How do the Harper and Levi Identities Constrain Belief Change?

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ABSTRACT. Belief revision is the process of revising beliefs in light of new information. Belief contraction is the process of giving up beliefs to make them consistent with new information. The Levi and Harper Identities provide constructions of revisions and contractions in terms of each other. This paper gives necessary and sufficient conditions for when revisions can be constructed from contractions, and contractions from revisions. I relate these conditions to other well-known principles for belief revision and contraction.

1 Minimal Belief Change and the Levi and Harper Identities

Belief revision is the process of incorporating new information into a body of extant beliefs. This process is clearly of central epistemic importance. Viewed in suitable generality, it encompasses scientific and inductive reasoning and many forms of everyday reasoning, for example causal and default reasoning. Another important epistemic process is belief contraction. Roughly, to contract one's beliefs on some proposition p is to "give $\neg p$ a hearing"—to transform one's beliefs, if necessary to weaken them, so that they not longer entail p. One reason why belief contraction has been interesting to philosophers of science is that it seems to model the situation of a scientist who holds a theory but opens her mind enough to consider alternatives [2].

Quine made famous the idea that rational belief revision should be minimal belief revision, the "minimal mutilation" of one's web of beliefs. Since then, philosophers and logicians have devoted much effort to determining general principles of minimal belief change [6]. One early proposal by Levi was to link the notion of minimal belief revision to belief contraction. He suggested that a theory change from the current theory T on new information p ought to proceed in two stages [4]. First, we may contract the theory T on the assertion $\neg p$. This yields a theory T' that does not entail $\neg p$, and hence is consistent with p. Then we add the new information p to T'. This recipe for deriving belief revision from belief contraction is known as the *Levi Identity*.

In this paper I investigate what constraints on belief revision result from the very idea of revision proceeding via contraction. Are there revision functions that violate Levi's proposal, in that they cannot be derived from contraction functions? If so, which revision functions are consistent with belief contraction and which aren't? I show that not all revision functions follow Levi's recipe. With mild assumptions about the operative notion of logical consequence, the following criterion is necessary and sufficient for a revision function to be derivable from a contraction function: the revision must be logically weaker than, or the same as, the result of simply adding the new information. In symbols, if $T * p$ represents the revision of a theory T given new information p, then $T \cup \{p\}$ must entail $T * p$. This is the AGM postulate K*3 [1, Ch.3.3]. Gärdenfors showed that any revision function that satisfies his "basic postulates for revision"—K*1 through K*6—can be constructed via the Levi Identity [1, p.71]. The result in this paper strengthens Gärdenfors' observation by providing necessary conditions as well as sufficient ones. We will see that in particular Gärdenfors' postulates K*5 (the "success postulate") and K*4 (the "preservation principle") are not necessary for satisfying the Levi Identity.

It turns out that there are strong independent grounds to hold that K*3, the main postulate for satisfying the Levi Identity, is a fundamental principle of minimal belief change. First, it is the characteristic condition of a constraint on belief change that I have called *Pareto-minimality*. Roughly, a belief revision $T * p$ is Pareto-minimal if it cannot be made more minimal by retracting fewer beliefs from T without adding more, or by adding fewer beliefs to T without retracting more.[1] Second, there is a standard way of translating belief revision postulates into axioms for conditionals known as the *Ramsey test*. Given the Ramsey test, the K*3 postulate corresponds exactly to a plausible and widely accepted principle of conditional logic, namely that $(p > q) \rightarrow (p \rightarrow q)$, where $\rightarrow$ stands for material implication and $>$ stands for a conditional connective ("if-then") [1, Lemma 7.3]. It is remarkable that four independently motivated ideas—the conditional axiom mentioned, Pareto-minimality, Levi's proposal of defining revision in terms of contraction, and the AGM postulate K*3—should amount to essentially the same constraint on belief revision.

Harper proposed a definition of belief contraction in terms of belief revision [2] that Gärdenfors refers to as the *Harper Identity* [1, p.70]. Briefly, the idea is that a belief q should be part of the contraction $T \dot{-} p$ just in case q is a member of T and of $T * \neg p$, the revision of T on the negation of p. As

[1] Pareto-minimality corresponds to Rott's symmetric-difference criterion [7].

with the Levi Identity and belief revision, we can ask what constraints the Harper Identity imposes on belief contraction. Which contraction functions can be derived from revision functions via the Harper Identity? It turns out that under mild assumptions about the operative notion of logical consequence, a contraction function is derivable from a revision function if and only if the contraction function satisfies the "recovery principle" [1, p.62]. Intuitively, the recovery principle asserts the following. Consider an agent who contracts her beliefs on a proposition p so that her contracted beliefs no longer entail p. Suppose that after giving $\neg p$ "a hearing", the agent decides to include p among her beliefs after all. Then according to the recovery principle, she should return to the epistemic state she was in before contracting her beliefs on p. In symbols, if $T \dot{-} p$ stands for the contraction of a theory T on an assertion p, then the recovery principle says that $T \dot{-} p \cup \{p\}$ entails $T \cup \{p\}$, and vice versa.

As with the Levi Identity, this result strengthens Gärdenfors' observation that his basic postulates for belief contraction suffice to guarantee that contraction functions can be constructed via the Harper Identity [1, p.71], by giving necessary as well as sufficient conditions for contraction functions to satisfy the Harper Identity.

So we find that two independently motivated ideas—the Harper Identity and the recovery principle—turn out to give equivalent constraints on belief contraction. One may take this equivalence to support both ideas. On the other hand, the recovery principle has been subject to some objections (see Section 6 below). It follows from the equivalence of the Harper Identity and the recovery principle that if we reject the recovery principle as a constraint on belief contraction, then we cannot assume that contraction functions are associated with revision functions as the Harper Identity stipulates, and we must be careful about "going back and forth between revision and contraction" (see also [5]).

The paper is organized as follows. I begin with the formal assumptions concerning the logical language and consequence relations. Then I introduce the Levi and Harper Identities. The next section characterizes the revision functions that satisfy the Levi Identity. I outline the related results concerning conditionals and Pareto-minimal belief change. The equivalence between the recovery principle and the Harper Identity is the final topic.

2 Theories

I begin with the representation of an agent's current beliefs as a deductively closed *theory* expressed in a formal language. As is usual in belief revision theory, my assumptions about the structure of the language in which an agent formulates her beliefs are sparse; essentially, all I assume is that the language features the usual propositional connectives. I take as given a suitable consequence relation between sets of formulas in the language, obeying the standard Tarskian properties. The formal presuppositions are as follows.

A **language** L is a set of formulas satisfying the following conditions.

1. L contains a **negation operator** $\neg$ such that if p is a formula in L, so is $\neg p$.

2. L contains a **conjunction connective** $\wedge$ such that if p and q are formulas in L, so is $p \wedge q$.

3. L contains an **implication connective** $\rightarrow$ such that if p and q are formulas in L, so is $p \rightarrow q$.

A **consequence operation** $Cn : 2^L \rightarrow 2^L$ represents a notion of entailment between sets of formulas from a language L. A set of formulas Γ **entails** another set of formulas Γ', written $\Gamma \vdash \Gamma'$, iff $Cn(\Gamma) \supseteq \Gamma'$. A set of formulas Γ entails a formula p, written $\Gamma \vdash p$, iff $p \in Cn(\Gamma)$. I assume that Cn satisfies the following properties, for all sets of formulas Γ, Γ'.

Inclusion $\Gamma \subseteq Cn(\Gamma)$.

Montonicity $Cn(\Gamma) \subseteq Cn(\Gamma')$ whenever $\Gamma \subseteq \Gamma'$.

Iteration $Cn(Cn(\Gamma)) = Cn(\Gamma)$.

A **theory** is a deductively closed set of formulas. That is, a set of formulas $T \subseteq L$ is a theory iff $Cn(T) = T$.

The entailment relation $\vdash$ is related to the propositional connectives as follows.

Modus Ponens If $\Gamma \vdash p$, $(p \rightarrow q)$, then $\Gamma \vdash q$.

Implication If $\Gamma \vdash q$, then $\Gamma \vdash (p \rightarrow q)$.

Deduction $\Gamma \cup \{p\} \vdash q$ iff $\Gamma \vdash (p \rightarrow q)$.

Conjunction $\Gamma \vdash (p \wedge q)$ iff both $\Gamma \vdash p$ and $\Gamma \vdash q$.

Consistency Suppose that $\Gamma \nvdash p$. Then $\Gamma \cup \{\neg p\} \nvdash p$.

Inconsistency $\{p \wedge \neg p\} \vdash L$.

Double Negation $\Gamma \vdash p$ iff $\Gamma \vdash \neg\neg p$.

For the remainder of this paper, assume that a language L and a consequence relation Cn (and hence an entailment relation $\vdash$) have been fixed that satisfy the conditions laid down above.

For example, suppose that we have a formal language for describing a very simple situation: there are three objects and one table. Our language has three propositional letters a, b, c. To provide some intuition, we interpret a to mean "the first object is on the table", b to mean "the second object is on the table", and c as "the third object is on the table". I will use the scenario throughout the paper to illustrate definitions.

Next, I state without proof two simple lemmas about theories and consequence relations that will be useful later.

LEMMA 1. *Let T_1, T_2 be two theories. Then $T_1 \cap T_2$ is a theory.*

We will have occasion to consider the logical consequences of adding a formula p to a theory, that is $Cn(T \cup \{p\})$. In belief revision theory, this operation is called **expansion**. Introducing a special symbol for expansion will simplify the notation in what follows.

DEFINITION 2. For all sets of formulas Γ and formulas p define $\Gamma + p = Cn(\Gamma \cup \{p\})$.

Note that in this notation, the Deduction Principle is expressed as $\Gamma \vdash p \rightarrow q$ iff $\Gamma + p \vdash q$.

A useful fact is that, given our assumptions about the consequence relation Cn, expansion distributes over the intersection of two theories.

LEMMA 3. *Let T_1, T_2 be two theories. For any formula p it is the case that $(T_1 \cap T_2) + p = (T_1 + p) \cap (T_2 + p)$.*

3 Belief Revision, Belief Contraction and The Levi and Harper Identities

A belief revision function represents the agent's disposition to change her beliefs in light of new evidence represented by a formula p.

DEFINITION 4. A belief revision function is a function $* : \mathbf{T} \times L \rightarrow \mathbf{T}$ such that for all formulas p, it is the case that $T * p \vdash p$.

A complete list of the AGM postulates K*1 through K*8 for revision functions may be found in [1, Ch.3.3]. In terms of the AGM axioms, Definition 4 restricts attention to belief revision functions satisfying K*1 and K*2.

A major part of the theory of minimal belief change is the analysis of belief contraction, which is formally represented by a belief contraction function.

DEFINITION 5. A belief contraction function $\dot{-}$ is a function $\dot{-} : L \rightarrow \mathbf{T}$

such that for all theories T and for all formulas p, it is the case that $T \vdash T \dot{-} p$.

Thus a belief contraction only retracts beliefs, but does not add any. Usually belief revision theorists require that a belief contraction on a formula p yields a theory that is consistent with the negation of p (provided that p is not a theorem). It turns out that we can characterize the content of the Harper and Levi Identities without that requirement, so to obtain the strongest possible result, I do not impose it.

Gärdenfors introduces eight postulates $\mathrm{K}^{-}1$ through $\mathrm{K}^{-}8$ for belief contraction, which may be found in [1, Ch.3.4]. In my usage, the postulates $\mathrm{K}^{-}1$ and $\mathrm{K}^{-}2$ define a belief contraction function.

One of the early ideas about belief revision was Levi's proposal for constructing revisions out of contractions [4]. Gärdenfors formalizes Levi's idea via the following definition [1, p.69].

DEFINITION 6 (The Levi Identity). Let $\dot{-}$ be a belief contraction function. The belief revision function $*$ associated with $\dot{-}$ is defined by $T * p = T \dot{-} \neg p + p$.

I write levi($\dot{-}$) to denote the belief revision function associated with $\dot{-}$. To illustrate the Levi Identity, consider again the belief contraction $Cn(\{a, \neg b\}) \dot{-} \neg b = Cn(\{a\})$. The associated belief revision is $Cn(\{a, b\}) * b = (Cn(\{a, b\}) \dot{-} \neg b) + b$, which is $Cn(\{a\}) + b = Cn(\{a, b\})$. In words, we can think of the revision as first withdrawing nothing but the negation of the new information b, and then adding the new belief b.

How should we define a belief contraction function given a revision function $*$ for a theory T? Harper made the following proposal (translated into our syntactic framework) [2]. Consider the revision $T * \neg p$. If $T * \neg p$ is a minimal revision of T on $\neg p$, then the difference between $T * \neg p$ and T is minimal, and so $T * \neg p$ has as much in common with T as is possible given the requirement of accommodating $\neg p$. Thus the overlap $T \cap T * \neg p$ ought to be as large as it can be while conforming with $\neg p$. This means that $T \cap T * \neg p$ is a plausible candidate for a minimal retraction of T that makes room for $\neg p$, that is, a contraction of T on p. Hence the following definition.

DEFINITION 7 (The Harper Identity). Let $*$ be a belief revision function for T. The contraction function associated with $*$ is defined by $T \dot{-} p = T \cap T * \neg p$.

As the Levi Identity yields a contraction function given a revision function, the Harper Identity defines a revision function from a contraction function; see Figure 1.

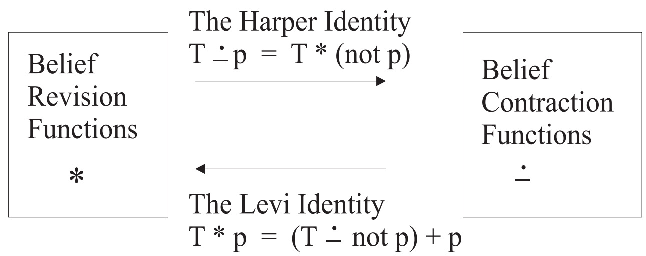

Figure 1. The Levi and Harper Identity

I write harper($*$) to denote the belief contraction function associated with $*$.

To illustrate the Harper identity, suppose that we have a revision $Cn(\{a, \neg b\}) * b = Cn(\{a, b\})$. (The agent initially believes that the first object is on the table and the second is not. On learning that the second is on the table, her new beliefs are that the first two objects are on the table.) Assume that the revision function treats b and $\neg\neg b$ identically, such that $Cn(\{a, \neg b\}) * \neg\neg b = Cn(\{a, b\})$. The associated belief contraction is $Cn(\{a, \neg b\}) \dot{-} \neg b = Cn(\{a, \neg b\}) \cap (Cn(\{a, \neg b\}) * \neg\neg b)$, which is $Cn(\{a, \neg b\}) \cap Cn(\{a, \neg b\}) * b = Cn(\{a, \neg b\}) \cap Cn(\{a, b\})$. It is possible to show that $Cn(\{a, \neg b\}) \cap Cn(\{a, b\}) = Cn(\{a\})$. All told, we have that $Cn(\{a, \neg b\}) \dot{-} \neg b = Cn(\{a\})$.

Note that in this case, the Levi and Harper Identities invert each other. If we start with the revision $Cn(\{a, \neg b\}) * b = Cn(\{a, b\})$, the Harper Identity yields the contraction $Cn(\{a, \neg b\}) \dot{-} \neg b = Cn(\{a\})$. And as we saw above, applying the Levi Identity to the contraction $Cn(\{a, \neg b\}) \dot{-} \neg b = Cn(\{a\})$ yields the revision $Cn(\{a, \neg b\}) * b = Cn(\{a, b\})$.

4 Necessary and Sufficient Conditions for Satisfying the Levi Identity

The Levi Identity stipulates a constraint on revision functions for minimal belief change by connecting them to belief contraction. What is the content of this constraint? That is, what properties must belief revision functions satisfy if they follow the Levi Identity? The answer is that the Levi Identity picks out those revision functions that satisfy K*3—the requirement that the expansion $T + p$ must be at least as strong as the revision $T * p$. Let us say that a function $*$ **satisfies the Levi Identity**, or is **generated by the Levi Identity**, if there is a belief contraction function $\dot{-}$ such that $*$ is the function associated with $\dot{-}$ (i.e., $* = \text{levi}(\dot{-})$) (cf. [5, p.386]). It is easy to see that if a belief revision function satisfies the Levi Identity, then it also satisfies K*3. (Proof omitted.)

LEMMA 8. *Let $\dot{-}$ be a belief contraction function with associated belief revision function $*$. Then for all theories T and for all formulas p, it is the case that $T + p \vdash T * p$.*

What about the converse of Lemma 8? The converse requires us to show that if a belief revision function $*$ satisfies K*3, then there is some contraction function $\dot{-}$ that generates $*$ via the Levi Identity. The obvious candidate for such a contraction function is the function harper($*$) that the Harper Identity associates with the revision operator. It turns out that indeed, applying the Levi Identity to harper($*$) yields the original belief revision function $*$; in other words, the Levi Identity inverts the Harper Identity, but only with some provisos. The first proviso is that $*$ must satisfy K*3, as Lemma 8 requires. The second is that $*$ must treat doubly negated formulas like unnegated formulas. Thus I say that a belief revision function $*$ for T respects **double negation** if for all formulas p, we have that $T * p = T * \neg\neg p$. Respect for double negation is much weaker than the AGM postulate K*6 which requires that the respective results of revising on logically equivalent formulas be the same. With these conditions in place, the postulate K*3 is a necessary and sufficient condition for the Levi Identity to invert the Harper Identity.

PROPOSITION 9. *Let $*$ be a belief revision function that respects double negation. Then the Levi Identity inverts the Harper Identity applied to $* \iff$ for all theories T and formulas p, it is the case that $T + p \vdash T * p$.*

From both a mathematical and a philosophical point of view, it is desirable to have this tight connection between the two identities. Gärdenfors puts it like this:

> But we also want the two definitions to be *interchangeable* in the sense that, if we start with one definition to construct a new contraction (or revision) function and after that use the other definition to obtain a revision (or contraction) function again, then we ought to get the original function back. If this can be proved, we will have shown that contractions and revisions are interdefinable in a strong sense. [1, p.70]

Proposition 9 characterizes the revision functions for which the Levi and Harper Identity are interchangeable in this sense. The proposition also immediately yields a characterization of the belief revision functions that are consistent with the Levi Identity.

COROLLARY 10. *A belief revision function $*$ that respects double negation can be generated by the Levi Identity $\iff$ for all theories T and formulas p, it is the case that $T + p \vdash T * p$.*

5 Discussion and Related Results

The fact that the postulate K*3 is equivalent to the Levi Identity, viewed as a constraint on belief revision, suggests that K*3 expresses a fundamental principle of minimal belief change. There are a number of other considerations that bring out the importance of K*3.

The Update Postulates. K*3 is one of the key principles of Katsuno and Mendelson's well-known *Update* theory of belief change, which they present as an account of belief change in a dynamically changing environment [3].

Conditional Logic. Gärdenfors introduced a formal connection between belief revision axioms and axioms for conditionals, known as the Ramsey test because Gärdenfors credited the basic idea to Frank Ramsey. The proposal is that an agent should accept a conditional $p > q$ just in case she accepts q after revising her beliefs on p; in symbols, the condition is that $T \vdash p > q \iff T * p \vdash q$. Gärdenfors showed that under the Ramsey test, K*3 corresponds exactly to the conditional axiom $(p > q) \rightarrow (p \rightarrow q)$, which is part of Lewis' system VC (for the details, see [1, Lemma 7.3]).

Pareto-minimal theory change. It turns out that K*3 characterizes another very plausible constraint on belief revision functions. For a given theory T and possible revision T', consider the symmetric difference $T \vartriangle T' = (T - T') \cup (T' - T)$. For example, if $T = Cn(\{a\})$ and $T' = Cn(\{a, b, c\})$, then $T \vartriangle T' = \emptyset \cup Cn(\{b, c\}) = Cn(\{b, c\})$.

Say that a theory T' satisfies the **symmetric difference criterion** for T iff for all theories T^* it is not the case that $T \vartriangle T^* \subset T \vartriangle T'$ ([7, Sec.II], [9, Sec.4]). In our example, $T' = Cn(\{a, b, c\})$ does *not* satisfy the symmetric difference criterion because if we take $T^* = Cn(\{a, b\})$, we have that $T \vartriangle T^* = Cn(\{b\})$, whereas $T \vartriangle T' = Cn(\{b, c\})$, so $T \vartriangle T^* \subset T \vartriangle T'$.

Roughly speaking, the symmetric difference criterion rules out a theory change $T * p$ if some other possible theory change T^* retracts less from T without adding more, or if T^* adds less to T without retracting more (see [9, Sec.4]). In our example, the revision $Cn(\{a\}) * b = Cn(\{a, b\})$ adds fewer beliefs to $Cn(\{a\})$ than $Cn(\{a, b, c\})$ does, and retracts no more, so $Cn(\{a, b, c\})$ is clearly not a minimal theory change. If we think of retractions and additions as a kind of cost to be minimized in theory change, this means that the symmetric difference criterion is an instance of the basic decision-theoretic principle of Pareto-optimality for choosing among objects with multiple attributes. That is, the symmetric difference criterion selects exactly those theory changes that are (weakly) Pareto-optimal with respect to the two "cost dimensions" retractions and additions. For this reason, Schulte [9, Sec.4] refers to theory changes satisfying the symmetric difference criterion as *Pareto-minimal theory revisions*. In the context of belief revision, we want to restrict attention to theory changes that entail a given piece of new information p. This leads to the following definition.

DEFINITION 11. Let T be a theory and let p be a formula. Then $T * p$ is a **Pareto-minimal** revision of T on $p \iff$

1. $T * p$ entails p, and
2. for all theories T' entailing p, it is not the case that $T \vartriangle T' \subset T \vartriangle T{*}p$.

It seems clear that Pareto-minimality is a necessary condition for a theory change $T * p$ to count as minimal. The next theorem gives an explicit characterization of Pareto-minimal belief revision functions; the proof is in [9, Th.5].

THEOREM 12. *Let T be a theory and let p be a formula. A theory revision $T * p$ is a Pareto-minimal revision of T on $p \iff$*

1. *$T * p \vdash p$, and*
2. *$T + p \vdash T * p$, and*
3. *if $T \vdash p$, then $T * p = T$.*

Clause 1 simply states the basic property of incorporating the new evidence, and Clause 3 says that if a theory T already incorporates the new evidence, no change at all should occur. Since no change is clearly the smallest change, Condition 3 is a trivial requirement for minimal belief change. Thus it is Clause 2 that captures the force of Pareto-minimality in theory change—and this condition is just the familiar postulate K*3.

So we have the striking result that three principles with independent strong motivations—the Levi Identity as a constraint on belief revision, the conditional axiom $(p > q) \rightarrow (p \rightarrow q)$, and Pareto-minimality—are basically equivalent to K*3.

6 Necessary and Sufficient Conditions for Satisfying the Harper Identity

This section investigates the content of the Harper Identity. Let us say that a function $\dot{-}$ **satisfies the Harper Identity**, or is **generated by the Harper Identity**, if there is a belief revision function $*$ such that $\dot{-}$ is the function associated with $*$ (i.e., $\dot{-} = \text{harper}(*)$). It is not hard to prove that the following condition is necessary for a belief contraction function to be generated by the Harper Identity. (Proof omitted.)

LEMMA 13. *Suppose that $*$ is a belief revision function, and that $\dot{-}$ is the contraction function associated with $*$. Then for all theories T and for all formulas p, it is the case that $T\dot{-}p + p = T + p$.*

To illustrate the lemma, let us consider an example of a contraction that does not satisfy the Harper Identity. For example, let $T = Cn(\{a, b\})$, and suppose that $T\dot{-}a = Cn(\emptyset)$ (to withdraw the belief that the first object is on the table, contract to being uncertain about all three objects). Then $T\dot{-}a + b = Cn(\{b\})$, which is different from $T + b = Cn(\{a, b\})$. Hence Lemma 13 entails that $T\dot{-}a$ does not satisfy the Harper Identity.[2]

In the case in which $T \vdash p$, the condition that $T\dot{-}p + p = T + p$ is essentially equivalent to Gärdenfors' postulate $K^{-}5$, viz. $T\dot{-}p + p \vdash T$. Since $T + p = T$ if $T \vdash p$, the condition of Lemma 13 entails $K^{-}5$. And since $T \vdash T\dot{-}p$ for any contraction function $\dot{-}$, it is immediate that $T + p \vdash T\dot{-}p + p$.

The postulate $K^{-}5$ is often referred to as a **recovery postulate** because it asserts that after first contracting on p and then adding p "back in", the agent recovers all of the beliefs in her original theory T. The condition of Lemma 13 is a slightly different formulation of the recovery principle. The intuition behind the recovery principle is this. To contract beliefs on p means to "give $\neg p$ a hearing", or to entertain the possibility that p may be false. If the agent gives $\neg p$ a hearing, but then finds that p is correct after all, the agent should restore confidence in any proposition q that he may have believed but called into doubt along with q.

Before establishing a converse to Lemma 13, I ask under what circumstances the Harper Identity inverts the Levi Identity, as before in the case of the Levi Identity. The recovery postulate turns out to be sufficient as well as necessary, provided that the consequence relation satisfies two more conditions.

First, as with belief revision functions, I say that a belief contraction function for a theory T **respects double negation** if for all formulas p, it is the case that $T\dot{-}\neg\neg p = T\dot{-}p$. Respect for double negation is an instance of Gärdenfors' postulate $K^{-}6$. Second, a consequence relation Cn **satisfies disjunctive syllogism** if for all sets of formulas Γ it is the case that if $\Gamma \vdash p \rightarrow q$ and $\Gamma \vdash \neg p \rightarrow q$, then $\Gamma \vdash q$. With these conditions in place, the recovery principle is a necessary and sufficient condition for the Harper Identity to invert the Levi Identity.

[2]To verify this fact directly, consider any revision $T * \neg a$ and apply Lemma 3 to $(T \cap T * \neg a) + a$.

PROPOSITION 14. *Assume that the consequence relation Cn satisfies disjunctive syllogism, and let $\dot{-}$ be a belief contraction function that respects double negation. Then the Harper Identity inverts the Levi Identity for the belief contraction function associated with $\dot{-}$ $\iff$ for all theories T and for all formulas p, it is the case that $T \dot{-} p + p = T + p$.*

Proposition 14 immediately yields a characterization of the belief revision functions that are consistent with the Harper Identity.

COROLLARY 15. *If the consequence relation Cn satisfies disjunctive syllogism, a belief contraction function $\dot{-}$ that respects double negation can be generated by the Harper Identity $\iff$ for all theories T and for all formulas p, it is the case that $T \dot{-} p + p = T + p$.*

Corollary 15 shows that two independently motivated principles for theory contraction, the Harper Identity and the Recovery Principle, turn out to be equivalent. There are several objections to the recovery principle, which by our result are also objections to the Harper Identity. It is not my purpose in this paper to adjudicate the status of the recovery principle; several authors have discussed the pros and cons of this principle—see [1, Ch.3.4], [5], [10], [8] and the references in these papers. The mathematical results apply whether one accepts the principle or not.

7 Conclusion

We may view the Levi Identity as a constraint on belief revision: revisions should be such that they can be constructed from a contraction function as directed by the Levi Identity. I showed that a revision function satisfies this constraint just in case it satisfies the AGM postulate K*3. Via the Ramsey test, the postulate K*3 in turn is equivalent to the conditional axiom $(p > q) \rightarrow (p \rightarrow q)$. Finally, K*3 is the characteristic axiom of Pareto-minimal theory change, a basic requirement for minimal belief revision. The fact that four independently motivated constraints on belief revision turn out to be essentially equivalent reinforces each of them, and provides strong evidence that K*3 is a basic principle of minimal theory change.

As with the Levi Identity, it is possible to view the Harper Identity as a constraint on belief contraction: contractions should be such that they can be constructed from a revision function as directed by the Harper Identity. I showed that a contraction function satisfies this constraint just in case it satisfies the recovery principle. The fact that the Harper Identity and the recovery principle turn out to be equivalent, though independently motivated, would seem to support both.

On the other hand, some belief revision theorists have objected to the recovery principle; if the recovery principle is objectionable, then so is the

Harper Identity, and there is only a limited extent to which we can make use of the Harper and Levi Identities to translate from revision to contraction and vice versa.

8 Proofs

Proof of Proposition 9. Let $\dot{-}$ be the belief contraction function $harper(*)$ defined by $T \dot{-} p = T \cap T{*}\neg p$. First we have that (a) $T \dot{-} \neg p + p = (T \cap T{*}\neg\neg p) + p = (T \cap T{*}p) + p$ by the assumption that $T{*}\neg\neg p = T{*}p$. By Lemma 3 we have that $(T \cap T{*}p) + p = T + p \cap T{*}p + p$, which is equal to $T + p \cap T{*}p$ since $T{*}p$ is a theory entailing p. Together with (a), this shows that (b) $T \dot{-} \neg p + p = T + p \cap T{*}p$. Thus $T \dot{-} \neg p + p = T{*}p$ if and only if $T + p \supseteq T{*}p$; in other words, if and only if $T + p \vdash T{*}p$.□

Proof of Proposition 14. Let $*$ be the belief revision function levi($\dot{-}$) defined by $T * p = T \dot{-} \neg p + p$.

($\Rightarrow$) If $\dot{-}$ is the result of applying the Harper Identity to the belief revision function $*$, it follows from Lemma 13 that for all formulas p, it is the case that $T \dot{-} p + p = T + p$.

($\Leftarrow$) Suppose that it is the case that $T \dot{-} p + p = T + p$. We want to show that $T \dot{-} p = T \cap T * \neg p$. By the definition of $*$, we must show that $T \dot{-} p = T \cap (T \dot{-} \neg\neg p + \neg p)$, which is equal to $T \cap (T \dot{-} p + \neg p)$ if $\dot{-}$ respects double negation. It is easy to see that $T \dot{-} p \subseteq T \cap (T \dot{-} p + \neg p)$. For if q is a formula in $T \dot{-} p$, then $q \in T$ since $T \dot{-} p \subseteq T$, and by Monotonicity $T \dot{-} p + \neg p \vdash q$. For the converse, let q be a formula in $T \cap (T \dot{-} p + \neg p)$. Then $q \in T + p$, and so by hypothesis $q \in T \dot{-} p + p$. Thus by Deduction, $T \dot{-} p \vdash p \rightarrow q$. Also $T \dot{-} p \vdash \neg p \rightarrow q$ since $q \in T \dot{-} p + \neg p$. So if Cn satisfies disjunctive syllogism, then $q \in T \dot{-} p$; since q is an arbitrary formula, this establishes that $T \dot{-} p = T \cap (T \dot{-} p + \neg p)$ and hence that $T \dot{-} p = T \cap T * \neg p$. Since this holds for any formula p, the Harper Identity inverts the Levi Identity for the belief contraction function $\dot{-}$, which was to be shown.□

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