Variants of Nash Equilibrium

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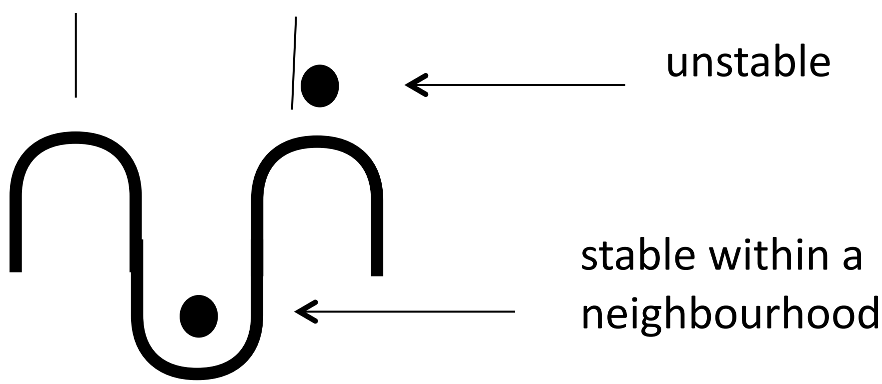
Equilibrium Refinements

- A complex game may have many Nash equilibria.
- Can we predict which one players will choose?
- If we can identify equilibria with special properties, we may be able to make more precise predictions.
- A definition of equilibrium that selects a nonempty subset of N.E.s is called an **equilibrium refinement**.

Stable vs. Unstable Equilibrium

A common idea is to select equlibria that are in some sense **stable** to small changes.



Formalizing Perturbations

- There are a number of ways to formalize the idea of a "small change".
  - Textbook uses convergent sequences, like modern calculus.
  - Blume, Brandenburger, Dekel used lexicographic probability systems.
  - Infinitesimal numbers are very convenient.
- An **infinitesimal** ε is a number s.t. $0<\varepsilon<x$ for all real numbers $x > 0$.
- Think of an infinitesimal as a "lower-tier" real number, or an "infinitely small" real. Yes Virginia they do exist.
- It's often convenient to write $x+\underline{\varepsilon}$ for the sum of a real number x and an infinitesimal.
- Note that $\varepsilon x < y$ for all real numbers x,y.

Trembling-Hand Equilibrium

- A mixed strategy profile $(s_1,\dots,s_n)$ is a **trembling-hand equilibrium** if there exist vectors of infinitesimals $(\varepsilon_1,\dots,\varepsilon_n)$ s.t. for each player *i*
  1. $\Sigma_j \varepsilon_{ij} = 0$.
  2. The vector $s_i + \varepsilon_i$ has only positive entries (i.e., each pure strategy has at least infinitesimal probability).
  3. The strategy s_i is a best reply to $(s_{-i}) + (\varepsilon_{-i})$.
- Intuitive interpretation: each player’s strategy s_i is optimal if the other player’s deviate with infinitely small probability. The deviation is viewed as a “mistake” or a “trembling hand”.

Trembling-Hand Example

	L	R
U	1,1	1,1
D	0,0	1,1

- (D,R) is N.E. but **not** trembling-hand perfect.
- (U,R) is perfect N.E.
- In coordination game, are pure NEs perfect?

Trembling-Hand Exercise

- Which pure N.E.s are trembling-hand perfect?

	A	B	C
A	0,0	0,0	0,0
B	0,0	1,2	2,0
C	0,0	0,2	2,2

Trembling-Hand Theorem

- There always exists a trembling-hand equilibrium.
- In a 2-player game, an equilibrium (s_1,s_2) is trembling-hand perfect if and only if neither s_1 nor s_2 is weakly dominated.

Hawk vs. Dove As A Population Game

	Hawk (H)	Dove (D)
Hawk	-2,-2	6,0
Dove	0,6	3,3

- Assume a large population of agents.
- Agents are either hawks (H) or doves(D).
- We randomly draw 2 at a time to play.
- The only fully mixed N.E. is s(H) = 3/5, s(D) = 2/5.

Evolutionarily Stable Strategies (ESS)

mixed population dist. = (1-ε) s* + εs

current dist s*	mutant dist s
HHHHHH DDDD	H D
10/12 = 1-ε	2/12 = ε

← mutant plays mutant: u(1/2,1/2; s)

← incumbent plays mutant: u(6/10,4/10; s)

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1. A distribution s* is an ESS ⇔ for all sufficiently small mutations s the incumbents in s* do better in the mixed population than the mutants.

2. A distribution s* is an ESS ⇔ for all infinitesimal ε for all mutations s ≠ s*
   u(s*; (1-ε) s* + εs) > u(s; (1-ε) s* + εs)

ESS Example

- Let s* be the N.E. strategy with s*(H) = 3/5.
- Consider mutation s(H) = 1.
- Then
  $u(s^*; (1-\varepsilon) s^* + \varepsilon s) = (1-\varepsilon)\, u(3/5;3/5) + \varepsilon u(3/5;1)$
  $u(s; (1-\varepsilon) s^* + \varepsilon s) = (1-\varepsilon)\, u(1;3/5) + \varepsilon u(1;1)$
- Now u(1;3/5) = u(3/5;3/5) (why?), so the difference between the incumbent and the mutant is given by
  $u(s^*; (1-\varepsilon) s^* + \varepsilon s) - u(s; (1-\varepsilon) s^* + \varepsilon s) =$
  $u(3/5;1) – u(1;1) = -2 \times 3/5 + 2 > 0$. So the mutants do worse.
- Intuitively, if the frequency of hawks increases, it’s worse to be a hawk = mutant.

ESS theorem

- Let s^* be an N.E. and consider mutation s of infinitesimal size ε.
  1. If s is not a best reply to s*, then s is not a successful mutation.
  2. If s is a best reply to s*, then s is successful iff $u(s;s) \geq u(s^*;s)$.
- Proof. Note that for any strategy s'
  $u(s'; (1-\varepsilon) s^* + \varepsilon s) = (1-\varepsilon) u(s';s^*) + \varepsilon u(s';s)$.
  So the real-valued part $u(s';s^*)$ dominates. If $u(s;s^*) = u(s^*,s^*)$, the infinitesimal term decides who does better in the mixed population.

ESS in Hawk-and-Dove

- With the ESS theorem it is fairly easy to establish that s*(H) = 3/5 is the unique ESS.
- Any mutation s is a best reply to s*, so we need only consider u(s;s*).
  - Case 1: s(H) > 3/5. Then D is the best reply to s, so u (s*;s*) > u(s;s*). Intuitively, if there are more hawks, it's better to place less weight on H.
  - Case 2: s(H) < 3/5. Then H is the best reply to s, so u (s*;s*) > u(s;s*). Intuitively, if there are fewer hawks, it's better to place more weight on H.

ESS Exercise 1

	L	R
L	1,1	0,0
R	0,0	1,1

- Which of the three Nash equilibria in the coordination game are ESS?
- Answer: only the two pure equilibria.

Coordination and Networking Effects

- IT markets often exhibit "networking effects": the more users a system has, the more attractive it becomes.
- A simple game-theoretic model of this is a coordination game: users don't care what system they use as long as they coordinate with other users.

	Check on E-bay	Check on Amazon
Post on E-bay	1,1	0,0
Post on Amazon auction	0,0	1,1

The ESS analysis suggests the following.

• Once a system/website is dominant, small groups of users cannot change this by deviating. (E.g., imagine Amazon tells its employees to user their own websites.)

• Coexistence of different systems (e.g.50%-50%) is not stable. A small shift in one direction will feed on itself.

ESS Exercise 2

	R	P	S
R	γ,γ	-1,1	1,-1
P	1,-1	γ,γ	-1,1
S	-1,1	1,-1	γ,γ

- For which values of γ with $0 \leq \gamma \leq 1$ does this game have an ESS?
- Answer: none, so not every game has an ESS.

Correlated Equilibrium (2 players)

- So far we have considered restrictions of NE. Correlated Equilibrium is a more general concept.
- Consider a distribution *p* over the set of all pairs of pure strategies, one strategy for each player.
- Given a pure strategy a_1 for player 1, we obtain a **conditional** distribution over player 2's strategies: $p(a_2|a_1) = p(a_1,a_2)/p(a_1)$. Ditto for player 2.
- A distribution *p* is a **correlated equilibrium** if

1. for each pure strategy a_1 with $p(a_1) > 0$, the action a_1 is a best reply against $p(\bullet|a_1)$.
2. Ditto for player 2.

Correlated Equilibrium: Intuition

- The best way to think about correlated equilibrium is to imagine that the players use a *shared* mutually accessible randomization device.
- The random device chooses a *pair* of strategies (a_1,a_2), one for each player, with probability $p(a_1,a_2)$.
- At equilibrium, each player cannot improve her utility by deviating from her "instruction" a_i.
- This model is formalized in the text.

Correlated Equilibrium: Example

	H	T
H	2,1	0,0
T	0,0	1,2

- Let $p(H_1,H_2) = p(T_1,T_2) = ½$.
- $p(H_2|H_1) = p(H_1|H_2) = 1$.
- In Nash equilibrium, choice of player 1 carries no information about choice of player 2 – no correlation.
- Any convex combination of N.E.s generates a correlated equilibrium.
- Gives players same expected utility (fair) and is pareto-optimal.

Correlated Equilibrium: Traffic Light Example

	Go now	Wait 1 min
Go now	-20,-20	0,-1
Wait 1 min	-1,0	-21,-21

- Traffic light system: if green, row player goes, else column player goes.
- Fair, and has 0 probability of collision, unlike mixed N.E.
- But need to invest in correlation device = traffic light.

Why Correlated Equilibrium is interesting for CS

- Easier to compute than Nash Equilibrium.
- Can produce fair Pareto-optimal outcomes → acceptable to users, programmers etc.
- Virtual or software correlation devices may be cheaper to set up than physical ones like traffic lights.
- Can you come up with a traffic light system for the internet? Compare with the TCP game.