Epistemology, Reliable Inquiry and Topology

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1 Introduction

From one perspective, the fundamental notions of point-set topology have to do with sequences (of points or of numbers) and their limits. A broad class of epistemological questions also appear to be concerned with sequences and their limits. For example, problems of empirical underdetermination—which of a collection of alternative theories is true—have to do with logical properties of sequences of evidence. Underdetermination by evidence is the central problem of Plato's Meno [Glymour and Kelly 1992], of one of Sextus Empiricus' many skeptical doubts, and arguably it is the idea in Kant's antinomies, for example of the infinite divisibility of matter [Kelly 1995, Ch.3]. Many questions of methodology, or of the logic of discovery, have to do with sequences and their limits, for example under what conditions Bayesian procedures, which put a prior probability distribution over alternative hypotheses and possible evidence and form conditional probabilities as new evidence is obtained, converge to the truth ([Savage 1954], [Hesse 1970], [Osherson and Weinstein 1988], [Juhl 1993]). Some analyses of "S knows that p" seem to appeal to properties of actual and possible sequences of something—for example Nozick's proposal that knowledge of p is belief in p produced by a method that would not produce belief in p if p were false and would produce belief in p if p were true. Even questions about finding the truth under a quite radical relativism, in which truth depends on conceptual scheme and conceptual schemes can be altered, have been analyzed as a kind of limiting property of sequences [Kelly *et al.* 1994].

One might think that is the end of the matter, a superficial analogy and nothing more. In a series of papers and a recent book [Kelly 1995], Kelly has shown otherwise: An array of epistemological and methodological questions about reliable inquiry translate literally into topological questions. The philosophical fruits can be highly rewarding: a host of ambiguities hidden behind philosophical questions can be revealed; puzzling issues and proposed resolutions become amenable to proof and disproof; spanking new epistemological questions arise.

This paper is a friendly introduction to the embedding of epistemological questions in topology and some of its results.[1]

2 A Topological Space for Methodology

We will study methods that infer general conjectures from observations. We adopt the convention of encoding discrete observations by natural numbers. Two examples will illustrate this idea. Consider an empirical generalization like "all swans are white". A hypothetical ornithologist may investigate this hypothesis by examining one swan after another. We encode the observation "this swan is not white" by 0 and "this swan is white" by 1; see figure 1. There is an ancient debate about whether matter has fundamental indivisible parts. The modern physicist's version of this question is "are there only finitely many (types of) elementary particles?". Let us take as our data annual reports from particle physicists as to whether they have discovered a new elementary particle or not. In figure 2, "no new particles this year" is encoded by 0, and "a new particle has been discovered" by 1.

If inquiry continues indefinitely, an infinite sequence of observations is produced, which we represent by an infinite sequence of natural numbers encoding the particular observations. We refer to infinite sequences of natural numbers as *data streams.* Data streams are denoted by lower case Greek letters ε, τ, etc. We say that a data stream ε is consistent with a finite sequence of observations e if ε begins with e. An *empirical hypothesis* is a proposition about what will be observed. The truth of an empirical proposition hence depends only on the data stream. For example, the hypothesis "all (observed) swans are white" is true on the infinite sequence of observations which features only white swans, which in our encoding is the everywhere 1 data stream.[2] The hypothesis "there

[1] The first four sections of this paper appeared previously as [Schulte and Juhl 1996].

[2] In what follows, we formulate claims about what will be observed, e.g. "all observed swans are white", more idiomatically without explicit reference to observations, e.g. "all swans are white". In effect, we assume that if there is a non-white swan, this fact will eventually be included in the scientist's data. Methodological issues that arise when a scientist's observations depend on the experiments she performs are examined in [Kelly 1995, Ch. 14]. A setting with theory-laden evidence, in which the scientist's theories influence what the scientist observes is investigated in [Kelly and Glymour 1992],[Kelly *et al.* 1994],[Kelly 1995, Ch.15]. See also our

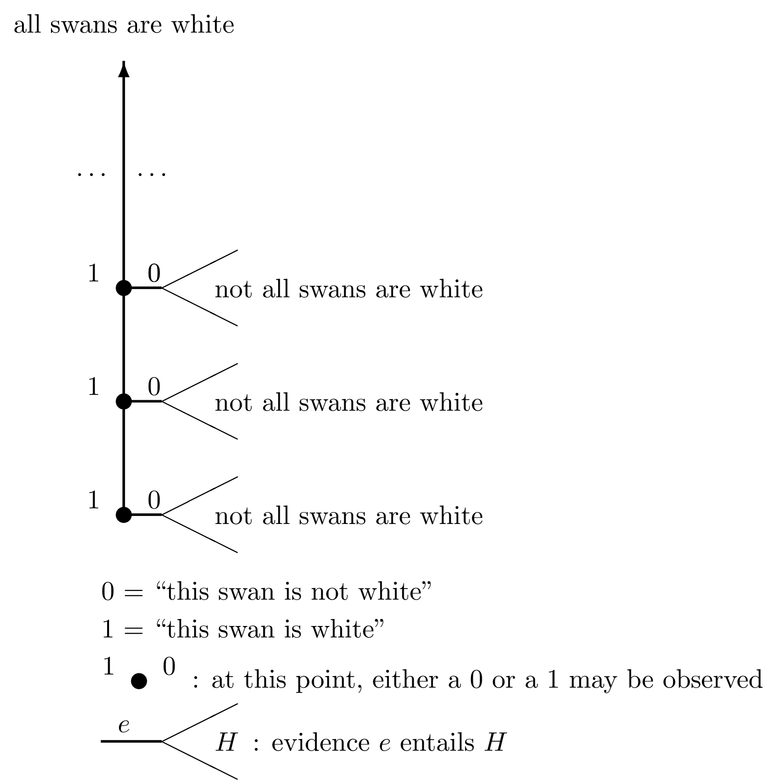
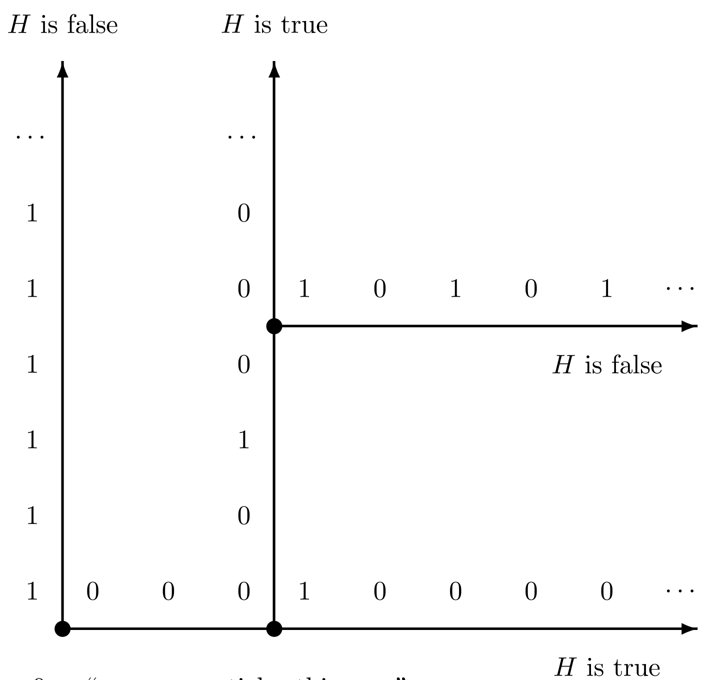


Figure 1: Encoding observations by natural numbers.



0 = "no new particles this year"
1 = "a new particle has been discovered"
H = "there are only finitely many particles"
0 1 : at this point, either a 0 or a 1 may be observed

Figure 2: How many particles are there?

are only finitely many elementary particles" is true just in case there is a time after which no new discoveries of particles are reported. In our encoding, this hypothesis is correct on any data stream which features only 0s after some finite time. We refer to the set of data streams on which a hypothesis is true as its *empirical content*. Since the truth of an empirical hypothesis depends only on the data stream, we identify empirical hypotheses with their empirical content, and treat such hypotheses as sets of data streams.

We think of the collection of all data sequences as the *space of empirical possibilities*. On this space we impose a topological structure which will allow us to study inductive inference with the tools of topology. A topological space can be built on one of three foundations: open sets, closed sets, or limit points. We begin with limit points.

Let H be an empirical hypothesis, that is, a set of data streams. A *limit point of* H is a data stream ε along which H is never refuted. This means that at any point along the data stream ε, there is an empirical possibility, i.e. a data stream τ, that is consistent with the data observed so far and makes H true. Figure 3 illustrates this concept.

In the example from figure 1, let H be the empirical content of the hypothesis "not all swans are white". In our encoding, H is the set of data streams along which a 0 appears at some point. The everywhere 1 data stream ("all swans are white") is a limit point of H, for no matter how many white swans (1s) have been observed, the next swan could be black. In the second example, let H be the empirical content of the hypothesis "there are only finitely many elementary particles". In our encoding, H is the set of data streams along which eventually only 0s appear. But no matter how many particles have been observed so far, it is possible that no new particles will be found, i.e. that the data stream obtained in inquiry continues with 0s only. This means that H is never refuted along any data stream ε. So *all* data streams are limit points of H.

An empirical hypothesis H is *closed* just in case H contains all of its limit points. The hypothesis "all swans are white" is closed. But "not all swans are white" is not closed, because the everywhere 1 data stream is a limit point of this hypothesis on which the hypothesis is false. The hypothesis "there are only finitely many particles" is not closed, and neither is "there are infinitely many particles": All data streams are limit points of these hypotheses, but not all data streams make these hypotheses true. The *complement* of an empirical hypothesis H is the set of data streams which are not in H, i.e. on which H is false. The particle example shows that sometimes neither H nor the complement of H is closed. An empirical hypothesis H is *open* just in case the complement of H is closed. The hypothesis "some swan is not white" is open, but its complement is not; "there are only finitely many particles" is not open, and neither is its complement.

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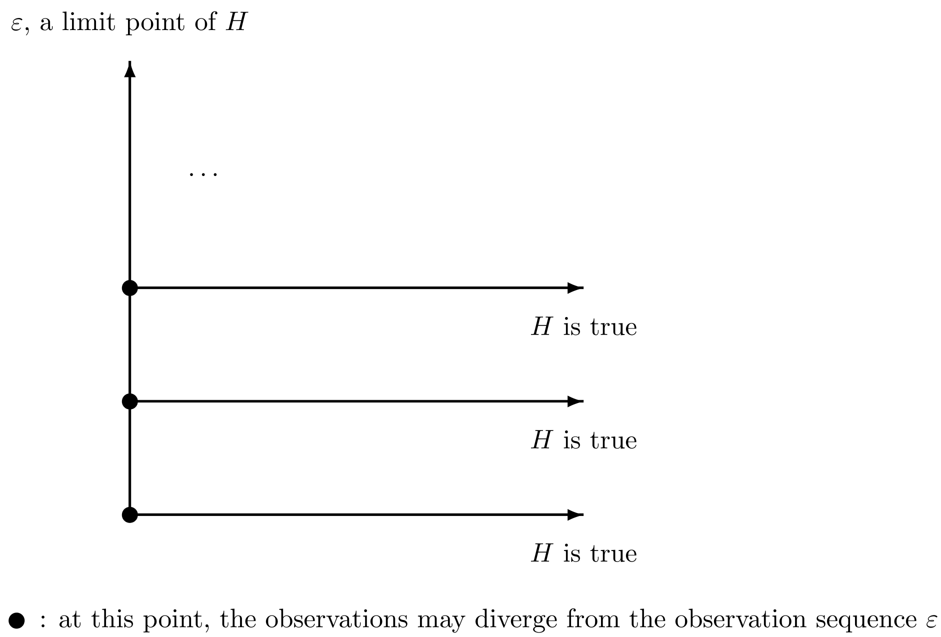

Figure 3: A limit point of H.

3 Universal Generalizations as Missing Limit Points

These topological structures played a role in philosophical debates long before topology was invented. Consider one of Sextus Empiricus' [1985] arguments for inductive skepticism from the second century:

> [The dogmatists] claim that the universal is established from the particulars by means of induction. If this is so, they will effect it by reviewing either all the particulars or only some of them. But if they review only some, their induction will be unreliable, since it is possible that some of the particulars omitted in the induction may contradict the universal. If, on the other hand, their review is to include all the particulars, theirs will be an impossible task ...

Figure 4 illustrates Sextus' scenario. The topological structure is the same as in figures 1 and 3: in each case, the universal generalization is a limit point of its complement. This is no accident. Limit points are essentially connected to the verifiability of empirical hypotheses; and there is a theorem that tells us how.

We say that an empirical hypothesis H is (logically) *entailed* by a finite data sequence e if all data streams consistent with e make H true, i.e. are members of H. So an empirical proposition like "e has been observed" entails another empirical proposition H just in case the former, viewed as a collection of data streams, is a subset of the latter. For example, any data sequence along which a black swan is observed entails the hypothesis "not all swans are white", and no finite data sequence entails "all swans are white". An empirical hypothesis H is *verifiable with certainty* if the hypothesis is eventually entailed by the evidence whenever it is true. This means that along any data stream ε that makes H true some finite number of observations from ε logically entails H. The hypothesis that not all swans are white is verifiable with certainty, but its negation is not. We say that a hypothesis H is (conclusively) *falsified* by a finite data sequence e if e entails the complement of H. A hypothesis is *refutable with certainty* if the hypothesis is conclusively falsified whenever it is false.

Suppose there is a limit point ε of an empirical hypothesis H that makes H false, i.e. ε is not in H. Then by the definition of a limit point, no finite number of observations from ε falsifies H. So if ε is the data stream obtained in inquiry, H is false but never falsified. This shows that if a hypothesis H is refutable with certainty, it must contain all of its limit points. In other words, H must be closed. Conversely, it is possible to prove [Kelly 1995, Proposition 4.6] that whenever a hypothesis H is closed, H is refutable with certainty. This leads to a *characterization* of the topological structure of refutable hypotheses.

Theorem 1 *An empirical hypothesis H is refutable with certainty if and only if H is closed.*

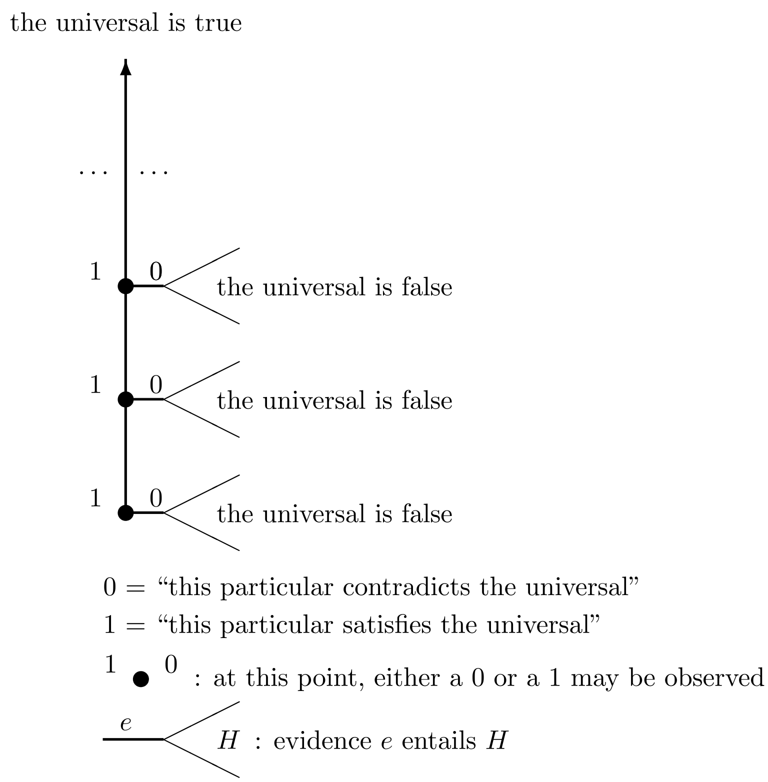

Figure 4: Sextus' argument that a universal generalization cannot be conclusively established by induction.

We observe that a hypothesis is verifiable with certainty just in case its complement is refutable with certainty. For a data sequence that entails a hypothesis H falsifies the complement of H, and vice versa. This observation yields a characterization of verifiable hypotheses.

Theorem 2 *An empirical hypothesis H is verifiable with certainty if and only if H is open.*

As noted above, the hypothesis that there is only a finite set of elementary particles is neither open nor closed. So by the characterization theorems, this hypothesis is neither verifiable nor refutable with certainty. On the other extreme, the investigation of hypotheses that are both closed and open will yield certainty whether they are true or false. Such hypotheses are said to be *decidable with certainty*. A trivial example is the a priori certain hypothesis "something will be observed", which is true on every data stream. This reflects the fact that in every topological space, the entire space and its complement, the empty set, are both closed and open. More generally, all hypotheses which are conjunctions, disjunctions or negations of propositions of the form "data sequence e will be observed" are decidable with certainty.

4 Falsifiability and Denseness

Karl Popper's *falsifiability criterion* [1968] has played a prominent role in 20-th century debates about methodology. According to this criterion, a hypothesis is a legitimate scientific hypothesis only if it is falsifiable. How might we formulate the falsifiability criterion for the kind of empirical hypotheses we have examined so far? An obvious interpretation is to require that an empirical hypothesis must be refutable with certainty. But this will hardly do for a falsificationist: the tautological hypothesis that is true no matter what is refutable with certainty, but is not subject to any 'critical tests'. It is true that the tautology is refuted whenever it is false—but it never is false. This suggests another reading: Say that an empirical hypothesis *has a critical test* if there is a finite sequence of observations which conclusively falsifies the hypothesis. Should we take the falsifiability criterion to aim at hypotheses with critical tests? This rules out the tautology, but still seems too lenient. For example, the hypothesis "the first swan will be white" may be refuted if the first swan is black; but after it passes this initial muster, the hypothesis tells us nothing more. A Popperian diagnosis would be that this hypothesis does not have enough content, where content is determined with respect to a probability measure: the more probable a hypothesis is, the less content it has. An alternative remedy is to require that there be no end to the critical tests: no matter how much muster a hypothesis has passed so far, there should be further observations which might falsify it. Accordingly, we define:

An empirical hypothesis H is *falsifiable* if and only if H has a critical test. An empirical hypothesis H is *always falsifiable* if and only if for every finite sequence of observations e, there are further observations e' which extend e and falsify H.

The fundamental topological notion of denseness corresponds exactly to these conceptions of falsifiability. An empirical hypothesis H is (everywhere) *dense* if every data stream is a limit point of H. A hypothesis H is *dense in a finite data sequence* e if every data stream that is consistent with e is a limit point of H. A hypothesis H is *nowhere dense* if H is not dense in any finite data sequence e. The hypothesis "all swans are white" is nowhere dense, and its complement is everywhere dense. The hypothesis "there are only finitely many elementary particles" is everywhere dense, and so is its complement.

Recall that by definition, a data stream ε is a limit point of an empirical hypothesis H if no finite sequence of observations from ε conclusively falsifies H. So if H has a critical test e, then no data stream that begins with e is a limit point of H, and H is not everywhere dense. Conversely, if there is no critical test for H, then H is never conclusively falsified along any data stream, and so all data streams of H are limit points of H. Thus we have:

Proposition 3 *An empirical hypothesis H is falsifiable if and only if H is not everywhere dense.*

Similarly, H can be conclusively falsified given a finite data sequence e if and only if some data streams that are consistent with e are not limit points of H; i.e. if and only if H is not dense in e.

Proposition 4 *An empirical hypothesis H is always falsifiable if and only if H is nowhere dense.*

There is no guarantee that the evidence will let an always falsifiable hypothesis "die in our stead" when it is false. A hypothesis may survive an eternal barrage of critical tests and yet be false. In topological terms, a nowhere dense set is not necessarily closed. Consider the hypothesis H "no swan is gray, but at least one swan is black". The empirical content of this hypothesis is illustrated in figure 5. Since the appearance of a gray swan may falsify the hypothesis at any point, H is always falsifiable. But if all swans are white, the hypothesis is false, although never refuted by the evidence. Propositions of the form "there are between 1 and n instances of X" are a broad class of always falsifiable hypotheses which may be false but never refuted by the data. Such hypotheses are always falsified by the observation of more than n instances of X, and false but never refuted when there are none. Some examples are "there is exactly one species of black swans", "there are exactly 37 elementary particles", "there are between one and three other intelligent life forms in the universe".

We have examined two properties of empirical hypotheses which the falsifiability criterion may be aimed at. On the one hand, one may want to be sure

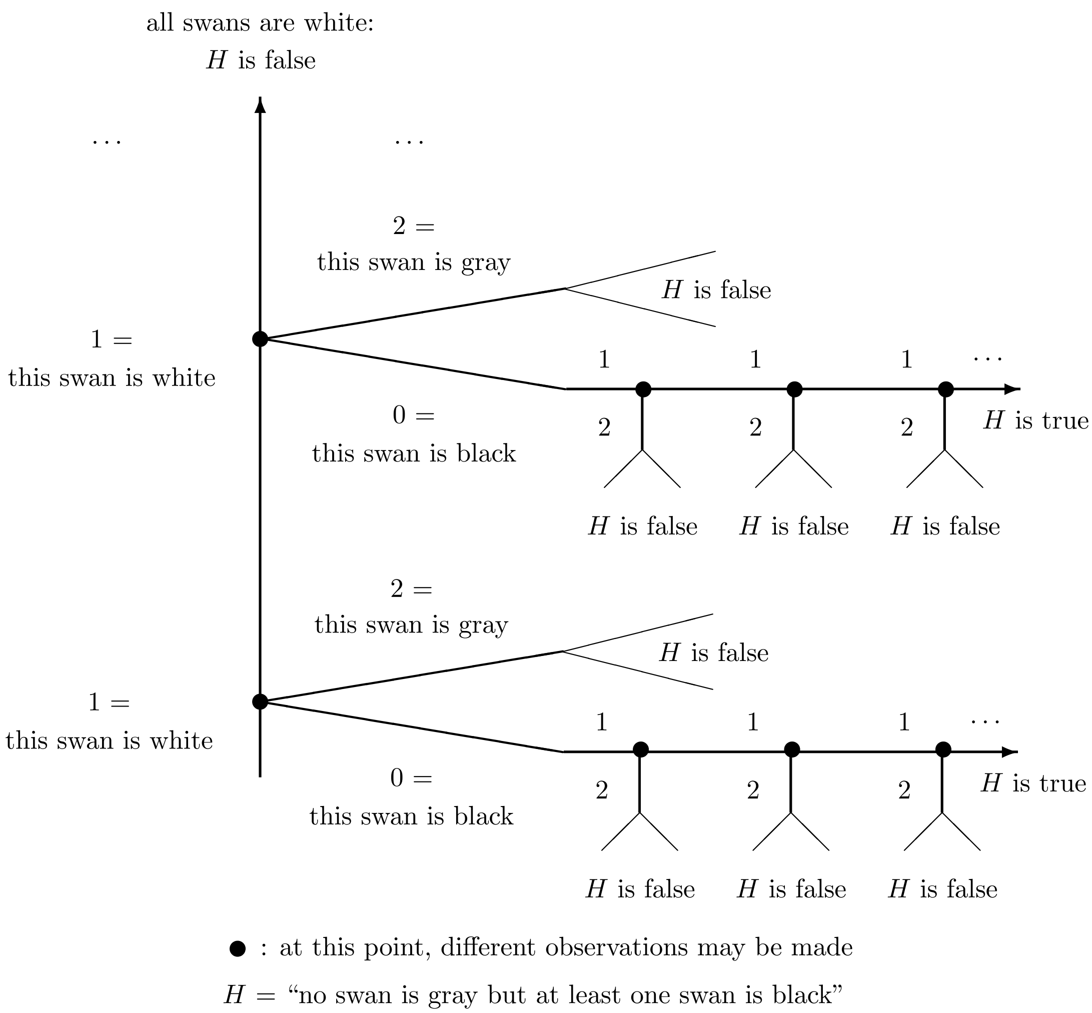

Figure 5: H is always falsifiable but not refutable with certainty.

that the data will logically refute a hypothesis whenever it is false. On the other hand, one may require the existence of critical tests (for eternity) for the hypothesis. These desiderata correspond to distinct topological properties; the first is satisfied by closed sets, the second by those hypotheses which are not everywhere (or nowhere) dense. Topology clarifies distinctions which unguided intuition tends to conflate.

5 Inquiry Without Certainty

Sextus Empiricus argued that we could never know with certainty that a universal hypothesis is true. The critical tests of falsificationists allow us to conclude with certainty that a hypothesis is false when it fails one of them. But science may arrive at the truth in the limit of inquiry without ever providing certainty about any of its hypotheses. This is the kind of empirical success that philosophers like Peirce, James, Putnam and Reichenbach have endorsed. William James [1948], for one, emphasized the difference between knowing and knowing with certainty that we know:

> We may talk of the *empiricist* and the *absolutist* way of believing the truth. The absolutists in this matter say that we not only can attain to knowing truth, but we can know when we have attained to knowing it; while the empiricists think that although we may attain it, we cannot infallibly know when. To *know* is one thing and to know for certain *that* we know is another.

This conception of empirical success has been formulated by philosophers and logicians in essentially the same way in a number of formal settings [Osherson *et al.* 1991], [Gold 1967], [Putnam 1965][3]. Kelly's [Kelly 1995] version is given in the topological framework of this paper. The key idea is that a scientist may eventually stabilize to true beliefs without ever knowing that she has done so. Consider again a universal generalization like "all swans are white". An ornithologist may conjecture that this hypothesis is true as long as all observed swans are white, and take the hypothesis to be refuted when a swan is found that is not white. If all swans are white, the epistemic situation of the ornithologist is just as described by James: The ornithologist always has the true belief that all swans are white, but she never knows this fact with certainty—for the next swan might be black, as Sextus pointed out. If it is not true that all swans are white, then our ornithologist will believe this after the first non-white swan

[3] Putnam's paper deals with success in the limit concerning *formal*, mathematical problems. Putnam recognized the applicability of his standard of success in inductive inference. This perspective reveals strong analogies between formal and empirical inquiry; the solvability of deductive problems is characterized by the same kind of structures as the solvability of inductive problems. [Kelly and Schulte 1995a] explores these analogies in some detail.

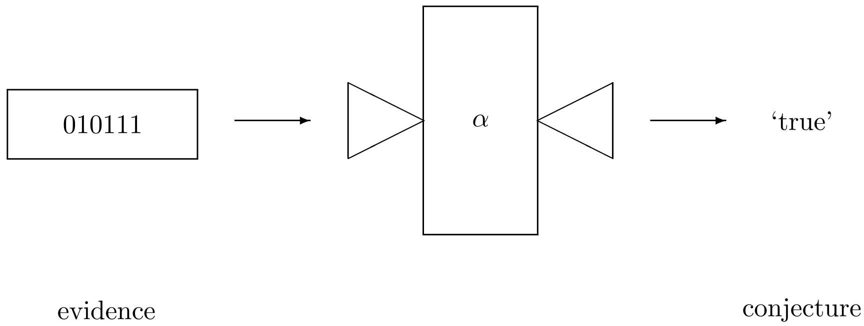

Figure 6: An assessment method.

has appeared (and in fact be certain of it). So no matter what is the case, the ornithologist will eventually always have only true beliefs.

In general, let H be an empirical hypothesis; an *assessment method* for H takes as input a finite data sequence and gives as output a conjecture 'true' or 'false'; see figure 6. An assessment method produces a conjecture after each new observation along an infinite data stream. We say that an assessment method *stabilizes to* 'true' ('false') along an infinite data stream ε just in case after some time the assessment method conjectures 'true' ('false') forever on the observations from ε; see figure 7. An assessment method α *decides* H *in the limit* on a data stream ε if

1. H is correct on ε and α stabilizes to 'true' along ε, or
2. H is not correct on ε and α stabilizes to 'false' along ε.

An assessment method α *reliably* decides H in the limit if α decides H in the limit on every data stream ε. Finally, an empirical hypothesis H is said to be *decidable in the limit* if there is an assessment method α for H that reliably decides H in the limit. A reliable assessment method is guaranteed to eventually arrive at the truth about H, but we may never know when it has done so.

As we noted, the ornithological assessment method described above reliably decides "all swans are white" in the limit. In fact, any hypothesis H that is refutable with certainty is reliably decidable in the limit: Conjecture 'true' until H is refuted by the evidence, and 'false' (with certainty) thereafter. If H is never logically refuted by the data, the hypothesis must be true, since

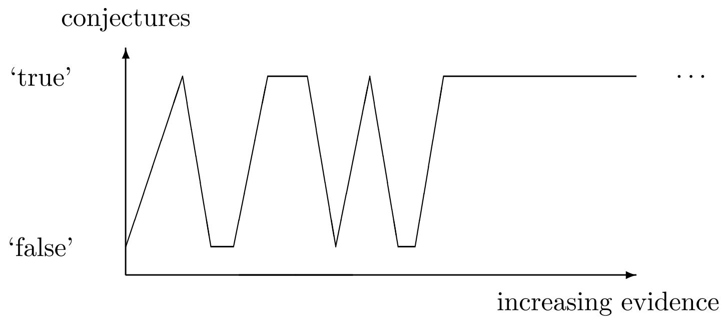

Figure 7: Conjectures stabilize to 'true'.

H is refutable with certainty; so this method reliably decides H in the limit. Similarly, verifiable hypotheses are decidable in the limit: conjecture 'false' until they are established, and 'true' thereafter. All of the hypotheses considered in the previous section are decidable in the limit, although they are neither refutable nor verifiable with certainty. For example, to reliably assess "there are between 1 and 3 other intelligent life forms in the universe", suppose we have launched an exhaustive space mission to seek out new forms of life. The method that conjectures 'false' while no new life forms have been found, 'true' if we have encountered at least one but no more than three, and 'false' (with certainty) after meeting four or more is guaranteed to eventually arrive at the truth about this hypothesis.

On the other hand, the hypothesis that there are only finitely many elementary particles is *not* decidable in the limit. For consider an arbitrary assessment method which aspires to reliably decide this hypothesis in the limit. An inductive kind of Cartesian demon might present the method with one new particle after another until the method conjectures that there are infinitely many particles. At that point, the demon can stop, having presented only finitely many particles, until our aspirant's confidence in the existence of infinitely many particles is shaken, and the method conjectures that there are only finitely many particles. Then the demon resumes the discovery of new particles, until the method changes its mind again to guess that there are infinitely many particles, etc. In order to decide the hypothesis in the limit, the method must eventually always conjecture 'true' or always conjecture 'false' during this interplay. If the method stabilizes to 'true', the demon presents new particles forever, so the method settles on a false belief. On the other hand, the demon stops presenting new particles when the method stabilizes to 'false', so in that case there are only finitely many particles, contrary to the opinion which the method arrives at. Hence *all* assessment methods fail to find the truth about the hypothesis that there are only finitely many particles on some possible data stream. This kind of argument against all possible methods is known as a *diagonal* argument [Putnam 1975][4].

So what must an empirical hypothesis be like if inquiry is to reliably find out whether it is true or false? As with verifiable and refutable hypotheses, we can give a topological characterization of which hypotheses are decidable in the limit. The key concept is that of a countable disjunction of refutable hypotheses.[5] An empirical hypothesis H is a *countable disjunction of refutable hypotheses* $H_1, H_2, \ldots$ if H is true just in case one of the H_i is. For example, the hypothesis "there are only finitely many elementary particles" is equivalent to the disjunction "there are no elementary particles or there is at most one or there are at most two or ...". Each hypothesis of the form "there are at most n elementary particles" is refutable with certainty—if it is false, eventually more than n particles will be reported. A verifiable hypothesis is a countable disjunction of decidable hypotheses. For example, "not all swans are white" is equivalent to "the first swan is not white or the second swan is not white or ...". In general, a verifiable hypothesis is true just in case one of the pieces of evidence which entail it is obtained. A refutable hypothesis is itself a trivial disjunction of refutable hypotheses. So any verifiable hypothesis and its complement, which is refutable, are countable disjunctions of refutable hypotheses, and the same is true of a refutable hypothesis and its verifiable complement. The next theorem tells us that this is the characteristic property of *all* hypotheses that are decidable in the limit.

Theorem 5 *An empirical hypothesis H is decidable in the limit if and only if both H and the complement of H are countable disjunctions of refutable hypotheses.*

The proof may be found in [Kelly 1995, Proposition 4.9]. It follows from the theorem and the diagonal argument given above that although "there are only finitely many particles" is a countable disjunction of refutable hypotheses, its complement "there are infinitely many particles" is not.

[4]Thus we have two forms of argument to establish whether an inductive problem can be solved by a reliable method: for a positive answer, we exhibit a reliable method, and for a negative answer, we give a diagonal argument. Whether this kind of question can always be settled this way, hinges, remarkably, on issues in the foundations of mathematics: [Juhl 1995a] shows that under the Axiom of Choice and assuming the Continuum Hypothesis, there is an empirical hypothesis H such that neither a reliable method nor a diagonal argument establishes whether H is decidable in the limit. From the Axiom of Determinacy, on the other hand, it follows that for every assessment problem there either is a reliable method or a diagonal argument against all reliable methods.

[5]Such sets are known in topology as G_δ sets.

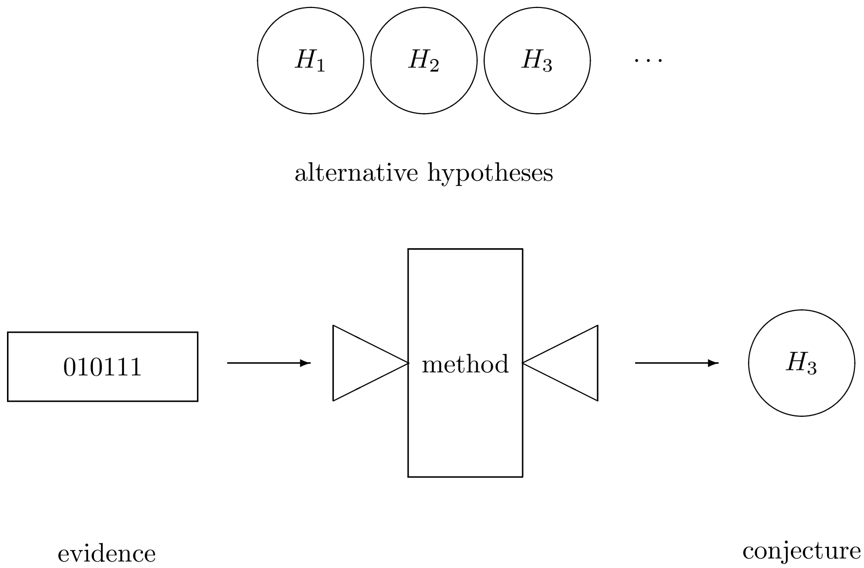

Figure 8: A discovery method.

6 Reliable Discovery

Suppose a scientist entertains a number of alternative hypotheses, and wants to find a true one among them. For example, the hypotheses could be theories of elementary particles that predict which particles exist and how they behave. Or we may think of different natural languages and a child trying to learn the language spoken in her environment. We refer to the task of finding a true hypothesis among a number of possibilities as a *discovery problem.* Popper's falsifiability criterion was proposed as part of a general *logic of discovery.* Do falsifiable hypotheses play a special role in scientific discovery? We will address this question through a model of discovery that is simplified but brings out some important philosophical points.

A *discovery method* for alternative hypotheses $H_1, H_2, \ldots$ takes as input a finite data sequence and conjectures one of the alternatives to be true; see figure 8. A discovery method *stabilizes* to a hypothesis H along a data stream ε if after some time the assessment method always conjectures H on the observations from ε. A discovery method *identifies a correct hypothesis* on a data stream ε just in case the method stabilizes to a correct hypothesis on ε. A discovery method *reliably* identifies a correct hypothesis from $H_1, H_2, \ldots$ just in case the method identifies a correct hypothesis from $H_1, H_2, \ldots$ on every data stream. In what follows, we make the simplifying assumption that the alternative hypotheses are exhaustive and mutually exclusive, so that on each data stream exactly one alternative is correct. Similar but somewhat more complicated results hold without this assumption (cf. [Kelly 1995, Chapter 9]).

What must the alternative hypotheses be like if it is possible to reliably identify the true alternative? It suffices if each hypothesis is refutable with certainty; in that case a Popperian *conjectures and refutations* method is guaranteed to identify a true hypothesis. Begin with the first hypothesis (which is the 'boldest' if the alternatives are ordered by audacity), and conjecture H_1 until H_1 is falsified by the evidence. Then move on to H_2, conjecture H_3 if H_2 is falsified, etc. To see that this works, let H_n be the true hypothesis. So $H_1, H_2, \ldots, H_{n-1}$ are false. Since every alternative is refutable with certainty, eventually each of the false alternatives $H_1, H_2, \ldots, H_{n-1}$ is conclusively falsified. After that point, our conjectures and refutation procedure always conjectures the true hypothesis H_n.

It may seem as though the alternative hypotheses *must* be refutable with certainty if inquiry can reliably identify the true one. For otherwise we run the danger of always maintaining a false conjecture which is never conclusively refuted by the evidence. But the next theorem tells us that this argument is fallacious; reliable discovery does not require that the alternatives under consideration are refutable with certainty, but only that they are decidable in the limit.

Theorem 6 *Let $H_1, H_2, \ldots$ be a collection of mutually exclusive and exhaustive empirical hypotheses. Then it is possible to reliably identify the true alternative from $H_1, H_2, \ldots$ if and only if each H_i is decidable in the limit.*

The proof may be found in [Kelly 1995, Corollary 9.12]. Indeed, when the alternatives in question are not refutable with certainty, the conjectures and refutations recipe may lead us far from the truth. In such cases the conjectures and refutation scheme is a hindrance rather than an aid to scientific discovery. To illustrate this point, let us consider the problem of estimating the probability of an event, for example the probability that a given toss of a coin will come up heads. Hans Reichenbach [1949] believed that all inductive inference could be reduced to such estimates. Reichenbach subscribed to the frequentist interpretation of probability. For a frequentist, to say that the probability of a coin coming up heads is, say, 1/2 means that as the coin is tossed more and more often, the rate of heads comes closer and closer to 1/2.[6] To make this idea precise, let e be a finite (non-empty) sequence of coin tosses. The *relative frequency of heads in e* is the number of heads occurring in e divided by the

[6] Provided the successive tosses are independent of each other.

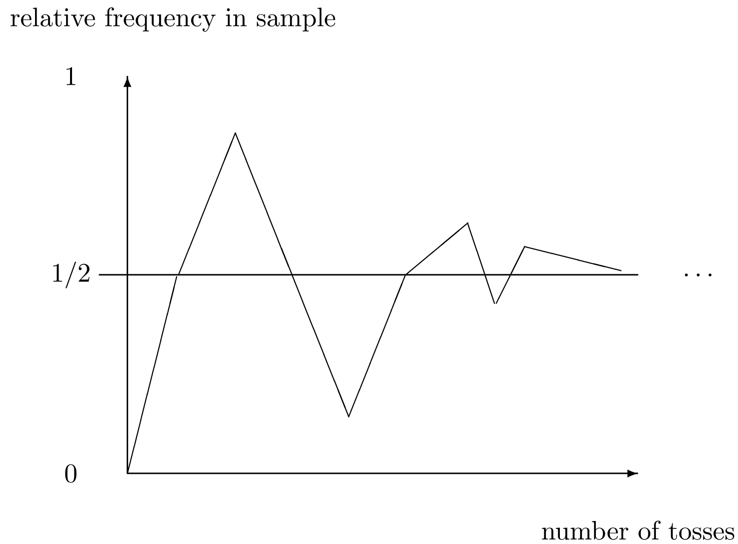

Figure 9: The limit of the relative frequencies is 1/2.

number of total tosses in e. Given an infinite sequence ε of coin tosses, the *limiting frequency of heads in* ε is, for example, 1/2 just in case for every ratio r different from 1/2, eventually the relative frequency of coin tosses in the finite initial sequences of ε is always closer to 1/2 than to r; see figure 9.

Now suppose we think the probability of a certain coin coming up heads or tails is either 1/4 or 3/4. On the frequentist interpretation of probability, this means that if we toss the coin indefinitely, the limiting frequency of heads in the resulting infinite sequence of tosses will be either 1/4 or 3/4. Given an infinite sequence of coin tosses, a natural procedure reminiscent of Reichenbach's 'straight rule' reliably identifies the true limiting relative frequency[7]: Conjecture "the probability is 1/4" if the relative frequency of heads in the tosses observed so far is closer to 1/4 than to 3/4, and conjecture "the probability is 3/4" otherwise. However, no finite sequence of coin tosses conclusively falsifies either of

[7] Reichenbach's straight rule tells us to conjecture that the limiting frequency of the event of interest is its relative frequency in the current sample. He pointed out that this rule is guaranteed to come arbitrarily close to the limiting frequency of the event if this limit exists, and took this fact to be a 'vindication' of the straight rule [Salmon 1991]. [Juhl 1994] shows that the straight rule approaches the correct limiting frequency as fast as possible (in a certain definite sense).

these hypotheses. So a conjectures and refutations method never changes its initial conjecture, and fails whenever its initial estimate is false. This example can be extended to any case in which the number of alternative probabilities under consideration is finite. Popper's reply to this problem [1968, Sections 65–68] was that in practice, statements about the probability of an event are "used" as falsifiable hypotheses. But as we have seen, hypotheses about limiting relative frequencies present no special challenge to empirical inquiry if the goal of empirical inquiry is to reliably find the true limiting relative frequency among a finite number of alternatives. The reason why such hypotheses pose methodological problems for Popper's falsificationism is not that there are difficulties in interpreting them, but that the conjectures and refutations method does not realize the full potential of reliable scientific discovery.

7 Related Results and More Philosophy

The mathematical study of inductive inference in the methodological framework of this paper is known as *learning theory*. In this concluding section we present a selective survey of further pertinent theorems and learning-theoretic approaches to philosophical issues.

The characterization theorems presented above provide methodological interpretations for a number of standard results in topology. We mention two more: In all of our examples, we assume that only a finite number of observations could arise at each stage (typically two, encoded by 0 or 1). The topological space that results when we restrict the space of possible data streams to those consistent with this assumption is *compact*. The *Heine-Borel theorem* implies that if an empirical hypothesis is decidable with certainty in a compact space, its truth or falsehood must be entailed by the data by a specific time t determinable a priori (cf. [Kelly 1995, Ex.4.14]). In other words: if all possible observations are drawn from a finite set, and we are guaranteed to eventually have certainty about a hypothesis, then there is a deadline by which the truth-value of the hypothesis will be settled, no matter what the actual data are. The idea behind this result is this: We can think of the set of possible data sequences in a compact space as forming a finitely branching tree. If there is an infinite branch in this tree (i.e. a data stream) along which a given hypothesis H is neither entailed nor falsified, H is not decidable with certainty. So if H is decidable with certainty, there cannot be such an infinite branch. But then the subtree comprising all data sequences that entail or falsify H is a finitely branching tree without an infinite branch, hence itself is finite and so has a longest branch. The length of this branch is the deadline by which the truth-value of H will be established.

Countable disjunctions of always falsifiable hypotheses are called *meager* sets in topology, because they are consistent only with a small[8] number of data

[8] Meagerness is a topological sense of 'small'. Meager sets can be 'big' if one takes the

streams, and hence have high empirical content. Another aspect of meager sets that suits falsificationism is that no meager hypothesis is ever entailed by the data; this follows from Baire's *category theorem.*

Learning theory has been applied to analyze various inductive settings of interest, for example language learning [Osherson *et al.* 1986], and the methodology of cognitive science [Glymour and Kelly 1990] and of cognitive neuropsychology [Glymour 1994], [Bub 1994]. In the study of *computable* learners, a fruitful combination of topological techniques with computability theory has been extensively investigated by philosophers, logicians and computer scientists [Angluin and Smith 1983]. For example, must a computer be able to derive the predictions of a theory in order to test the theory? Not necessarily: there is an empirical hypothesis H for which we can write a computer program that eventually pronounces H false if (and only if) H is false, even though the predictions of H are infinitely uncomputable[9] [Kelly and Schulte 1995b]. Is the point of discovery to find theories which allow us to make predictions? There is a collection of alternative theories and a computer program that reliably finds a true predictive theory among them, such that no computer can derive the predictions of the theories it discovers [Blum and Blum 1975, 131, example 2]; see also [Kelly 1995, Proposition 11.13].

An apparently banal prescription of methodology is to reject a hypothesis when it is falsified by the evidence. This principle is endorsed by such accounts of 'inductive rationality' as Bayesian conditionalization, theories of the confirmation of hypotheses by the evidence [Hempel 1965, I.1], and 'minimal change belief revision' theories along the lines of [Gärdenfors 1988]. We refer to this as the *consistency* principle, and to methods which satisfy it as consistent. Innocuous as it may seem, the consistency principle severely restricts the scope of successful inquiry for agents who are not logically omniscient. In particular, there is an empirical hypothesis for which there exists a (non-consistent) computer program that eventually declares H false if (and only if) H is false—but no computable consistent method can even decide H in the limit (and this would still be true if consistent methods had access to an oracle for all arithmetical questions) [Kelly and Schulte 1995b]; see also [Juhl 1993].

Many influential approaches to methodology are based on probabilistic concepts. [Kelly 1995, Ch.13] examines measure theory, the mathematical foundation of statistics, from the reliabilist perspective on induction presented in this paper; see also [Juhl and Kelly 1994].

We have seen that the conjectures and refutations scheme can stand in the way of finding the truth. What about other proposed 'principles of rationality' for inductive inference, such as Bayesian conditionalization and 'minimal change' belief revision? [Juhl 1995b] shows that when only two observations are possible at each stage of inquiry, for any empirical hypothesis H that is

size of a set to be determined by a probability measure: there are meager sets with Lebesque measure 1 (cf. [Royden 1988, Ch.7.8]).

[9] By this we mean that the predictions of H are not definable in arithmetic.

decidable in the limit, a Bayesian's degrees of belief can be tailored to the problem at hand so that the conditionalizer decides H in the limit. It is an open question whether this is always possible when any number n of observations, or infinitely many, may arise at each stage. On the other hand, it follows from our discussion of the consistency principle above that conditionalization can prevent agents who are not logically omniscient from finding the truth; for when a hypothesis H is falsified by evidence e, the probability of H conditional on e is 0, and hence Bayesian updaters are consistent. Principles of 'minimal change' belief revision [Gärdenfors 1988] can be formulated for inductive methods. The results obtained so far indicate that these principles do not interfere with an agent's ability to find the truth [Martin and Osherson 1995], but open questions remain [Kelly *et al.* 1995].

The inductive methods discussed in this paper are reliable in the sense that they are logically guaranteed to succeed (given appropriate background assumptions)—an idea that originated with Reichenbach's 'vindication of induction' [Salmon 1991]. A weaker, *subjunctive*, notion of reliability can be derived from Nozick[1981]'s discussion of methods which produce knowledge (see [Kelly 1995, Ch.4]). For example, a method α *subjunctively decides an empirical hypothesis H with certainty* if and only if the following two counterfactuals are true:

1. if H were true, then α would eventually declare H as certainly true, and

2. if H were false, then α would eventually declare H as certainly false.

Other subjunctive criteria of success for inductive methods may be defined similarly. If we take a *possible empirical world* to be a data stream along with a specification of the current stage of inquiry, we can apply possible world semantics [Lewis 1973] to define the truth of the relevant counterfactuals in terms of neighborhood relations among data streams. Whether inductive problems can be solved by subjunctively reliable methods then depends on which data stream is the actual one, which other data streams are 'closest to' the actual one, and the topological structure of the hypotheses under investigation—an intriguing blend of techniques from the theory of counterfactuals and topology. One result is this: If a universal generalization H like "all swans are white" is true, then H is subjunctively decidable with certainty just in case there is a fixed deadline t such that the counterfactual "if H were false, then the data would falsify H by time t" is true [Kelly 1995, Ex. 4.15 (ii)]. Subjunctive reliability awaits further philosophical and mathematical exploration.

[Kelly 1995, Ch. 14] examines an *experimental setting* in which the object of study is a system with discrete consecutive states that can be manipulated by the scientist. Experimental science is more complicated than the non-interfering observational methods we have studied so far. One difficulty is that some of the system's dispositions may never be observed in the course of nature left to itself, because the behavior in question only occurs under conditions that are not realized without the scientist's intervention. But in an indeterministic system, even the most ingenious experiments might not reveal all the behavior the system is capable of: It might just happen that no matter how often the scientist puts the system into state s, the system never goes from state s to state r even though this transition is possible in principle. This problem does not arise in systems that satisfy the following principle of *plenitude:* if a state s occurs infinitely often, and it is possible that state r follows state s, then r follows s infinitely often. Another problem of experimental science is *experimental interference;* when a scientist intervenes to determine what is and is not possible in the system under study, she may permanently alter the system in such a way that other facts of interest can no longer be ascertained. For systems that satisfy the plenitude principle and are free from experimental interference, reliable inquiry about the system's behavior can be characterized with topological concepts much in the same way as passive observation can (cf. [Kelly 1995, Ch. 14]). This throws an interesting light on the idea, found in physics and metaphysics from ancient atomism to modern science, that the course of nature is determined through the interaction of ideal entities with immutable dispositions, for example atoms or elementary particles. Since the dispositions of the fundamental entities do not change, no human act makes a difference to what is in principle possible in nature, so experimental interference is no longer a danger to scientific inquiry. If we add the assumption that the fundamental entities satisfy the plenitude principle, an important epistemological consequence of introducing such immutable fundamental entities is to eliminate serious methodological difficulties that arise when science not only observes nature but changes it.

Ever since Kuhn's seminal work [Kuhn 1970], philosophers of science have emphasized that a scientist's observations as well as the truth of her theories may depend on her conceptual scheme. [Kelly and Glymour 1992], [Kelly 1995, Ch.15] and [Kelly *et al.* 1994] examine *relativistic* settings in which "truth, meaningfulness and what count as data all depend not only on the way things are, but also on something else associated with the scientist" [Kelly 1995, p. 385]. Let us call this "something else" the scientist's *conceptual scheme.* Inductive success may be defined in a number of ways for relativistic inquiry. For example, we may require a scientist first to settle on a conceptual scheme, and then find the truth relative to that conceptual scheme. Or we may allow the scientist to keep sifting through conceptual schemes forever, provided he eventually always believes the truth relative to his current conceptual scheme.

When a scientist's observations depend on the scientist's theories, the act of adopting a theory may bar the scientist from important observations much as an infelicitous physical manipulation might. So the problem of experimental interference arises in science with theory-laden data as in experimental science. In a relativistic setting problems of experimentation manifest themselves in a second way, because not only the scientist's evidence but also the truth of his hypotheses depends on his theories. This means that a scientist may have to perform 'semantic experiments' in order to find out how the truth (and meaning) of the hypotheses under investigation depends on his choice of conceptual scheme. [Kelly 1995, Ch.15] describes an example in which a scientist has to go through an infinite series of conceptual revolutions in order to arrive at beliefs that may change but are eventually always true relative to his current conceptual scheme. The venerable assumption that the truth of our beliefs does not depend on us obviates the need for such semantic experiments. If in addition we assume that theory-laden evidence satisfies a plenitude principle (cf. [Kelly and Glymour 1992]), relativistic inquiry can be characterized with topological concepts much in the same way as passive observation can. In this perspective metaphysical assumptions are seen to eliminate methodological difficulties in relativistic inquiry. The epistemologically rich subjunctive, experimental and relativistic settings for inductive inference await further philosophical and mathematical exploration.

Given the right formal framework, topology can be directly applied to questions about inductive inference. We hope to have illustrated how the epistemological connection to topology turns this fundamental branch of mathematics into a fecund source of philosophical insight.

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