Simon Fraser University

Oliver Schulte

Computational Game Theory

The Logic of Contracts: Cooperative Games

Cooperative Game Theory

- The fundamental assumption of cooperative game theory is that **agents can make binding agreements**.

- The fundamental question of cooperative game theory is, **which** agreement will agents make?

- Cooperative game theory applies in **bargaining situations**, broadly construed. These include:

  1. International Relations (Treaties, Secession)
  2. Labour Contracts
  3. Constitutional Debates

Cooperative Games: Examples

1. Treasure Hunt: An expedition of *n* people have found a treasure in the mount; each pair of them can carry out one piece, but not more. How will they pair up?

2. A council has 4 members on it. They are considering how to allocate one unit of a divisible good ($1 million for example). The council makes its decisions by majority rule, with one exception: Player 4, also known as the big guy, has a **veto**. This means that if Player 4 rejects a proposal, then the proposal does not go through, and the council has to reconsider.

3. We have 1 seller who has an indivisible good for sale, and 2 buyers who want to buy it. The buyers are each willing to pay up to $1 for the good, the seller will sell it for 1 cent or more.

Cooperative Games: Definition

- Let $N = \{1,2,...,n\}$ be a finite set of players.
- A **coalition** is a nonempty subset of N, i.e., a coalition is a nonempty set of players.
- A **cooperative game** is a pair (N,v) where v is a function that assigns a number to each coalition C.
- The number $v(C)$ is called the **worth** of coalition C.

Example

Treasure Hunt: An expedition of n people have found a treasure in the mount; each pair of them can carry out one piece, but not more. How will they pair up?

We can model treasure hunt as the game (N,v) where $N = \{1,2,..,n\}$ and $v(C) = |C|/2$ if $|C|$ is even and $v(C) = (|C|-1)/2$ if $|C|$ is odd. ($|C|$ is the number of players in coalition C.)

Thus: $v(\{1\}) = v(\{2\}) = (1-1)/2 = 0$
$v(\{1,2\}) = v(\{2, 3\} = v(\{2,4\}) = 2/2 = 1$
$v(\{1,2,3\}) = (3-1)/2 = 1$

Cooperative Games: Exercises

1. A council has 4 members on it. They are considering how to allocate one unit of a divisible good ($1 million for example). The council makes its decisions by majority rule, with one exception: Player 4, also known as the big guy, has a **veto**. This means that if Player 4 rejects a proposal, then the proposal does not go through, and the council has to reconsider.

What is the set of players? Write down 5 coalitions.
What would the worth of a coalition be (what is *v*?). Write down what the worth of the 5 coalitions from above is.

2. We have 2 sellers that have an indivisible good for sale, and 4 buyers who want to buy it. The buyers are willing to pay up to $1 for the good, the sellers will sell it for 1 cent or more.

What is the set of players? Write down 5 coalitions.
What would the worth of a coalition be (what is *v*?). Write down what the worth of the 5 coalitions from above is.

The Core: Definition

- The most important concept for predicting what agreements players will make in a cooperative game is the **core**.
  We will give an informal definition of the core.

- An allocation **x** = ($x_1, x_2, ..., x_n$) assigns a payoff to players 1 through n; player 1 gets x1, player 2 gets x2 and so on.

- A coalition C is **effective** for an allocation $x_1, x_2, ..., x_n$ if $v(C) \geq)\sum_i x_i$.

- We say that an allocation **x** dominates an allocation **y** *via* coalition C iff

  1. C is effective for **x**, and
  2. every member of C prefers **x** to **y**.

- An allocation **y** is in the **core** of a cooperative game iff there is no other allocation **x** and coalition C such that **x** dominates **y** via C.

Effective Coalitions and the Core: Example

1. In the 3-player treasure hunt game: The coalition {1,2} can earn one piece of the treasure, so that coalition is effective for the allocation (1/2, ½, 0), as it is for (1,0,0), (0,1,0), (2/3, 1/3, 0). A 1-player coalition has 0 worth, so in all allocations that a 1-player coalition can effect, that player gets 0 payoff.

2. In the Treasure Hunt game, the core depends on whether there is an even or odd number of players. If there are 4 players, then the core is {(1/2,1/2,1/2,1/2)}. That is, we predict that the four players will team up in pairs and divide the piece of treasure equally. If the number of players is odd, the core is empty. If there are three players, the "fair" allocation (1/3,1/3,1/3) is not in the core because the coalition of 1 and 2 unanimously prefers the feasible allocation (1/2,1/2,0). But that payoff is not in the core either because the coalition of 1 and 3 unanimously prefers the feasible allocation (2/3,0,1/3).

When the core is empty, we predict that negotations will go around and around inconclusively.

Effective Coalitions and the Core: Exercises

1. A council has 4 members on it. They are considering how to allocate one unit of a divisible good ($1 million for example). The council makes its decisions by majority rule, with one exception: Player 4, also known as the big guy, has a veto. This means that if Player 4 rejects a proposal, then the proposal does not go through, and the council has to reconsider.

Write down 5 coalitions, and for each of them 1 allocation that the coalition can effect. Try to find 1 allocation that is in the core of this game.

2. We have 1 seller who has an indivisible good for sale, and 2 buyers who want to buy it. The buyers are each willing to pay up to $1 for the good, the seller will sell it for 1 cent or more.

Write down 3 coalitions, and for each of them 1 allocation that the coalition can effect. Try to find 1 allocation that is in the core of this game.

Veto Players and The Core

In certain types of cooperative games, it is easy to determine the core.

Definition

A **veto player** is a player without whom no coalition can achieve anything. That is, p is a veto player iff for all coalitions C, if p is not a member of C, then $v(C) = 0$.

Theorem
Suppose that a cooperative game (with divisible payoffs) has some veto players. Then in any allocation in the core, the non-veto players get 0 and the veto players divide the value of the game among themselves.

For example, in the council game the core predicts that the veto player will get everything. In the buyer-seller game, the core predicts that the seller will get everything.

(Can you think of other veto players in familiar situations?)