Sequential Data - Part 2

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Bishop PRML Ch. 13
Russell and Norvig, AIMA

Outline

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Inference Tasks

- Filtering: $p(z_t|x_{1:t})$
  - Estimate current unobservable state given all observations to date
- Prediction: $p(z_k|x_{1:t})$ for $k > t$
  - Similar to filtering, without evidence
- Smoothing: $p(z_k|x_{1:t})$ for $k < t$
  - Better estimate of past states
- Most likely explanation: $\arg\max_{z_{1:N}} p(z_{1:N}|x_{1:N})$
  - e.g. speech recognition, decoding noisy input sequence

Sequence of Most Likely States

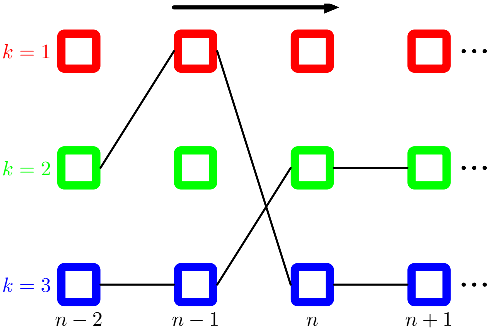
- Most likely sequence is not same as sequence of most likely states:

$$\arg\max_{z_{1:N}} p(z_{1:N}|x_{1:N})$$

versus

$$\left(\arg\max_{z_1} p(z_1|x_{1:N}), \ldots, \arg\max_{z_N} p(z_N|x_{1:N})\right)$$

Paths Through HMM



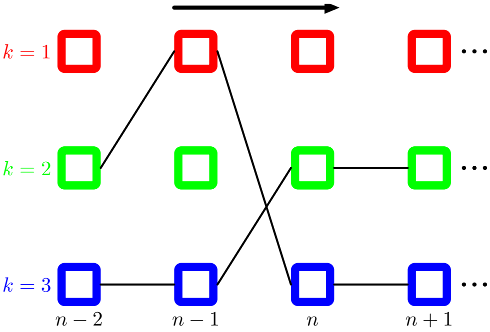
- There are K^N paths to consider through the HMM for computing

$$\arg\max_{z_{1:N}} p(z_{1:N}|x_{1:N})$$

- Need a faster method

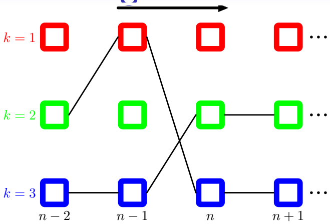
Viterbi Algorithm

Demo: `http://cmble.com/viterbi.jsp`



- Insight: for any value k for z_n, the best path $(z_1, z_2, \ldots, z_n = k)$ ending in $z_n = k$ consists of the best path $(z_1, z_2, \ldots, z_{n-1} = j)$ for some j, plus one more step
  - Don't need to consider exponentially many paths, just K at each time step
  - Dynamic programming algorithm – Viterbi algorithm

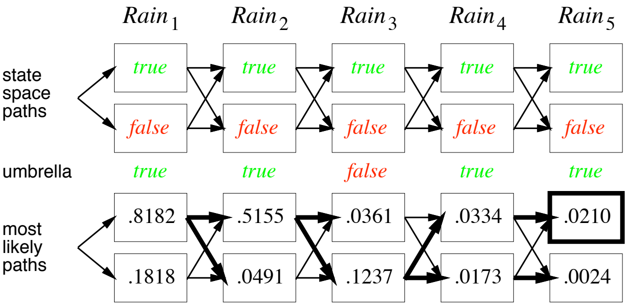
Viterbi Algorithm - Math



- Define message
  $w(n, k) = \max_{z_1,\ldots,z_{n-1}} p(x_1, \ldots, x_n, z_1, \ldots, z_n = k)$
- The max probability for ending up in state k at time n.
- From factorization of joint distribution:

$$
\begin{aligned}
w(n,k) &= \max_{z_1,\ldots,z_{n-1}} p(x_1,\ldots,x_{n-1},z_1,\ldots,z_{n-1})p(x_n|z_n=k)p(z_n=k|z_{n-1}) \\
&= \max_{z_{n-1}} \max_{z_1,\ldots,z_{n-2}} p(x_{1:n-1},z_{1:n-1})p(x_n|z_n=k)p(z_n=k|z_{n-1}) \\
&= \max_j w(n-1,j)p(x_n|z_n=k)p(z_n=k|z_{n-1}=j)
\end{aligned}
$$

Viterbi Algorithm - Example



R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(rain_1 = true) = 0.5$$

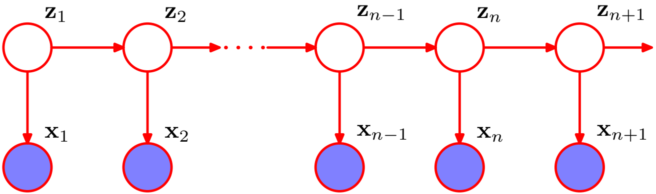
$$\begin{aligned} w(n,k) &= \max_{z_1,\dots,z_{n-1}} p(x_1,\dots,x_n,z_1,\dots,z_n = k) \\ &= \max_j w(n-1,j)p(x_n|z_n = k)p(z_n = k|z_{n-1} = j) \end{aligned}$$

Viterbi Algorithm - Complexity

- Each step of the algorithm takes $O(K^2)$ work
- With N time steps, $O(NK^2)$ complexity to find most likely sequence
- Much better than naive algorithm evaluating all K^N possible paths

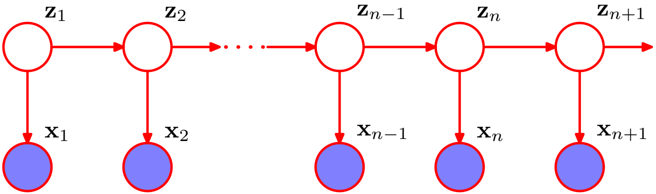
Outline

Continuous State Variables



- In HMMs, the state variable z_t is assumed discrete
- In many applications, z_t is continuous
  - Object tracking
  - Stock price, gross domestic product (GDP)
  - Amount of rain
- Can either discretize
  - Large state space
  - Discretization errors
- Or use method that directly handles continuous variables

Linear Dynamical Systems



- As in the HMM, we require model parameters – transition model and sensor model
- Unlike HMM, each of these is a conditional probability density given a continuous-valued z_t
- One common assumption is to let both be linear Gaussians:

$$
\begin{aligned}
p(z_t|z_{t-1}) &= \mathcal{N}(z_t; Az_{t-1}, \Sigma_z) \\
p(x_t|z_t) &= \mathcal{N}(x_t; Cz_t, \Sigma_x)
\end{aligned}
$$

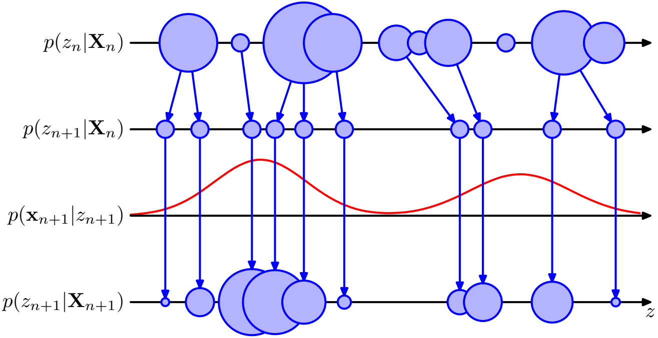
Continuous State Variables - Filtering

- Recall the filtering problem $p(z_t|x_{1:t})$ distribution on current state given all observations to date
- As in discrete case, can formulate a recursive computation:

$$p(z_{t+1}|x_{1:t+1}) = \alpha p(x_{t+1}|z_{t+1}) \int_{z_t} p(z_{t+1}|z_t)p(z_t|x_{1:t})$$

- Now we have an integral instead of a sum
- Can we do this integral exactly?
  - If we use linear Gaussians, yes: Kalman filter
  - In general, no: can use particle filter

Particle Filter Illustration



Particle Filter Example

Okuma et al. ECCV 2004

Conclusion

- Readings: Ch. 13.2.5, 13.3
- Most likely sequence in HMM
  - Viterbi algorithm – $O(NK^2)$ time, dynamic programming algorithm
- Continuous state spaces
  - Linear Gaussians – closed-form filtering (and smoothing) using Kalman filter
  - General case – no closed-form solution, can use particle filter, a sampling method