Sequential Games

CMPT 882
Computational Game Theory
Simon Fraser University
Spring 2010
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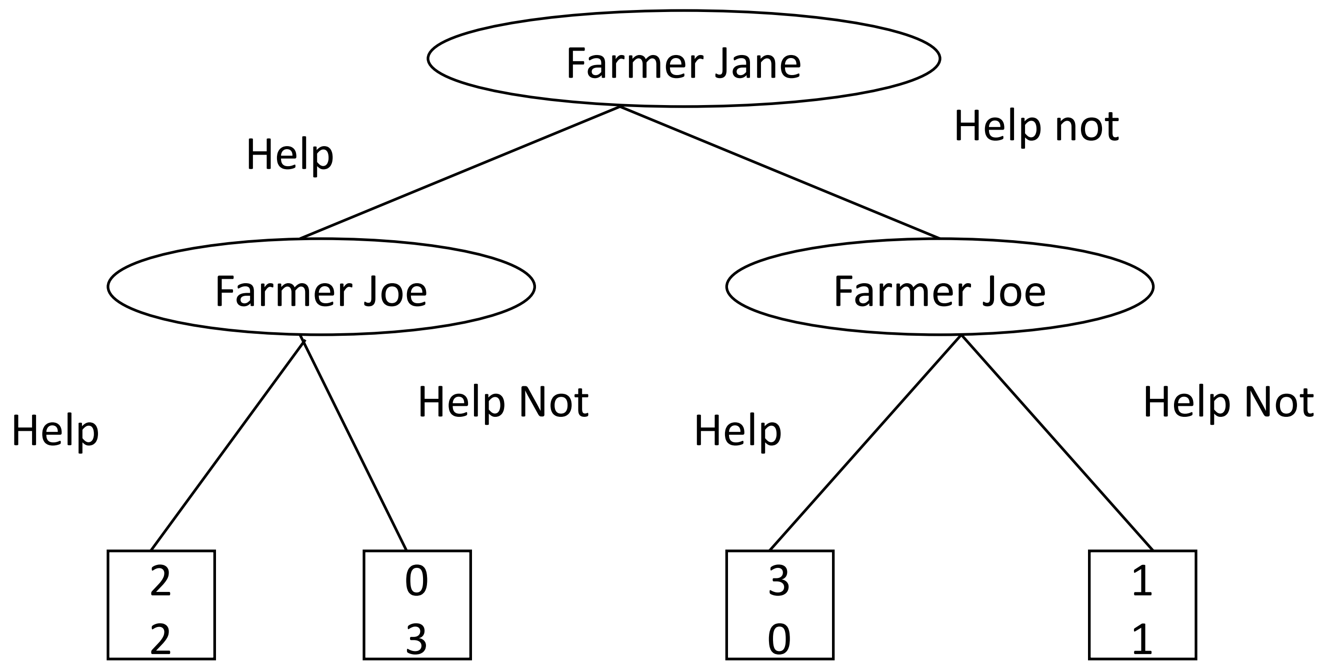
Motivation

- Many strategic situations, including most parlour games, involve turn taking and repeated play where agents respond to the choices of other agents.
- These situations can be modelled using tree structures.

Game Trees: An Example



Definition

- A finite perfect information game in extensive form is a tree with the following structure.
- The leaves or **terminal nodes** of the tree are denoted *Z*.
- The nonleafs or **choice nodes** of the tree are denoted *H*.
- $\chi(h)$ assigns a **set of possible actions** to choice node *h*.
- *player(h)* assigns a player to *h*.
- *succ(h,a)*in *H* specifies the node following *h* after action possible action *a* in $\chi(h)$ is taken by *player(h)*.
- $u_i(z)$ assigns a **payoff** to player *i* when terminal node *z* is reached.

The Centipede Game

- The game is played for *T* periods, *T* even.
- Each player in turn decides whether to stop or go on.
- Each player prefers to stop at *t* rather than have the opponent stop at *t+1*.
- But also each player prefers to stop at *t+2* rather than at *t*.
- How should the players choose?

Backward Induction (Zermelo 1913)

- The following “look ahead, reason backwards” procedure can be used to make predictions in a game tree. We recursively label each node with the payoffs of an optimal choice.

1. *BI(z) = u(z)* for terminal node *z*.
2. For choice node *h* of height *k+1*, assume BI has been computed for successor nodes of height *k*. Let *i* be the player moving at *h*. Find an action *a* such that *BI(succ(h,a)*) is maximal. Assign *BI(h) := BI(succ(h,a)*).
3. Repeat until root of tree is reached.

BI returns a unique answer if there are no ties in the game tree, or if the game is a 2-player zero-sum game. (why?).

The Dollar Auction

- I as the auctioneer auction off $1. You can bid less or more than $1. Less say bids have to be in units of dimes ($1.10, $1.20,…).
- The highest bidder gets the dollar. *But* the second-highest bidder also has to pay their bid as a participation fee, although they get nothing.
- How would you play?

Backward Induction and the Dollar Auction

- To apply backward induction, let’s assume that we have two bidders, Eli and John, each bidder has exactly $2.50 in their pockets, and each bidder knows this fact.
- What is the BI solution?
- To get started, think about this: what if John has bid $1.40 and Eli bids $2.40?

The Normal Form of A Game Tree

- Motivation: At first glance it seems that we cannot apply concepts like Nash equilibrium to a game tree. Von Neumann and Morgenstern made a bold and brilliant proposal to address this gap, called the *normal form* of the game tree.
- Let T be a game tree. A pure strategy s for player i assigns a possible action $s(h)$ to every choice node h at which i moves.
- In a 2-player game, a strategy pair s_1,s_2 defines a unique terminal node z that is reached when we match s_1,s_2. Define $u_i(s_1,s_2) = u_i(z)$.
- The matrix game defined by the set of pure strategies and payoff functions u_i is called the **normal form** of T.

Interpretation of the Normal Form

- The normal form can be interpreted in computer science terms as follows.
- Imagine that each player *i* writes a program p_i that outputs a possible action for each choice node *h* belonging to *i*.
- Intuitively, instead of responding dynamically to moves by the other players, a player decides *in advance* how they are going to respond in each situation.
- If we run the programs p_1, p_2, each program keeps outputting moves until a final position is reached.
- The payoffs to each player are then assigned according to the final position reached by their programs.

Exercise

- Find the normal form of Hume's Farmer Game. What are the Nash equilibria?

Subgame Perfect Equilibrium

- A Nash equilibrium need not be consistent with backward induction. A typical case in which this happens arises if the NE involves player 1 making a "threat" to punish player 2, even though the threat to punish is inconsistent with BI because player 1 would be punished as well.
- However, there is always at least one NE that is consistent with BI. To define this formally, say that the subtree rooted at node *h* defines a **subgame** T_h.
- A strategy pair (s_1,s_2) is a **subgame-perfect** N.E. if for every node *h*, the restriction of (s_1,s_2) to T_h is an N.E. for T_h.
- Intuitively, (s_1,s_2) is a subgame-perfect N.E. if each strategy is a best response to the other no matter where we "start the play".

Exercises

- Find the subgame-perfect equilibria of Hume's Farmer Game.
- Find the subgame-perfect equilibria of the Centipede game.

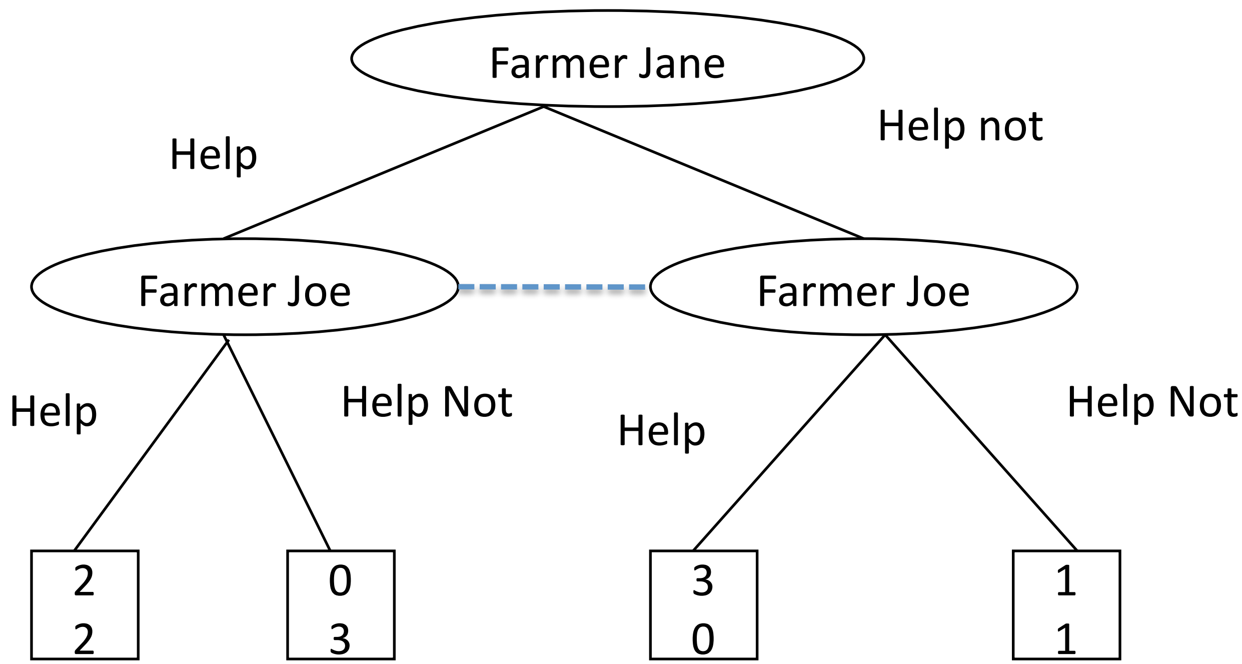
Games of Imperfect Information

- In many situations, players are not completely aware of the entire game history when they move.
- For example, in a card game there is an initial “move by nature” that assigns cards to each player.
- If players are not aware of previous moves by other players, or themselves, we have an **imperfect information game**.

Information Sets

- To represent a player's uncertainty about the game history, we augment the game with an additional structure called **information sets**.
- A game with imperfect information is a game tree together with a partition I_i for each player *i*, where
  1. every history *h* belonging to *i* is contained in exactly one information set I_{ij} in I_i.
  2. If *h,h'* are both in info set *I*, then the same actions are possible at *h* and *h'*.
- This is a model of S5 modal (epistemic) logic.

Example: Imperfect Information



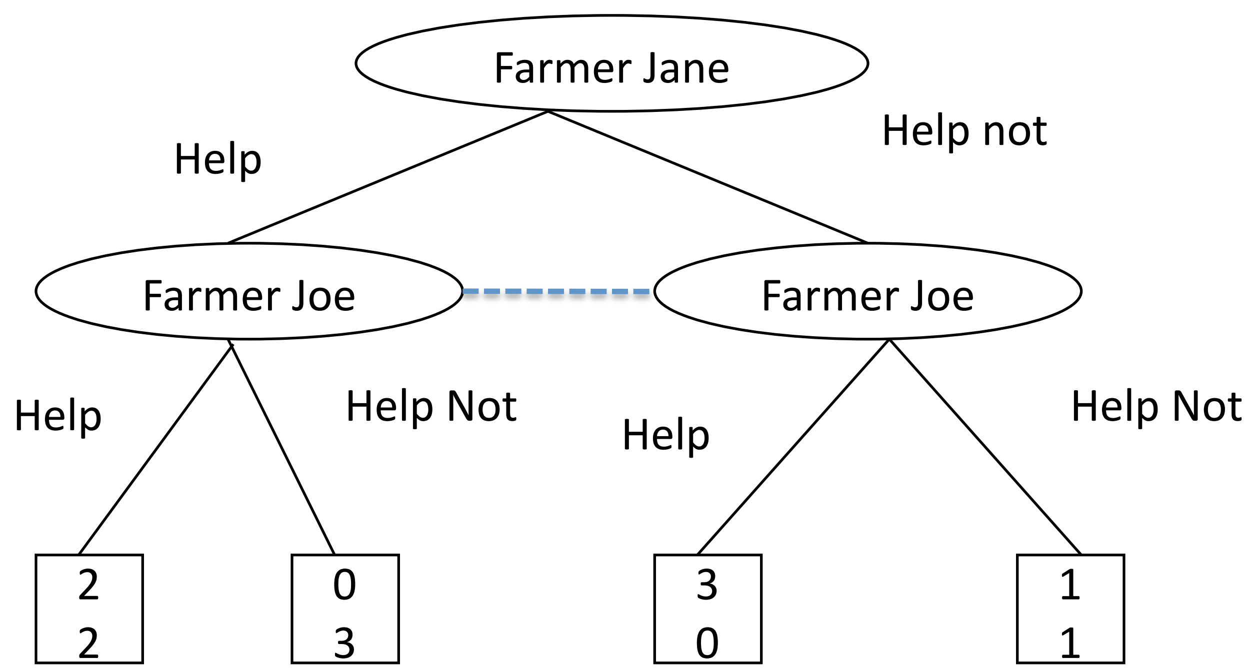
The information set means that Farmer Joe does not find out Farmer Jane's choice before making a move.

Strategies in Imperfect Information Games

- Since a player does not know which node in the info set she is at, her choice must be the same at all nodes.
- So a pure strategy for a game tree with imperfect information is a function *s* from nodes to possible actions such that if *h,h'* are in the same information set, then *s(h) = s(h')*.
- One can also consider strategies that choose an action at an info set *I* with some probability. Such strategies are called **behavioural strategies** (in CS, they are often called policies).

Exercise

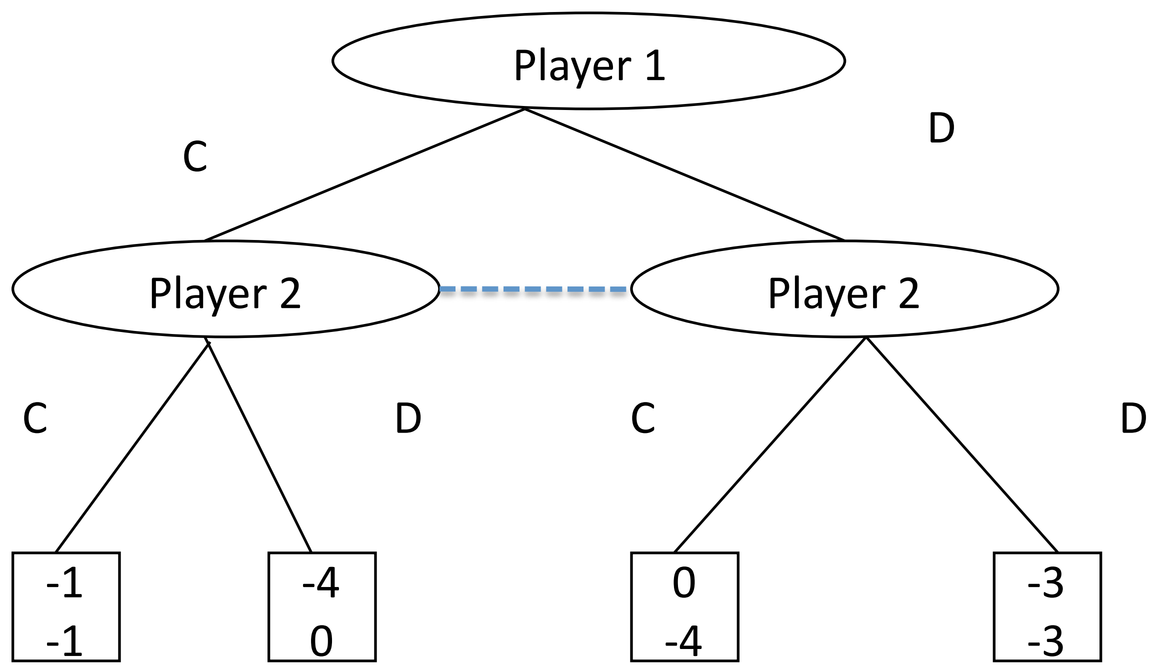
- Find the normal form for the imperfect info version of Hume's farmer game. What are the equilibria?



Simultaneous Move Games

- Just as we can translate game trees into matrix games, we can represent matrix games as simple game trees.
- In a 2-player game, we would have player 1 move at the root and choose an action, then it's player 2's turn.
- The information set of player 2 contains all of her nodes.

Example: PD as a game tree



Repeated Games

- A simple but important example of sequential games are repeatedly played matrix games.
- We start with the case where the game is repeated T times, where T is known to all players.
- After each round, players find out their respective choices and receive a payoff.
- The payoff for all T games is thus the sum of the payoffs in the individual games.

Repeated Games: Representation

- It is possible in principle to draw a game tree representing a repeated game. (Think about how.) But a more common simpler representation is to show a chain of matrices of length *T*, like this.

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

Subgame Perfect in Repeated Games

- A subgame in a repeated game is a finite sequence of plays of the repeated games.
- Note that although a repeated game is a game tree with imperfect information, each subgame *G* corresponds to an information set that contains just a single history h_G (belonging to player 1).
- A pair of pure strategies (s_1,s_2) for the repeated game is a subgame-perfect equilibrium if it is a Nash equilibrium in each subgame h_G.

Exercise: SPNE in repeated games

- Find the SPNE in the three-period PD below.

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

Infinite Games

- Infinite games are interesting mathematically.
- They are useful to model a situation in which players don’t know when their interaction is going to end, even if they do know that it will end at some point.
- They are useful to analyze learning strategies, as asymptotic analysis with infinitely large sample sizes is useful for statistical methods.
- Formally, an infinite game tree contains infinitely long branches, corresponding to infinitely long play sequences.
- The utility functions for the players are extended to maximal sequences (infinitely long branches) in the tree.

Infinitely Repeated Games

- The new issue for infinitely repeated game is how to define a total payoff for an infinite sequence of payoffs.