Introduction to Computational Game Theory

CMPT 882

Simon Fraser University

Oliver Schulte

# Matrix Games and Nash Equilibrium

## Matrix Games: Definition

- We have $n$ players $1,2,\ldots,n$.
- Each player $i$ has a strategy set $S_i$. Members of $S_i$ are denoted by $s_i$ or $a_i$.
- Each player $i$ has a utility function $u_i$: $S_1$ x ... x $S_n \rightarrow$R. Let $(s_1,..,s_n)$ be a tuple of strategies, one for each player. Then $u_i(s_1,\ldots,s_n)$ specifies the payoff for player $i$.
- A matrix game $G$ is **finite** if each strategy set $S_i$ is finite.
- In the case of two players and a finite game, we can represent the game in a **game matrix**: the rows are labelled with the strategies for player 1, the columns with the strategies of player 2, and the matrix entry $M_{ij}$ is the pairs of payoffs for the strategy pair $(s_i,s_j)$.
- The syntactic format of this definition is very simple. However, as we will see, matrix games can model strategic phenomena of arbitrarily high complexity.
- Equivalent Games:
  - Note that the labels of the strategies (rows or columns) do not matter.
  - Also remember that payoff functions can be rescaled by positive linear transformations.

## Examples

| | C | D |
|---|---|---|
| C | *a,a* | *b,c* |
| D | *c,b* | *d,d* |

- Any game where $c > a > b > d$ is a **Prisoner's Dilemma.**
- An alternative definition: A PD is a 2x2 game where each player has a strictly dominant strategy, and the outcome where both players choose their dominated strategy strongly Pareto-dominates the outcome where both players choose their dominant strategy.

## The TCP backoff game

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0,-4 | -3,-3 |

In the TCP protocol, you are supposed to back off from sending packets if congestion occurs. Interpret C as “back off” and D as “don’t back off”. If you don’t back off and other people do, your message experiences no delay. The game matrix is a possible model of delays in this scenario.

- Note the similarity to the fire exit problem.
- Like the fire exit, this is best treated as a population game. We’ll come back to this.

## Exercise

Write down a game matrix for the game of Rock, Paper, Scissor.

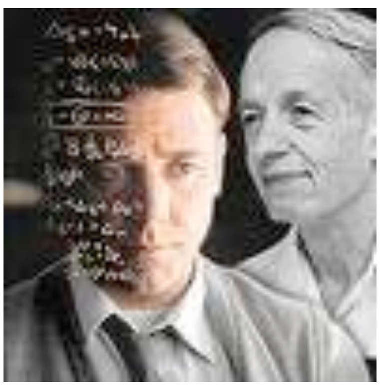
**Game Theory vs. Decision Theory**

- From an agent's point of view, a game is a special decision problem where States of the World = Choices of Other Players.
- Conversely, a single agent decision problem under uncertainty can be viewed as a special case of a 2-player game.
- Let player 1 be the agent. Let player 2 be "nature" or the "environment".
- The strategy set of the agent is the set of options in the choice problem. The strategy set of Nature is the set of states of the world.
- Nature is indifferent among all outcomes. (How do you represent this with a utility function?). Therefore we can omit listing Nature's payoffs.
- The fact that single-agent decision problems are a special game means that theorems of game theory also apply to decision problems.

| | Nature | |
|---|---|---|
| Agent | Heads | Tails |
| Bet 1 | 100 | 60 |
| Bet 2 | 70 | 100 |

## Solution Concepts

- A definition of how players will play a certain game is called a **solution concept**. We will consider a number of solution concepts.
- The most famous is the Nash equilibrium.



**Nash Equilibrium** (pure or deterministic strategies)

Consider a 2-player game.

- A strategy $s_1$ for player 1 is a **best response** against a strategy $s_2$ for player 2 iff there is no strategy $s'_1$ that does better against $s_2$ than $s_1$ does. In symbols, there can't be a strategy $s'_1$ such that $u_1(s'_1,s_2) > u_1(s_1,s_2)$.
- Similarly, a strategy $s_2$ for player 2 is a **best response** against a strategy $s_1$ for player 1 iff there is no strategy $s'_2$ such that $u_2(s_1,s'_2) > u_2(s_1,s_2)$.
- Note that a player can have more than one best response against another player's strategy.
- A **pair of strategies** ($s_1$, $s_2$) is a **Nash equilibrium** iff $s_1$ is a best response against $s_2$ **and** $s_2$ is a best response against $s_1$.

# Examples of Nash Equilibrium

| | Column | |
|---|---|---|
| Row | C | D |
| C | 0, 0 | -2, 2 |
| D | 2, -2 | -1, -1 |

The Prisoner's Dilemma has a unique N.E.: (D,D)

| | Column | |
|---|---|---|
| Row | L | R |
| L | 1, 1 | 0, 0 |
| R | 0, 0 | 1, 1 |

The pure Coordination Game has two N.E.'s: (L,L) and (R,R)

## Exercises for Nash Equilibrium

Find the Nash Equilibria of the following games.

| | Column | |
|---|---|---|
| Row | L | R |
| L | 2, 1 | 0, 0 |
| R | 0, 0 | 1, 2 |

The Battle of the Sexes

| | Column | |
|---|---|---|
| Row | Straight | Turn |
| Straight | -15, -15 | 4, 0 |
| Turn | 0, 4 | 1, 1 |

Chicken

What about Rock, Paper, Scissors?

## Justifications of Nash equilibrium

Economists predict routinely that players will play Nash equilibria. There are two main justifications for this.

1. **The steady state interpretation.** If players were to play the game repeatedly, they would keep improving their responses until they are each playing a best reponse.

2. **The self-enforcing agreement interpretation.** Suppose the players agreed in advance how they were going to play. Then they could trust each other to keep their agreement iff that agreement is a Nash equilibrium.

# Mixed Strategies