Mixed Strategies

CMPT 882
Computational Game Theory
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Motivation

- Some games like Rock, Paper, Scissor don’t have a Nash equilibrium as defined so far.
- Intuitively, the reason is that there is no steady state where players have *perfect knowledge* of each other’s actions: knowing exactly what the other player will do allows me to achieve an optimal payoff at their expense.
- Von Neumann and Morgenstern observed that this changes if we allow players to be **unpredictable** by choosing **randomized strategies.**

Definition

- Consider a 2-player game (A_1, A_2, u_1, u_2).
- The members of A_i are the **pure strategies** for player i.
- A **mixed strategy** is a probability distribution over pure strategies.
- Or, if we have *k* pure strategies, a mixed strategy is a *k*-dimensional vector whose nonnegative entries sum to 1.
- Note: this is the notation from the text. More usual are Greek letters for mixed strategies, e.g. σ.

Example: Matching Pennies

	Heads: 1/2	Tails: 1/2
Heads: 1/3	1,-1	-1,1
Tails: 2/3	-1,1	1,-1

Mixed Strategies ctd.

- Suppose that each player chooses a mixed strategy s_i.
- The expected utility of pure strategy a_1 is given by $EU_1(a_1,s_2) = \Sigma_k u_1(a_1,a_k) \times s_2(a_k)$.
  where a_j ranges over the strategies of player 2.
- The expected utility of mixed strategy s_1 is the expectation over the pure strategies:
  $EU_1(s_1,s_2) = \Sigma_j EU_1(a_j,s_2) \times s_1(a_j)$
  $= \Sigma_j \Sigma_k u_1(a_j,a_k) \times s_1(a_j) \times s_2(a_k)$.
- We also write $u_1(s_1,s_2)$ for $EU_1(s_1,s_2)$. Then we have in effect a new game whose strategy sets are the sets of mixed strategies, and whose utility functions are the expected payoffs.

Nash equilibrium in mixed strategies

- The definition of best response is as before: a mixed strategy s_1 is a best response to s_2 if and only if there is no other mixed strategy s'_1 with $u_1(s'_1, s_2) > u_1(s_1, s_2)$.
- Similarly, two mixed strategies (s_1, s_2) are a **Nash equilibrium** if each is a best response to the other.

Examples and Exercises

- Find a mixed Nash equilibrium for the following games.
  - Matching Pennies.
  - Coordination Game.
  - Prisoner’s Dilemma.
  - Battle of the Sexes or Chicken.
- Can you find **all** the Nash equilibria?

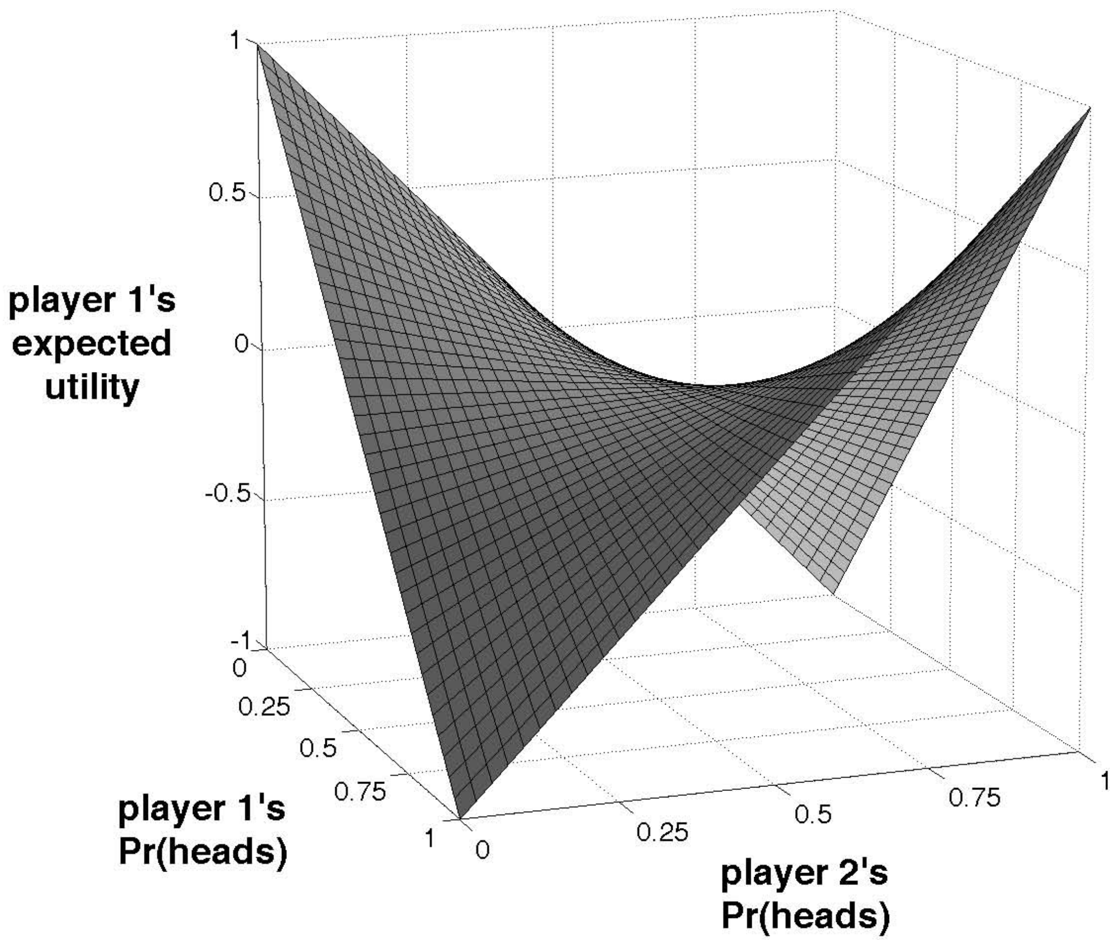
Visualization of Equilibria

- In a 2x2 game, we can graph player 2's utility as a function of p, the probability that 1 chooses strategy 1. Similarly for player 1's utility.

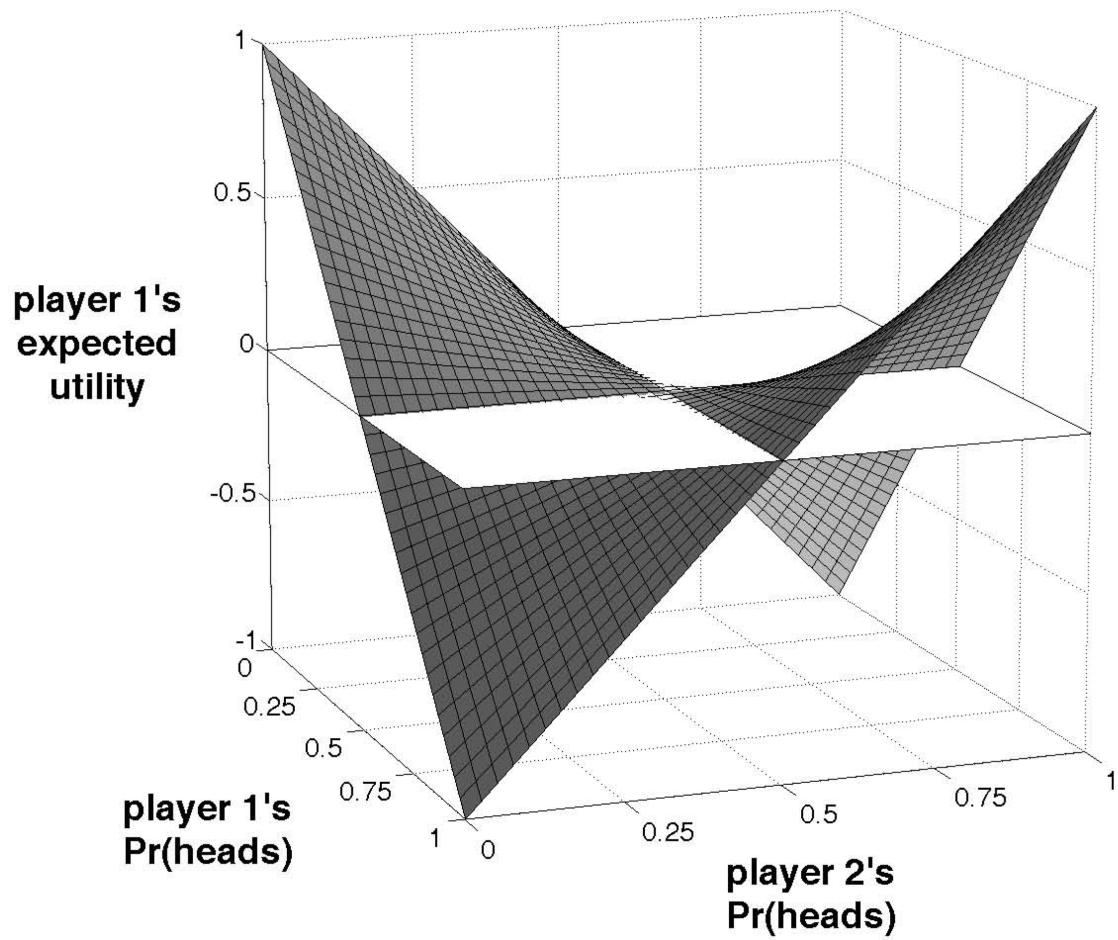
Strategies and Topology

- Consider mixed strategy profiles as k-dimensional vectors, where k is the total number of pure strategies for each player. This set is convex and compact (why?).

Visualization of Matching Pennies



Visualization of M.P. equilibrium



Computation of Equlibria

- **Definition** The **support** of a probability measure *p* is the set of all points *x* s.t. *p*(*x*)>0.
- **Proposition**. A mixed strategy pair (s_1,s_2) is an N.E. if and only if for all pure strategies a_i in the support of s_i, the strategy s_i is a best reply so s_{-i}.
- **Corollary.** If (s_1,s_2) is an N.E. and a_1,b_1 are in the support of s_1, then $u_1(a_1,s_2) = u_1(b_1,s_2)$. That is, player 1 is indifferent between a_1 and b_1. Ditto for a_2,b_2 in the support of s_2.

A general NP procedure for finding a Nash Equilibrium

1. Choose support for player 1, support for player 2.
2. Check if there is a Nash equilibrium with those supports.

Existence of Nash Equilibrium

- Theorem (Nash 1950). In any finite game (any number of players) there exists a Nash equilibrium.
- Short proof by Nash.

Existence Proof (1)

- Transform a given pair of mixed strategies (s_1,s_2) as follows.
  For each pure strategy a_1 of player 1, set
  $c(a_1) := max(0,[u_1(a_1,s_2) – u_1(s_1,s_2)])$.
  $s'_1(a_1) := [s_1(a_1) + c(a_1)]/ [1+ \Sigma_b c(b)]$.
  Ditto for player 2.
- This defines a **continuous** operator $f(s_1,s_2) = (s'_1,s'_2)$ on the space of mixed strategy pairs.
- A strategy a is a best reply if and only if $c(a) = 0$.
- So if (s_1,s_2) is an N.E., then c(a) = 0 for all a, *so* $f(s_1,s_2) = (s_1,s_2)$.

Existence Proof (2)

- The generalized Brouwer fixed point theorem states that if *K* is convex and compact, and *f: K*➔ *K* is continuous, then *f* has a fixed point *f(x) = x*.
- If (s_1,s_2) is a fixed point of the mapping on the previous slide, then (s_1,s_2) is an N.E.
- Proof outline: Some action a_1 with $s_1(a) > 0$ must be a best reply against any s_2. Therefore
  $c(a_1) = 0$.
  Since we have a fixed point,
  $1+ \Sigma_b\, c(b) = 1$.
  This implies that $c(b) = 0$ for each b.

Illustration in Excel

- Illustrate construction in Excel.
- Note that the update operation can be viewed as a local computation method, and even as a learning method!

Maxmin and Minmax

Consider a 2-player game.

- A **maxmin** strategy for player 1 solves $max_{s_1} min_{s_2}\ u_1(s_1,s_2)$. Ditto for player 2.
- Interpretation: *Conservatively* choose strategy against *worst-case adversary*.
- The value $\max_{s_1} \min_{s_2} u_1(s_1,s_2)$ is called the **security level of player 1.**
- A **minmax** strategy for player 1 solves $min_{s_1} max_{s_2}\ u_2(s_1,s_2)$. Ditto for player 2.
- Interpreation: *Punish* the other player by minimizing the best payoff she can get.

N.E. in Zero-Sum Games: The Minimax Theorem (von Neumann 1928).

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Consider a 2-player **zero-sum** game.

1. s_i is a maxmin strategy if and only if s_i is a minmax strategy for i = 1,2.
2. For both players, the maxmin value = minmax value.
3. If s_1,s_2 are each maxmin (minmax) strategies, then (s_1,s_2) is a Nash equilibrium.
4. If (s_1,s_2) is an N.E., then s_1 and s_2 are maxmin (minmax) strategies.

Interpretation of mixed N.E.

N.E. in Population Games