wedge S_j(\bar{b}_m)$. We call $R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v)$ the body of p with notation $body(p)$.

Definition 8. $\Pi_{\sigma, \psi} = (MX_{\sigma, \psi}, \mathcal{P})$ is called a General Prioritized Model Expansion Problem where $\mathcal{P} = \{p_1, \dots, p_n\}$ is a set of conditional preferences.

A translation of $\Pi_{\sigma, \psi}$ into a standard Prioritized Model Expansion problem $\Pi_{\sigma, \psi^*}^* = (MX_{\sigma, \psi^*}, P^*)$ is as follows: First, we add each element of Dom as a constant symbol to τ . We assume that for all $a \in \tau$ such that $a \in Dom$ and for all optimal expansions $\mathcal{A}$, $a^{\mathcal{A}} = a$. For conditional preference p , let us introduce two new (ground) auxiliary atoms T and T' . Consider formulas $\psi_1 : (R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v) \wedge S_i(\bar{b}_l)) \supset T$ and $\psi_2 : (R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v) \wedge S_j(\bar{b}_m)) \supset T'$. Set $\psi^* = \psi \wedge \psi_1 \wedge \psi_2$, $T \sqsupset_{P^*} T'$, and $\tau^* = \tau \cup \{T, T'\}$. It is clear that an expansion τ^* -structure $\mathcal{A}$ is preferred to another expansion τ^* -structure $\mathcal{B}$ with respect to p if $\mathcal{A}$ and $\mathcal{B}$ satisfy $R_q(\bar{a}_u) \wedge \dots \wedge R_s(\bar{a}_v)$, $\mathcal{A} \models S_i(\bar{b}_l)$, and $\mathcal{B} \models S_j(\bar{b}_m)$. The binary relation $\geq_{P^*}^s$ is constructed from $\mathcal{P}$ based on the preference semantics *Weak Pareto*, *Upper Bound Dominance*, and *Element Dominance* similarly to before.

Example 4. For a graph three-colouring problem with conditional preferences $\Pi_{\{E\}, \psi} = (MX_{\{E\}, \psi}, \mathcal{P})$, consider the following conditional preferences: red v_1 is preferred to blue v_2 if v_5 is green which is expressed as $p : R(v_1) \supset (B(v_2) \sqsupset_p G(v_5))$. We introduce new atoms T_1 and T_2 such that $\psi_1 : (R(v_1) \wedge B(v_2)) \supset T_1$ and $\psi_2 : (R(v_1) \wedge G(v_5)) \supset T_2$. Also, we set $\psi^* = \psi \wedge \psi_1 \wedge \psi_2$ and $T_1 \sqsupset_p T_2$. For an input graph $\mathcal{G} = (V, E^{\mathcal{G}})$ where $V = \{v_1, \dots, v_5\}$ and $E^{\mathcal{G}} = \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_1, v_3), (v_3, v_1), (v_2, v_4), (v_4, v_2), (v_4, v_5), (v_5, v_4), (v_3, v_5), (v_5, v_3)\}$, three-colouring $\mathcal{A}$ where $R^{\mathcal{A}} = \{v_1, v_4\}$, $B^{\mathcal{A}} = \{v_2, v_5\}$, and $G^{\mathcal{A}} = \{v_3\}$ is preferable based on the *Weak Pareto* semantics.

4 Relation to Other Preference-Based Declarative Approaches

4.1 Preference-based SAT

3-SAT is a canonical NP-complete problem. Let φ be a Conjunctive Normal Form (CNF) boolean formula and $X = \{x_1, \dots, x_n\}$ be the set of boolean variables appearing in φ . A truth assignment $\mathcal{I}$ is a mapping from X to $\{\text{true}, \text{false}\}$. The problem of deciding whether φ is satisfiable (i.e., there is a truth assignment $\mathcal{I}$ that satisfies φ) can be converted into a Model Expansion problem $\text{MX}_{\sigma, \psi}$ with an input σ -structure $\mathcal{I}$. The general idea is to represent formula φ by $\mathcal{I}$. The domain of $\mathcal{I}$ is X . Also, the expansion vocabulary (i.e., $\tau \setminus \sigma$) includes unary predicates T and F that specify each boolean variable as true or false. An interpretation of the expansion vocabulary represents a truth assignment to the boolean variables in φ . Formula ψ specifies the notion of satisfying a boolean CNF formula by a truth assignment.

Preference-based SAT, which is related to the problem of max-SAT [3], is the problem of finding truth assignments satisfying a boolean formula when some variables are favourite to be assigned the value true. A preference-based SAT problem for a boolean formula φ and the set X of boolean variables appearing in φ is defined as a pair $(\varphi, (X, \geq))$ where $\geq$ is a preorder on X that specifies a preference over variables in X . Let $\mathcal{I}$ and $\mathcal{I}'$ be truth assignments that satisfy φ . We say $\mathcal{I}$ is preferred to $\mathcal{I}'$ if all variables assigned the value true by $\mathcal{I}$ are preferred to all variables that $\mathcal{I}'$ maps to true.

Theorem 2. *The problem of deciding whether there is a preferred truth assignment $\mathcal{I}$ satisfying φ that maps all variables in some $Y \subset X$ to true is Σ_2^P -complete.*

Proof: Consider a Prioritized NP-complete Model Expansion problem $\Pi_{\sigma, \psi} = (\text{MX}_{\sigma, \psi}, P)$. All NP-complete problems, obviously, can be reduced to SAT in polynomial time. Also, by considering the Weak Pareto semantics in the Prioritized Model Expansion framework, preference relation $\geq$ in the preference-based SAT framework is matched with $\sqsupseteq_P$. For each relation $R(\bar{a}) \sqsupseteq_P S(\bar{b})$, we consider $x_i \geq x_j$ in the preference-based SAT framework where x_i and x_j are boolean variables representing ground atoms $R(\bar{a})$ and $S(\bar{b})$, respectively. In order to reduce an NP-complete $\text{MX}_{\sigma, \psi}$ problem to SAT, some auxiliary propositional variables are introduced that are considered to be equally preferred with respect to relation $\sqsupseteq_P$. Therefore, Prioritized NP-complete Model Expansion problem $\Pi_{\sigma, \psi}$ is reduced to a preference-based SAT S in polynomial time. Thus, the *Goal-oriented Optimal Expansion* problem for $\Pi_{\sigma, \psi} = (\text{MX}_{\sigma, \psi}, P)$ where $\text{MX}_{\sigma, \psi}$ is an NP-complete problem can be reduced (in polynomial time) to deciding whether there is a preferred truth assignment $\mathcal{I}$ satisfying φ that maps all variables in some $Y \subset X$ to true, which is, based on Theorem 1, Σ_2^P -complete. ■

4.2 Logic Programs with Preferences

Logic programming with stable model semantics is one of the main declarative approaches for specifying problems in NP. Let P be a program which is defined as a set of rules of the form $r : c_1, \dots, c_l \leftarrow a_1, \dots, a_m, \text{not } b_1, \dots, \text{not } b_n$ where a_i for $i \in [1, m]$, b_j for $j \in [1, n]$, and c_k for $k \in [1, l]$ are propositional atoms. Let us define

vocabulary symbols $Rule$, $Stable$, $Body^+$, $Body^-$, and $Head$. Symbol $Stable(x)$ denotes that x belongs to a stable model of program P , $Rule(y)$ means that y is a rule of program P , $Body^+(y, x)$ indicates that x is a positive atom in the body of rule y and, similarly, $Body^-(y, x)$ points out that x is an atom in the negative part of the body of y . Also, $Head(y, x)$ means that atom x is in the head of rule y . For $\sigma = \{Rule, Stable, Body^+, Body^-, Head\}$, a σ -structure $\mathcal{P}$ specifies a program by interpreting predicate symbols in σ . A first-order formula ψ characterizes the stable model semantics as $\psi = \forall x (\exists y (Rule(y) \wedge Head(x, y) \wedge \forall z [Body^+(y, z) \supset Stable(z)] \wedge \forall w [Body^-(y, w) \supset \neg Stable(w)]) \supset Stable(x)) \wedge \forall x (\exists y (Rule(y)) \wedge Head(x, y) \wedge \forall z [\neg Body^+(y, z)] \wedge \forall w [\neg Body^-(y, w)] \supset Stable(x))$.

A stable model of program P is represented by an expansion structure $\mathcal{M}$ that expands input σ -structure $\mathcal{P}$, which represents program P , and satisfies ψ .

Preferred Models

Prioritized logic program (PLP) [26] is one of the impactful frameworks proposed for logic programming with preferences. A PLP program is a pair (Pr, Φ) where Pr is a general extended disjunctive logic program with answer set semantics and Φ is a set of preference relations among propositional atoms of the form $a \succeq b$ that means a is preferred to b . The transitive closure of Φ is denoted by Φ_c . The reflexive transitive binary relation $\sqsubseteq$ among answer sets of Pr is defined as: 1) $X_1 \sqsubseteq X_1$, 2) if there exist $a \in X_1 - X_2$ and $b \in X_2 - X_1$ where $(a \succeq b) \in \Phi_c$ and there is no $d \in X_1 - X_2$ such that $(b \succ d) \in \Phi_c$, then $X_1 \sqsubseteq X_2$, and 3) if $X_1 \sqsubseteq X_2$ and $X_2 \sqsubseteq X_3$, then $X_1 \sqsubseteq X_3$. X is called a preferred answer set if there is no answer set Y such that $Y \sqsubset X$. One could examine conditions 1 and 2 in polynomial time in the size of the input (the number of propositional atoms). However, condition 3 requires possibly an exponential number of comparisons over the answer sets of Pr . The complexity results of the decision problems associated to a PLP program are based on the assumption that deciding whether X is preferable to Y is in polynomial time that is not accurate due to condition 3. The role of condition 3 is to make relation $\sqsubseteq$ transitive. On the other hand, relation $>_P^s$ in the Prioritized Model Expansion framework is not necessarily transitive for some preference semantics (e.g., the Element Dominance semantics).

Theorem 3. Let $\Pi_{\sigma, \psi} = (MX_{\sigma, \psi}, P)$ be a Prioritized Model Expansion problem with an input σ -structure $\mathcal{I}$ and $\Gamma = (Pr, \Phi_c)$ be a PLP program. Assume ψ characterizes the stable model semantics, $\mathcal{I}$ represents Pr , and P specifies Φ_c . If there are expansion structures $\mathcal{A}$ and $\mathcal{B}$, then Pr has answer sets M_1 that is represented by $\mathcal{A}$ and M_2 that is represented by $\mathcal{B}$ such that if $\mathcal{A} >_P^{ed} \mathcal{B}$, then $M_1 \sqsubset M_2$.

Proof: Φ_c can be viewed as a preference expression in the PMX framework and finding answer sets of generalized extended disjunctive program Pr can be expressed by an $MX_{\sigma, \psi}$ problem. As discussed in the previous subsection, there is a correspondence between the expansion structures of $MX_{\sigma, \psi}$ and the answer sets of Pr . Each expansion structure of $MX_{\sigma, \psi}$ can be one-to-one mapped in polynomial time to an answer set of Pr and vice versa. The relation $>_P^{ed}$ with the Element Dominance semantics is a subset of relation $\sqsubseteq$ in the PLP because it satisfies conditions 1 and 2 in the definition of $\sqsubseteq$ while it does not guarantee the transitivity of the preference relation among

the expansion structures. So, for structures $\mathcal{A}$ and $\mathcal{B}$, Pr has answer sets M_1 and M_2 such that $M_1 \sqsupset M_2$ and M_1 and M_2 are represented by $\mathcal{A}$ and $\mathcal{B}$, respectively. ■

4.3 Answer Set Optimization

An Answer Set Optimization (ASO) program [12] is a pair (P_g, R) where P_g is a generating normal logic program and R is a set of rules of the form $r : C_1 > \dots > C_k \leftarrow a_1, \dots, a_n, \text{not } b_1, \dots, \text{not } b_m$. In each rule, a_i and b_i are literals. Also, C_i is defined as a combination of atoms integrated by conjunction, disjunction, default negation (*not*) and strong negation ($\neg$) that must appear only before atoms. $C_i > C_j \leftarrow \text{body}$ means that if *body* is satisfied, C_i is preferred to C_j . Given a set of l rules $r_1, \dots, r_l$, each answer set M of P_g is associated with a satisfaction vector $d(M) = \langle d_1(M), \dots, d_l(M) \rangle$ where $d_i(M)$ is called the satisfaction degree of M in r_i . Satisfaction degree denotes the minimum j of C_j s in r_i that are satisfied by M whenever M satisfies *body*, and 1 in other case. Let M_1 and M_2 be two answer sets of P_g . M_1 is preferred to M_2 with respect to R (notation $M_1 \succeq M_2$) if for all $i \leq l$, $d_i(M_1) \leq d_i(M_2)$. When in each rule $r \in R$, C_i s in the head of r are literals, the relation between ASO and Prioritized Model Expansion is formulated as follows:

Theorem 4. *Let $ASO = (P_g, R)$ be an ASO program where P_g is a normal logic program and R is a set of preference rules. There is a Prioritized Model Expansion problem $\Pi_{\sigma, \psi} = (MX_{\sigma, \psi}, P)$ with an input $\mathcal{I}$ such that each preferred answer set of ASO is represented by an optimal expansion of $\Pi_{\sigma, \psi}$.*

Proof. Let P_g^* be a logic program such that $P_g^* = P_g \cup R^*$ where R^* is a set of r_1^* and r_2^* rules that are constructed as follows: For each rule r in R of the form $C_1 > C_2 \leftarrow \text{body}(r)$, we introduce auxiliary atoms n_1 and n_2 and define r_1^* as $n_1^* : n_1 \leftarrow C_1, \text{body}(r)$ and r_2^* as $n_2^* : n_2 \leftarrow C_2, \text{body}(r)$. Normal logic program P_g^* with stable model semantics can be cast as Model Expansion problem $MX_{\sigma, \psi}$ such that ψ specifies stable model semantics and $\mathcal{I}$ represents P_g^* . Also, let us define preference expression P such that for auxiliary atoms n_1 and n_2 , we have $n_2 \sqsupseteq_P n_1$. We consider all other ground atoms in P_g as equally preferred. Let M_1 and M_2 be answer sets of P_g . Assume $\mathcal{A}$ and $\mathcal{B}$ are expansion structures of $MX_{\sigma, \psi}$ that represent M_1 , and M_2 , respectively. If $M_2 \succ M_1$ in ASO, for each C_i in the head of each rule $r \in R$ such that $M_1 \models C_i$, there is C_j such that $M_2 \models C_j$ and $C_j > C_i$. This is equivalent to say that for each auxiliary atom n_i satisfied by $\mathcal{A}$, there is an auxiliary atom n_j satisfied by $\mathcal{B}$ such that $n_j \sqsupseteq_P n_i$ that is matched with the Upper Bound Dominance semantics. So, $\mathcal{A} \succ_P^{ubd} \mathcal{B}$ and the result follows. □

We showed that an ASO program can be viewed as a Prioritized Model Expansion. Also every Prioritized NP-complete Model Expansion problem with the Upper Bound Dominance semantics can be encoded as an ASO program. The problem of deciding the existence of a stable model of a normal program is NP-complete and, obviously, an NP-complete problem can be reduced in polynomial time to another NP-complete

problem. Preference expression P is encoded by a single rule $r \in R$ of the form $a_1 > a_2 \dots > a_n \leftarrow$ where a_i s are ground atoms and $a_i > a_j$ if $a_i \sqsupset_P a_j$.

As a result, *Goal-oriented Prioritized NP-complete Model Expansion* problems can be reduced to deciding whether some answer sets of an ASO satisfy a formula in polynomial time. The existence of an answer set that satisfies a set of ground atoms is NP-complete (i.e., brave reasoning in logic programs with stable model semantics), and determining the existence of a solution of ASO that satisfies a goal formula ϕ is Σ_2^P -complete [12]. Likewise, for an NP-complete $\text{MX}_{\sigma,\psi}$, the problem of deciding whether there is an optimal expansion of $\Pi_{\sigma,\psi}$ that satisfies a goal ϕ , based on Theorem 1, is Σ_2^P -complete.

5 Conclusion

We proposed a novel language-independent preference framework for characterizing preference-based computational decision and search problems. We demonstrated that adding preferences raises the computational complexity of deciding the existence of an expansion structure satisfying a goal. Our proposal can also be related to a variety of other preference frameworks, such as CP-nets [5] that model conditional preferences. A CP-net can be approximated by a Prioritized Model Expansion problem such that if an outcome o_1 is preferred to an outcome o_2 in a CP-net, then for associated expansion structures $\mathcal{A}$ and $\mathcal{B}$ in the Prioritized Model Expansion problem, $\mathcal{A}$ is preferred to $\mathcal{B}$. Moreover, finding preferred repairs of a database that violates some integrity constraints [27] can be translated into a Prioritized Model Expansion problem using the Upper Bound Dominance semantics. One possible future direction is to devise an algorithm that solves Prioritized Model Expansion problems using generic solvers empowered by propagators. The solver would provide symbolic explanations for rejecting and accepting models to prune the search space.

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