Common Reasoning About Admissibility

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Abstract

We analyze common reasoning about admissibility in the strategic and extensive form of a game. We define a notion of sequential proper admissibility in the extensive form, and show that, in finite extensive games with perfect recall, the strategies that are consistent with common reasoning about sequential proper admissibility in the extensive form are exactly those that are consistent with common reasoning about admissibility in the strategic form representation of the game. Thus in such games the solution given by common reasoning about admissibility does not depend on how the strategic situation is represented. We further explore the links between iterated admissibility and backward and forward induction.

1 Introduction

A well known problem with non-cooperative game theory is that Nash equilibria are seldom relevant for predicting how the players will play. The equilibria of a game do not represent all the possible outcomes. Rather, they represent the set of self-enforcing agreements: had the players known their respective choices before playing the game, then they must have constituted an equilibrium. Some game theorists have argued that predictability must involve what Binmore (1987/88) has called an "eductive" procedure. When asking how the players' deductive processes might unfold, one must usually specify some basic principles of rationality, and then examine what choices are consistent with common knowledge of the specified principles. The advantage of this approach is that it is possible to refine our predictions about how players might choose without assuming that they will coordinate on a particular equilibrium. Principles such as iterated strict dominance or rationalizability (Pearce 1984), (Bernheim 1984) are examples of how it is possible to restrict the set of predictions using rationality arguments alone. In this paper we embrace the eductive viewpoint, and examine the game-theoretic implications of adopting the classic admissibility postulate of decision theory. An admissible choice is a choice that is not weakly dominated, and we take rationality to coincide with admissibility. However, a player might be indifferent between two strategies, one of which is weakly dominated by the other, if she assigns probability zero to the state on which the weakly dominant act is strictly preferred. To guarantee that a player will always eliminate a weakly dominated strategy, we have to impose a strict coherence requirement (Shimony 1955), meaning that—when a player starts deliberating about a game—each player must assess positive probability for each of the other players' strategies. The admissibility principle thus follows from combining Savage's sure thing principle (1954, pp.21–26) and strict coherence. We assume rationality to be common knowledge, and describe players' common reasoning about admissibility in the strategic and extensive forms of a game. Common reasoning about admissibility in the extensive form leads to iterated elimination of weakly dominated strategies (IWD). In the last part of the paper we explore the relationship between IWD in the extensive and strategic forms of a game. One commonly held disadvantage of IWD is that—unlike iterated strict dominance—different orders of deletion can result in different solutions. A standard solution to this problem is to delete at each round all weakly dominated strategies of all players (Rochet 1980), (Moulin 1986), (Harper 1991). We support this view by arguing that order-independent elimination of weakly dominated strategies captures common reasoning about admissibility in the strategic form. In the extensive form of a game, a strategy may prescribe choices in parts of the tree that will never be reached if that strategy is played. If we evaluate strategies only with respect to information sets that are consistent with them (i.e., information sets that can be reached if the strategy is played), we are led to the concept of sequential proper admissibility: A strategy is sequentially properly admissible in a game tree just in case the strategy is admissible at each information set that is consistent with the strategy. A striking result of our paper is that, for finite extensive form games with perfect recall, the strategies that are consistent with common reasoning about sequential proper admissibility in the extensive form are exactly those that are consistent with common reasoning about admissibility in the strategic form representation of the game. Thus in these games, the solution given by common reasoning about admissibility does not depend on how the strategic situation is represented.

Like iterated strict dominance and rationalizability, application of iterated weak dominance (IWD) has the advantage that it does not require advanced computation of equilibria. It is therefore a more global condition than backward and forward induction principles, some of whose features IWD is held to capture. Though backward and forward induction principles are understood to be local conditions, in that they provide a test which can only be applied after the equilibria of a game have been computed, we think that our characterization of IWD captures some crucial features of both principles. For example, we show that, in generic finite game of perfect information, common reasoning about weak admissibility yields exactly the backward induction solution. And in finite games of imperfect information, common reasoning about admissibility yields typical forward induction solutions. Thus backward and forward induction seem to follow from one principle, namely that players' choices should be consistent with common reasoning about admissibility. This result may seem questionable, as it is also commonly held that backward and forward induction principles are mutually inconsistent. That is, if we take backward and forward induction principles to be restrictions imposed on equilibria, then they lead to contradictory conclusions about how to play. We show that the problem with the examples one finds in the literature is that no constraints are set on players' forward induction "signals". We define a credible forward induction signal in an extensive game as a signal consistent with common reasoning about sequential admissibility. Thus the examples in the literature which purport to show the conflict between backward and forward induction principles involve forward induction signals that are not credible.

2 Extensive Form Games

We introduce the basic notions for describing games in extensive form. Note that our formalization is limited to finite games, and that we restrict players to only play pure strategies. A finite extensive form game for players $N = 1, 2, ..., n$ is given by a *game tree* T with finitely many nodes V, root r, payoff functions u_i which assigns a payoff to each player i at each terminal node in T, and information sets I_i for each player i. For each node x in T, $I(x)$ is the information set containing x. A pure *strategy* s_i for player i in a game tree T assigns a unique action, called a *move*, to each information set I_i of player i in T. We denote the set of i's pure strategies in T by $S_i(T)$ (in what follows, the term "strategy" always refers to pure strategies.) A *strategy profile* in T is a vector $(s_1, s_2, ..., s_n)$ consisting of one strategy for each player i. We denote the set of pure strategy profiles in T by $S(T)$; i.e. $S(T) = \times_{i \in N} S_i(T)$. We use '$s$' to denote a generic strategy profile. It is useful to denote a vector of length $n-1$ consisting of strategy choices by player i's opponents by s_{-i}. We write $S_{-i}(T)$ for the set of strategy profiles of i's opponents, i.e. $S_{-i}(T) = \times_{j \in N - \{i\}} S_j(T)$.

Given a strategy profile s, we use $s[i]$ to denote the strategy of player i in s, and $s[-i]$ to denote the strategy profile of i's opponents in s.

In the games we consider, the root is the only member of its information set (i.e. $I(r) = \{r\}$), so that a strategy profile s in T determines a unique maximal path $< r, x_1, x_2, ..., x_n >$ from the root r to a terminal node x_n; we refer to this path as the *play sequence* resulting from s, and denote it by $play(s)$. When a strategy profile s in T is played, each player receives as payoff the payoff from the terminal node reached in the play sequence resulting from s. With some abuse of notation, we use u_i to denote both a function from strategy profiles to payoffs for player i, as well as a function from terminal nodes to a payoff for player i, and define $u_i(s) = u_i(x)$, where x is the terminal node in the play sequence $play(s)$. For a finite game tree T, the *height* of a node x in T is denoted by $h(x)$, and defined recursively by $h(x) = 0$ if x is a terminal node in T, and $h(x) = 1 + \max\{h(y) : y \text{ is a successor of } x \text{ in } T\}$ otherwise.

An important part of players' deliberation about which strategy to choose in a given game consists of ruling out possibilities about how the game might be played. Though players may use different principles to exclude some plays of the game, any such reasoning will result in a game tree restricted to those possibilities consistent with the application of a given principle. The following definitions allow us to describe this notion precisely.

DEFINITION 1 *Restricted Game Trees*

- *Let T be a finite game tree for $N = 1, 2, \ldots, n$ players.*
- *$T|V$ is the restriction of T to V, where V is a subset of the nodes in T. All information sets in $T|V$ are subsets of information sets in T.*
- *T_x is the game tree starting at node x (i.e. T_x is the restriction of T to x and its successors.) If $I(x) = \{x\}$, then T_x is called a* subgame.
- *If s_i is a strategy for T and T' is a restriction of T, $s_i|T'$ is the strategy that assigns to all information sets in T' the same choice as in T. Formally, $s_i|T'\ (I'_i) = s_i(I_i)$, where I_i is the (unique) information set in T that contains all the nodes in I'_i. Note that $s_i|T'$ is not necessarily a strategy in T'; for the move assigned by s_i at an information set I_i in T may be not possible in T'.*
- *If s is a strategy profile in T and T' is a restriction of T, $s|T'$ is the strategy vector consisting of $s[i]|T'$ for each player i.*
- *Let $S \subseteq S(T)$ be a collection of strategy profiles in a game tree T with players N. Then a node x is* consistent with *S if and only if there is a strategy profile s in S such that x is part of the play sequence resulting from s, i.e. $x \in range(play(s))$. The restriction of T to nodes consistent with S is denoted by $T|S$. We observe that $T|S(T) = T$.*
- *A node x is* consistent with *a strategy s_i by player i in T just in case there is a strategy profile s_{-i} in T such that x appears in the play sequence $play(s_i, s_{-i})$.*

3 Common Reasoning About Rationality

We may assume that in deliberating players use some principle to rule out plays of the game that are inconsistent with that principle. One such principle is rationality. In the next sections we explore the consequences of adopting two rationality criteria: weak admissibility, which follows from sure-thing (Savage 1954) and coherence principles, and admissibility, which follows from sure-thing and strict coherence principles. In the first case, a player never plays a strictly dominated strategy, whereas in the second case also weakly dominated strategies are eliminated. Assuming strict coherence (Shimony 1955) is crucial, since a player might be indifferent between two strategies, one of which is weakly dominated by the other, if she assigns probability zero to the state on which the weakly dominant act is strictly preferred. To guarantee that a player will always eliminate a weakly dominated strategy, we have to impose a strict coherence requirement, meaning that – at the beginning of a game – each player must assess positive probability to each of the other players' strategies.

A player who is reasoning, say, with the help of admissibility would not go very far in eliminating plays of the game inconsistent with it, unless he assumes that the other players are also applying the same principle. In the game of Figure 1, for example, player 1 could not eliminate *a priori* any play of the game unless he assumed player 2 never plays a dominated strategy. [1] In general, even assuming that other players are rational might not be enough to rule out possibilities about how a given game might be played. Players must reason about other players' reasoning, and such mutual reasoning must be common knowledge. Unless otherwise specified, we shall assume that players have common knowledge of the structure of the game and of rationality, and examine how common reasoning about rationality unfolds.

3.1 Strict Dominance and Subgame Perfection

This section explores in detail the implications of common reasoning about weak admissibility, the requirement that players should avoid strictly dominated actions. We show that in finite games of perfect information, common reasoning about weak admissibility gives exactly the same results as Zermelo's backward induction algorithm, which in finite games of perfect information corresponds to Selten's notion of *subgame perfection* [2]. We then show by examples that the tight connection between common reasoning about weak admissibility and subgame perfection breaks down in games of imperfect information.

We define a strategy to be sequentially weakly admissible in a game tree T if it is weakly admissible at each information set in T. A strategy s_i for player i is not weakly admissible at a given information set I_i if the strategy is strictly dominated at I_i. This means that there is some other strategy s'_i that yields i a better outcome than s_i at every node x in I_i. For example, in the game of Figure 1, playing right ('R') at 2's information set is strictly dominated by playing left ('L').

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[1] Here and elsewhere, the payoff at a terminal node is given as a pair (x, y), where x is the payoff for player 1 and y is the payoff for player 2.

[2] cf. (Osborne and Rubinstein, 1994, Ch. 6).

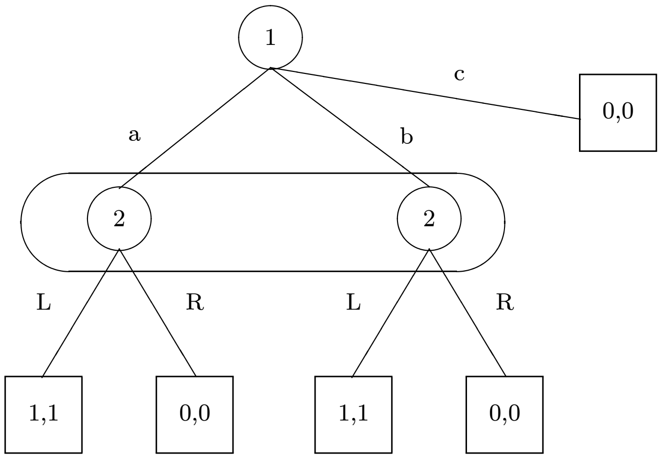

Figure 1: Weak Admissibility

The formal definition of sequential weak admissibility is the following.

DEFINITION 2 *Strict Dominance and Weak Admissibility in Extensive Form Games*

- *Let T be a finite game tree for $N = 1, 2, \ldots, n$ players.*
- *We define the payoff to player i from strategy s_i and strategy profile s_{-i} at x, written $u_i(s_i, s_{-i}, x)$, to be $u_i(s_i, s_-i, x) = u_i(s_i|T_x, s_{-i}|T_x)$.*
- *A strategy s_i is* strictly dominated *by another strategy s_i' at an information set I_i belonging to i in T just in case for all strategy profiles s_{-i} in T, and for all y in I_i, $u_i(s_i, s_{-i}, y) < u_i(s_i', s_{-i}, y)$.*
- *A strategy s_i is* weakly admissible at an information set *I_i in T just in case s_i is not strictly dominated at I_i.*
- *A strategy s_i is* sequentially *weakly admissible in T if and only if s_i is weakly admissible at each information set I_i in T that belongs to player i.*

Our procedure for capturing common reasoning about sequential weak admissibility in T is the following. First, eliminate at each information set in T all moves that are inconsistent with weak admissibility, i.e. strictly dominated choices. The result is a restricted game tree T'.

Repeat the pruning procedure with T' to obtain another restricted game tree, and continue until no moves in the resulting game tree are strictly dominated. Note that the recursive pruning procedure does not start at the final information sets. Our procedure allows players to consider the game tree as a whole and start eliminating branches anywhere in the tree by applying weak admissibility. To illustrate the procedure, look at the game of figure 1. R is eliminated at 2's information set in the first iteration, and then c is eliminated for player 1 because, after R is eliminated, either a or b yield player 1 a payoff of 1 for sure, while c yields 0. The pruning procedure is formally defined as follows. For a given game tree T, let $Weak-Ad_i(T) = \{s_i \in S_i(T) : s_i$ is sequentially weakly admissible in $T\}$, and let $Weak-Ad(T) = \times_{i \in N} Weak-Ad_i(T)$.

DEFINITION 3 *Common Reasoning about Sequential Weak Admissibility*

- *Let T be a finite game tree for $N = 1, 2, \ldots, n$ players.*
- *The strategies in T* consistent with common reasoning about sequential weak admissibility *are denoted by $CR_{WA}(T)$, and are defined as follows:*
  1. $WA^0(T) = S(T)$.
  2. $WA^{j+1}(T) = Weak - Ad(T|WA^j(T))$.
  3. $s \in CR_{WA}(T) \Longleftrightarrow \forall j : s|[T|WA^j(T)] \in WA^{j+1}(T)$.

If T is a finite game tree, the set of strategies for player $i, S_i(T)$ is finite, and our procedure will go through only finitely many iterations. To be precise, let max $= \sum_{i \in N} |S_i| - 1$; then the procedure will terminate after max iterations, i.e. for all $j \geq \max, WA^j(T) = WA^{j+1}(T)$.

We introduce the concept of Nash equilibrium and one of its refinements, subgame perfection, for generic finite games in extensive form. A strategy s_i in a game tree T is a *best reply* to a strategy profile s_{-i} of i's opponents if there is no strategy s_i' for player i such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. A strategy profile s is a *Nash equilibrium* if each strategy $s[i]$ in s is a best reply against $s[-i]$. A strategy profile s is a *subgame perfect equilibrium* if for each subgame T_x of $T, (s|T_x)$ is a Nash equilibrium of T_x. We say that a strategy s_i in T is *consistent with subgame perfection* if there is a subgame perfect strategy profile s of which s_i is a component strategy, i.e. $s_i = s[i]$. We denote the set of player i's strategies in T that are consistent with subgame perfection by $SPE_i(T)$, and define the set of strategy profiles consistent with subgame perfection by $SPE(T) = \times_{i \in N} SPE_i(T)$. Note that not all strategy profiles that are consistent with subgame perfection are subgame perfect equilibria. In figure 2, all strategy profiles are consistent with subgame perfection, but L, ba' and R, ab' are not equilibria, since in equilibrium 1 must be playing a best reply to 2's strategy.

Finally, T is a game of *perfect information* if each information set I of T is a singleton. The game in Figure 2 is a game of perfect information.

A standard approach to finite games of perfect information is to apply Zermelo's backwards induction algorithm which yields the set of strategy profiles that are consistent with subgame perfection, i.e. $SPE(T)$ [3]. Common reasoning about weak admissibility, as defined by the procedure WA, does not follow Zermelo's backwards induction algorithm. For example, suppose that in a game tree a move m at the root is strictly dominated by another move m' at the root for the first player. Common reasoning about weak admissibility rules out m immediately, but the backwards induction algorithm eliminates moves at the root only at its last iteration. Nonetheless, our first result is that in games of perfect information, the final outcome of the two procedures is the same: In these games, the strategies that are consistent with common reasoning about sequential weak admissibility are exactly those consistent with subgame perfection.

PROPOSITION 1 *Let T be a finite game tree of perfect information. Then a strategy s_i is consistent with common reasoning about sequential weak admissibility in T if and only if s_i is consistent with subgame perfection. That is, $CR_{WA}(T) = SPE(T)$.*

In games of imperfect information, the equivalence between strategies consistent with subgame perfection and those consistent with common reasoning about sequential weak admissibility fails in both directions. Figure 1 shows that a strategy profile s may be a subgame perfect equilibrium although s is not consistent with common reasoning about sequential weak admissibility: The strategy profile (c, R) is a subgame perfect equilibrium, but R and (hence) c are not consistent with common reasoning about sequential weak admissibility. And in figure 3, a is not strictly dominated for player 2, but a is neither a best reply to L nor to R. Although a is

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[3] cf. (Osborne and Rubinstein, 1994, Ch. 6.2).

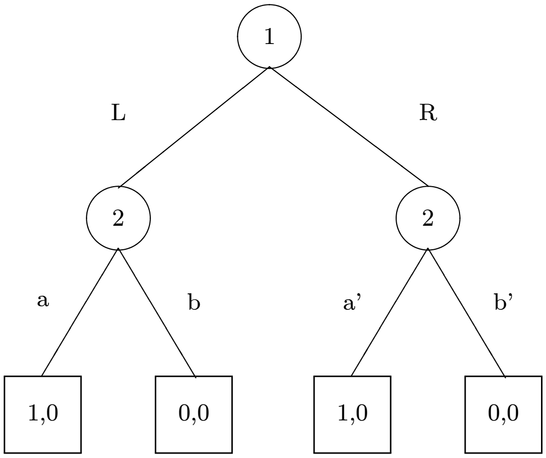

Figure 2: A game of perfect information.

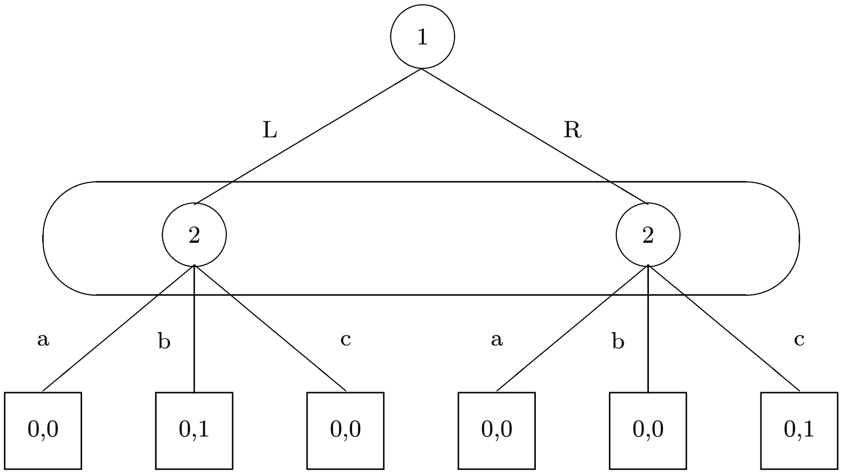

Figure 3: Subgame Perfection vs. Weak Admissibility

not strictly dominated, a seems like a bad choice because it never gives player 2 a better payoff than the alternatives and sometimes gives her less. In other words, a is *weakly dominated.* In the remainder of this paper, we investigate how players might reason about a game on the assumption that no player will choose a weakly dominated strategy.

4 Sequential Weak Dominance and Forward Induction

4.1 Weak Dominance

Informally, a strategy s_i is weakly dominated by another strategy s_i' at an information I_i in a game tree T if s_i' never yields less to i at I_i than s_i does, and sometimes yields more. For example, in the game of Figure 3, a is weakly dominated at 2's information set. And in the game of Figure 4, choosing b is weakly dominated for 2 because a yields player 2 the payoff 2 for sure, while b may yield only 0 if player 1 plays R_2. As in the case of weak admissibility, we call a strategy s_i sequentially admissible just in case s_i is admissible at each information set belonging to player i.

Definition 4 *Weak Dominance and Admissibility in Extensive Form Games*

- *Let T be a finite game tree for $N = 1, 2, \ldots, n$ players.*
- *A strategy s_i is* weakly dominated *by another strategy s_i' at an information set I_i belonging to i in T just in case*
    1. *for all strategy profiles s_{-i} in T, and for all y in $I_i, u_i(s_i, s_{-i}, y) \leq u_i(s_i', s_{-i}, y)$, and*
    2. *for some strategy profile s_{-i} and some node y in $I_i, u_i(s_i, s_i, y) < u_i(s_i', s_{-i}, y)$.*
- *A strategy s_i is* admissible at an information set *I_i in T just in case s_i is not weakly dominated at I_i.*
- *A strategy s_i is* sequentially admissible *in T if and only if s_i is admissible at each information set I_i in T that belongs to i.*

We define a procedure to capture common reasoning about sequential admissibility analogous to common reasoning about sequential weak admissibility. To illustrate the procedure, consider figure 4. Common reasoning about admissibility rules out b as a choice for player 2 because b is weakly dominated. Then given that only a remain at 2's decision node, R_1 (strictly) dominates L_1 for player 1. So the only play consistent with common reasoning about sequential admissibility is for player 1 to play R_1 and end the game. Note however that common reasoning about sequential weak admissibility, i.e. the standard backwards induction procedure, is consistent with both R_1 and the play sequence L_1, b, L_2. So even in games of perfect information, common reasoning about sequential admissibility may lead to stronger results than common reasoning about sequential weak admissibility.

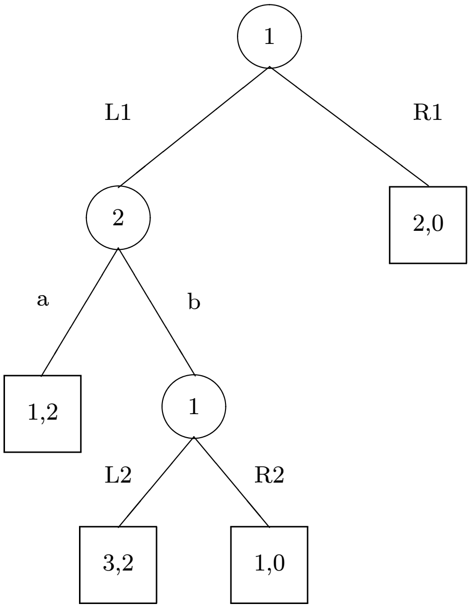

Figure 4: Admissibility in a game of perfect information.

For a given game tree T, let $Seq - Ad_i(T) = \{s_i \in S(T) : s_i$ is sequentially admissible in $T\}$, and let $Seq - Ad(T) = \times_{i \in N} Seq - Ad_i(T)$.

Definition 5 *Common Reasoning about Sequential Admissibility*

- *Let T be a finite game tree with players $N = 1, 2, ...n$.*
- *The strategies in T* consistent with common reasoning about sequential admissibility *are denoted by $CR_{Seq}(T)$, and are defined as follows:*
  1. $Seq^0(T) = S(T)$.
  2. $Seq^{j+1}(T) = Seq - Ad(T|Seq^j(T))$.
  3. $s \in CR_{Seq}(T) \Longleftrightarrow \forall j : s|[T|Seq^j(T)] \in Seq^{j+1}(T)$.

We have seen that common reasoning about sequential admissibility can lead to stronger results than common reasoning about sequential weak admissibility; we next show that the former never leads to weaker results than the latter. The key is to observe that if a strategy s_i is strictly dominated in a game tree T, s_i will be strictly dominated in a restriction of T. The next lemma asserts the contrapositive of this observation: If a strategy s_i is admissible in a restriction of T, s_i is not strictly dominated in T.

Lemma 2 *If T is a restriction of T' and s_i is sequentially admissible in T, then there is an extension s'_i of s_i to T' such that s'_i is sequentially weakly admissible in T'.*

This means that our procedure Seq yields, at each stage j, a result that is at least as strong as that of common reasoning about weak admissibility, the procedure WA. Hence we have the following proposition.

proposition 3 *Let T be a finite game tree. If a play sequence is consistent with common reasoning about sequential admissibility in T, then that play sequence is consistent with common reasoning about sequential weak admissibility. That is, $\{play(s) : s \in CR_{Seq}(T)\} \subseteq \{play(s) : s \in CR_{WA}(T)\}$.*

4.2 Forward Induction

It is commonly held that iterated weak dominance (i.e., iterated sequential admissibility) captures some of the features of backward and forward induction. Fudenberg and Tirole (1993, p.461) thus state that: "Iterated weak dominance incorporates backward induction in games of perfect information: The suboptimal choices at the last information sets are weakly dominated; once these are removed, all subgame-imperfect choices at the next-to-last information sets are removed at the next round of iteration; and so on. Iterated weak dominance also captures part of the forward induction notions implicit in stability, as a stable component contains a stable component of the game obtained by deleting a weakly dominated strategy".

Indeed, we have previously shown that, in finite game of perfect information, common reasoning about weak admissibility yields exactly the backward induction solution. In this section we show how, in finite games of imperfect information, common reasoning about admissibility yields typical forward induction solutions. Thus backward and forward induction seem to follow from one principle, namely that players' choices should be consistent with common knowledge of (and common reasoning about) admissibility. This result may seem questionable, as it is also commonly held that backward and forward induction *principles* are mutually inconsistent (Kohlberg and Mertens 1986), (Myerson 1991). That is, if we take backward and forward induction principles to be restrictions imposed on equilibria, then they may lead to contradictory conclusions about how to play.

A backward induction principle states that each player's strategy must be a best reply to the other players' strategies, not only when the play begins at the initial node of the tree, but also when the play begins at any other information set.[4] A forward induction principle says that players' beliefs should be consistent with sensible interpretations of the opponents' play. Thus a forward induction principle restricts the range of possible interpretations of players' deviations from equilibrium play. Deviations should be constructed as 'signals' (as opposed to mistakes), since players should privilege interpretations of the opponents' play that are consistent with common knowledge of rationality. The typical example of a contradiction between backward and forward induction principles would be a game of imperfect information, where one may apply forward induction in one part of the tree, and then use the conclusion for a backward induction argument in a different part of the tree (Kohlberg 1990).

The game of Figure 5 is taken from (Kohlberg 1990, p.10). Since player I, by choosing y, could have received 2, then by forward induction if he plays n he intends to follow with T; but for the same reason II, by choosing D, shows that she intends to play R, and hence—by backward induction— I must play B. What seems to be at stake here is a conflict between different but equally powerful intuitions. By playing D, player II is committing herself to follow up with R, and thus player I would be safe to play y. On the other hand, once player I's node has been reached, what happened before might be thought of as strategically irrelevant, as I now has a chance—by choosing n—of signaling *his* commitment to follow with T. Which commitment is firmer? Which signal is most credible?

We must remember that players make their choices about which strategy to adopt after a process of deliberation that takes place $before$ the game is actually played. During deliberation, we have argued, players will employ some shared principle that allows them to rule out some plays of the game as inconsistent with it. A plausible candidate is admissibility. Let us now see how the ex ante deliberation of the players might unfold in this game by applying the procedure $Seq(T)$ to the strategies UL, UR, DL, DR and yT, yB, nT, nB. Note that if we recursively apply

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[4] This principle corresponds to subgame perfection.

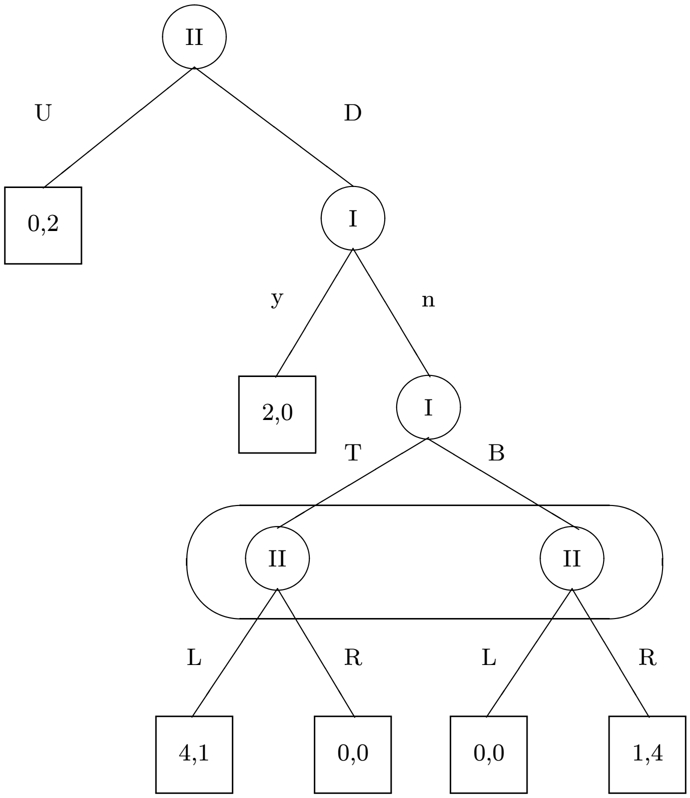

Figure 5: Backward vs. Forward Induction Principles

to this game the concept of sequential admissibility presented in the previous section, we must conclude that the only strategies consistent with common reasoning about sequential admissibility are UR, and yT. Indeed, common reasoning about sequential weak admissibility alone yields this result. For during the first round of iteration, the strategy nB of player I is eliminated because this strategy is strictly dominated by any strategy that chooses y at I's first choice node. Similarly, the strategy DL of player II is immediately eliminated because this strategy is strictly dominated by any strategy that chooses U at the root. So after the first round of elimination, II's second information set is restricted to the node reached with nT, and her choices at this information set are restricted to R only. This means in turn that y now strictly dominates nT at I's first information set, and U strictly dominates DR at the root. Finally, the strategies yB and UL are not strategies in the restricted tree obtained after the first round of elimination, and therefore they are eliminated. After the second round of elimination, only UR and yT survive. Thus we predict that players who deliberate according to a shared admissibility principle will expect U to be chosen at the beginning of the game.

A brief comment about the intuitive plausibility of our procedure is now in order. Note that the procedure we propose does not allow the players to discount whatever happens before a given information set as strategically irrelevant. For example, if player II were to choose D, player I should not keep playing as if he were in a new game starting at his decision node. We rather suggest that I should expect II to follow with R, if given a chance. In which case he should play y and player II, who can replicate I's reasoning, will in fact never play D. On the other hand, playing D to signal that one wants to continue—if given a chance—with R would make little sense, since II must know that nB is never going to be chosen, and R makes sense only if it follows nB. In other words, D is not a rational move for player II. Similar reasoning excludes nB as a rational strategy for player I.

The problem with Kohlberg's and similar examples is that no constraints are set on players' forward induction "signals". We define the notion of a credible signal in an extensive form game, and show that the credible signals are the signals consistent with common reasoning about sequential admissibility (much as Selten's subgame-perfect equilibria characterize "credible threats"). Thus the examples in the literature which purport to show the conflict between backward and forward induction principles involve forward induction signals that are not credible.

The following definition formulates the notion of a forward induction signal in general, and a credible forward induction signal in particular. The idea is this: Let us consider a move m at a given information set I_i, and ask what *future* moves of player i at lower information sets I_i' are consistent with sequential admissibility and the fact that m was chosen at I_i. If there are future moves that are consistent with sequential admissibility and the fact that m was chosen at I_i, then we take the move m at I_i to be a signal that player i intends to follow with one of those moves at I_i'. But we argue that in order for this signal to be credible to i's opponents, at least one of the future admissible moves must be consistent with common reasoning about sequential admissibility in T.

We say that an information set I_i' in a game tree T is reachable from another information set I_i *with a strategy* s_i if there is are nodes $x \in I_i, y \in I_i'$ such that some play sequence that is consistent with $s_i|T_x$ contains y.

DEFINITION 6 *Let T be a game tree with information set I_i. Let $T|I_i$ denote the restriction of T to nodes in I_i and successors of nodes in I_i.*

- *A strategy s_i is* consistent with forward induction *at I_i if s_i is sequentially admissible at I_i.*
- *A move m at an information set I_i is a* forward induction signal *for S_i^* at a lower information set I_i' (written $< I_i : m, I_i' : S_i^* >$), where $s_i \in S_i^* \iff$*
  1. *$s_i(I_i) = m$;*
  2. *I_i' is reachable from I_i with s_i;*
  3. *s_i is consistent with forward induction at I_i.*
- *A forward induction signal $< I_i : m, I_i' : S_i^* >$ is* credible *if some strategy s_i in S_i^* is consistent with common reasoning about sequential admissibility in T, i.e. $s_i \in CR_{Seq}(T)_i$.*

Let us illustrate these concepts in the game of figure 5. According to our definitions, the only strategy that chooses n at I's first information set and is consistent with forward induction is nT. So $< I_{\mathrm{I}}^1 : n, I_{\mathrm{I}}^2 : \{nT\} >$ is a forward induction signal, where I_{I}^1 denotes I's first information set and I_{I}^2 denotes I's second information set. However, $< I_{\mathrm{I}}^1 : n, I_{\mathrm{I}}^2 : \{nT\} >$ is not a credible signal. For nT is inconsistent with common reasoning about sequential admissibility, since such reasoning rules out L at II's second information set. Similarly for player II, $< I_{\mathrm{II}}^1 : D, I_{\mathrm{II}}^2 : \{DR\} >$ is a forward induction signal. But it is not a credible signal, since DR is inconsistent with common reasoning about sequential admissibility. Hence neither forward induction signal is credible, as "sending" either signal is inconsistent with common reasoning about sequential admissibility as defined by CR_{Seq}.

In terms of reasoning about admissibility, the difference between Kohlberg's and our analysis is this. Kohlberg applies admissibility once to argue that D is a forward induction signal for R and n is a forward induction signal for T. But if we assume that admissibility is common knowledge among the players, then neither D nor n are credible signals. Indeed, common knowledge is not even needed to get to this conclusion: it is sufficient to apply admissibility twice to get the same result.

5 Common Reasoning about Admissibility in the Extensive and Strategic Forms

A game G in *strategic form* is a triple $\langle N, S_{i \in N}, u_{i \in N} \rangle$, where N is the number of players and, for each player $i \in N$, S_i is the set of pure strategies available to i, and u_i is player i's utility function. Given a strategy profile $s = (s_1, \ldots, s_n), u_i(s)$ denotes the payoff to player i when players follow the strategies $(s_1, \ldots, s_n)$. Consider the set of strategy profiles $S = S_1 \times S_2 \times ... \times S_n$, and two strategies $s_i, s_i' \in S_i$ of player i. Player i's strategy s_i is *weakly dominated* by her strategy s_i' given S just in case:

1. for all $n-1$-tuples s_{-i} chosen by i's opponents that are consistent with S, $u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$ and
2. for at least one $n-1$-tuple s_{-i} consistent with S, $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$.

A strategy s_i is *weakly dominated given* S just in case there is a strategy s_i' consistent with S such that s_i' weakly dominates s_i given S. A strategy s_i is *admissible in* S just in case s_i is not weakly dominated given S. We denote the strategic form of an extensive form game T by the collection $S(T)$ of strategies in T, with payoffs defined as in T.

Our goal in this section is to determine what reasoning in the strategic form of a game corresponds to common reasoning about sequential admissibility. To this end we characterize what properties a strategy s_i must satisfy in the extensive form T of a game in order to be admissible in the strategic form $S(T)$. The key idea is to evaluate a strategy only with respect to information sets that can be reached by the given strategy. For example, in the game of figure 4, the strategy $(R_1 R_2)$ for player 1 yields the same payoff as $(R_1 L_2)$. Hence neither strategy weakly dominates the other in the normal form, although $(R_1 L_2)$ is sequentially admissible and $(R_1 R_2)$ is not. Evaluating strategies only with respect to information sets that are consistent with them leads to what we call *proper weak dominance*, and *proper admissibility*. So in the game of figure 4, $(R_1 R_2)$ is properly admissible.

We say that an information set I in a game tree T is reachable with a strategy s_i if some node in I is consistent with s_i.

DEFINITION 7 *Sequential Proper Admissibility*

- *Let T be a finite game tree.*
- *A strategy s_i is* properly weakly dominated *at an information set I_i belonging to i in T just in case I_i is reachable with s_i and s_i is weakly dominated at I_i.*
- *A strategy s_i is* properly admissible *at an information set I_i just in case s_i is not properly weakly dominated at I_i.*
- *A strategy s_i is* sequentially properly admissible *in T if and only if s_i is properly admissible at each information set I_i in T that belongs to player i.*

We define the result of common reasoning about sequential proper admissibility in the by now familiar way. For a given game tree T, let $Seq - PA_i(T) = \{s_i \in S_i(T) : s_i$ is sequentially properly admissible in $T\}$, and let $Seq - PA(T) = \times_{i \in N} Seq - PA_i(T)$.

Definition 8 *Common Reasoning About Sequential Proper Admissibility*

- *Let T be a game tree, with players $N = 1, 2, ...n$.*
- *The strategies in T* consistent with common reasoning about sequential proper admissibility *are denoted by $CR_{PSeq}(T)$, and are defined as follows:*
  1. $PSeq^0(T) = S(T)$.
  2. $PSeq^{j+1}(T) = Seq - PA(T|PSeq^j(T))$.
  3. $s \in CR_{PSeq}(T) \iff \forall j : s|[T|PSeq^j(T)] \in PSeq^{j+1}(T)$.

The two notions of sequential admissibility are equivalent in terms of their predictions about how the game will be played. That is, exactly the same play sequences are consistent with both restrictions.

Lemma 4 *Let T be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is, $\{play(s) : s$ is sequentially admissible in $T\} = \{play(s) : s$ is sequentially properly admissible in $T\}$.*

From this fact it follows immediately that common reasoning about sequential admissibility yields the same predictions as common reasoning about proper sequential admissibility.

proposition 5 *Let T be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is, $\{play(s) : s \in CR_{Seq}(T)\} = \{play(s) : s \in CR_{PSeq}(T)\}$.*

However, it is not always the case that a strategy that is admissible in the strategic form of a game is properly admissible in an extensive form of the game. For example, in the game of figure 6, the strategy L is properly weakly dominated for player 2 at her information set: at node y, R yields a higher payoff than L, and starting at node x, both choices yield the same. On the other hand, node y cannot be reached when 2 plays L, so that L is admissible in the strategic form of the game, yielding 2's maximal payoff of 1. The game in figure 6 has the strange feature that if 2 plays R at x to arrive at y, she has 'forgotten' this fact and cannot distinguish between x and y. Indeed, this is a game without perfect recall. Perfect recall is defined as follows.

Definition 9 (Kuhn) *Let T be a finite game tree. Then T is an extensive game with* perfect recall *if and only if for each information set I_i belonging to player i, and each strategy s_i in T, all nodes in I_i are consistent with s_i if any node in I_i is.*

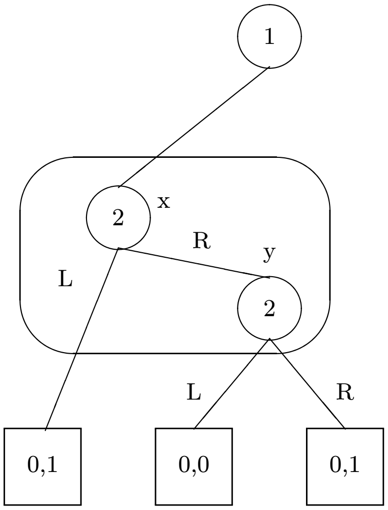

Figure 6: A Game Without Perfect Recall

We note that if T is a game with perfect recall, then all restrictions of T satisfy perfect recall. The next proposition shows that in extensive form games with perfect recall, the notion of proper weak dominance coincides exactly with admissibility in the strategic form.

PROPOSITION 6 *Let T be a finite game tree with perfect recall. Then a strategy s_i for player i is admissible in the strategic form $S(T)$ if and only if s_i is sequentially properly admissible in T.*

Consider a game G in strategic form. We define an *order-free* iterative procedure for eliminating weakly dominated strategies. If S is a set of strategy profiles, let $Admiss_i(S)$ be the set of all strategies s_i for player i that are consistent with S and admissible given S, and let $Admiss(S) = \times_{i \in N} Admiss_i(S)$.

DEFINITION 10 *Common Reasoning About Admissibility in the Strategic Form*

- Let the strategic form of a finite game G be given by $\langle N, S_{i \in N}, u_{i \in N} \rangle$, and let $S = S_1 \times S_2 \times ... \times S_n$ be the set of strategy profiles in G.
- The strategies in S *consistent with common reasoning about admissibility* are denoted by $CR_{Ad}(S)$, and are defined as follows.

1. $Ad^0(S) = S$.
2. $Ad^{j+1}(S) = Admiss(Ad^j(S))$.
3. $CR_{Ad}(S) = \bigcap_{j=0}^{\infty} Ad^j(S)$.

The procedure goes through at most $\sum_{i \in N} |S_i - 1|$ iterations; that is, for all $j \geq \sum_{i \in N} |S_i - 1|, Ad^j(S) = Ad^{j+1}(S)$.

For example, consider the game in figure 7. In the first iteration, player 1 will eliminate c, which is weakly dominated by b, and player 2 will eliminate R, which is dominated by L and M. Since admissibility is common knowledge, both players know that the reduced matrix only contains the strategies a, b and L, M. Common reasoning about admissibility means that both players will apply admissibility to the new matrix (and know that they both do it), and since now L dominates M, both will know that M is being eliminated. Finally, common reasoning about admissibility will leave b, L as the unique outcome of the game.

Our main result is that in games with perfect recall, iterated sequential proper admissibility and order-free elimination of inadmissible strategies in the strategic form yield exactly the same result.

THEOREM 7 *Let T be a finite game tree with perfect recall. A strategy profile s is consistent with common reasoning about sequential proper admissibility in T if and only if s is consistent with common reasoning about admissibility in the strategic form of T. That is, $CR_{PSeq}(T) = CR_{Ad}(S(T))$.*

It is noteworthy that if the order-free elimination of inadmissible strategies in the normal form yields a unique solution, then that solution is a Nash equilibrium (Bicchieri 1994).

General existence is now easy to establish.

	L	M	R
a	1,3	3,2	1,2
b	2,2	2,0	0,0
c	2,1	1,2	0,0

Figure 7: Order-Free Elimination of Weakly Dominated Strategies

PROPOSITION 8 *For all finite games G with pure strategy profiles S,* $CR_{Ad}(S) \neq \emptyset$.

6 Proof of Results

For the proof of proposition 1, we rely on the well-known one-deviation property of subgame perfect equilibrium: If it is possible for one player to profitably deviate from his subgame perfect equilibrium strategy s_i, he can do so with a strategy s_i' that deviates from s_i only once.

LEMMA 0 *Let* T *be a finite game tree of perfect information. Then* s *is a subgame perfect equilibrium in* T *if and only if for each node* x*, for each player* $i, u_i(s[i], s[-i], x) \geq u_i(s_i', s[-i], x)$*, whenever* $s[i]$ *and* s_i' *differ only at* x.

Proof. See (Osborne and Rubinstein 1994, Lemma 98.2).

For the next proposition, we note that if T is finite, then our iterative procedure goes only through finitely many iterations. In particular, this means that if a strategy s_i is strictly dominated given $CR_{WA}(T)$, then s_i is not in $CR_{WA}(T)$.

PROPOSITION 1 *Let* T *be a finite game tree of perfect information. Then a strategy* s_i *is consistent with common reasoning about sequential weak* *admissibility in T if and only if s_i is consistent with subgame perfection. That is, $CR_{WA}(T) = SPE(T)$.*

Proof. We prove by induction on the height x of each node that $CR_{WA}(T_x) = SPE(T_x)$. The proposition follows when we take x to be the root of T.

Base Case, $h(x) = 1$. Then all successors of x are terminal nodes. Let player i be the player to move at x. Let $\max(x)$ be the maximum payoff player i can achieve at x (i.e. $\max(x) = \max\{u_i(y) : y \text{ is a successor of } x\}$). Then $s_i|T_x$ is consistent with subgame perfection at x if and only if $s_i(x)$ yields i the maximum payoff $\max(x)$, which is exactly when $s_i|T_x$ is not strictly dominated at x.

Inductive Case: Assume the hypothesis in the case when $h(y) < h(x)$ and consider x.

($\Rightarrow$): Let s be a strategy profile consistent with common reasoning about sequential weak admissibility (i.e. $s \in CR_{WA}(T_x)$). Suppose that it is player i's turn at x. For each player $j, s[j]|T_y$ is consistent with subgame perfection in each proper subgame T_y of T_x, by the inductive hypothesis and the fact that $s[j]$ is consistent with common reasoning about sequential weak admissibility in T_x. So the implication ($\Rightarrow$) is established if we show that $s[i]$ is consistent with subgame perfection in T_x. Let y be the successor of x that is reached when i plays $s[i]$ at x. Let $\max(y)$ be the maximum that i can achieve given common reasoning about sequential weak admissibility when he follows $s[i]$ (i.e. $\max(y) = \max\{u_i(s[i], s_{-i}, x) : s_{-i} \text{ is consistent with } CR_{WA}(T_x)\}$). For each y' that is a successor of x, let $\min(y')$ be the minimum that i can achieve given common reasoning about sequential weak admissibility when he follows $s[i]$ in $T_{y'}$. Then we have (*) that $\max(y) \geq \min(y')$ for each successor y' of x. For otherwise player i can ensure himself a higher payoff than $s[i]$ can possibly yield, by moving to some successor y' of x and continuing with $s[i]$. That is, the strategy s_i^* which moves to y' at x and follows $s[i]$ below y' strictly dominates $s[i]$ in $T_x|CR_{WA}(T_x)$. But since T and hence T_x is finite, this contradicts the assumption that $s[i]$ is consistent with $CR_{WA}(T_x)$. Now by inductive hypothesis, $CR_{WA}(T_{y'}) = SPE(T_{y'})$ for each successor y' of x. So there is a subgame perfect equilibrium $s_{\max}$ in T_y which yields i the payoff $\max(y)$ in T_y and in which player i follows $s[i]$ (i.e. $s[i] = s_{\max}[i]$). Again by inductive hypothesis, for each successor node y' of x there is a subgame perfect equilibrium $s^{y'}_{\min}$ in $T_{y'}$ which gives player i the payoff $\min(y')$ and in which player i follows $s[i]$ in $T_{y'}$. Now we define a subgame perfect equilibrium s^* in T_x in which player i follows $s[i]$:

1. $s^*[i](\{x\}) = s[i](\{x\})$,
2. in T_y,s^* follows $s_{\max}$,
3. in $T_{y'}, s^*$ follows $s^{y'}_{\min}$, where y' is a successor of x other than y. By our observation (*), there is no profitable 1-deviation from s^* for player i at x, and hence by lemma 0, s^* is a subgame perfect equilibrium in T_x.

($\Leftarrow$) Let s be consistent with subgame perfection in T_x. Let i be the player moving at x. Consider any strategy $s[j]$ in s, where $j \neq i$. Since j is not moving at $x, s[j]$ is consistent with common reasoning about sequential weak admissibility in T_x if and only if $s[j]|T_y$ is consistent with common reasoning about sequential weak admissibility in each subgame T_y of T_x. Since s is consistent with subgame perfection in T_x, there is a subgame perfect equilibrium s^* in T_x in which j follows $s[j]$. Since s^* is subgame perfect, $s^*|T_y$ is subgame perfect in T_y. Hence $s[j]|T_y = s^*[j]|T_y$ is consistent with subgame perfection in T_y. By inductive hypothesis, this entails that $s[j]|T_y$ is consistent with common reasoning about sequential weak admissibility in T_y. Since this is true for any subgame T_y of $T_x, s[j]$ is consistent with common reasoning about sequential weak admissibility in T_x. Next, consider $s[i]$, the strategy followed by the player who is moving at x. We just established that for each iteration $WA^j(T)$ of common reasoning about weak sequential admissibility, $s^*[-i]$ is consistent with $WA^j(T)$. Since s^* is a subgame perfect equilibrium in T_x, $s^*[i]$ is a best reply against $s^*[-i]$ in T_x and each subgame of T_x. So in each subgame T_y of T_x (including T_x) and at each iteration $WA^j(T), s^*[i]$ is a best reply against some strategy profile of his opponents consistent with $WA^j(T)$, namely $s^*[-i]|T_y$, and hence $s^*[i]$ is sequentially weakly admissible given $WA^j(T)$. Since $CR_{WA}(T) = WA^k(T)$ for some k, because T is finite, $s^*[i]$ is consistent with common reasoning about sequential weak admissibility. This shows that all strategies in the strategy profile s are consistent with common reasoning about sequential weak admissibility in T_x, and completes the proof by induction.□

LEMMA 2 *If T is a restriction of T' and s_i is sequentially admissible in T, then there is an extension s_i' of s_i to T' such that s_i' is sequentially weakly admissible in T'.*

Proof. We construct s_i' as follows. At each information set I_i in T' such that I_i contains a node in $T, s_i' = s_i$. At all other information sets I_i, s_i' follows a strategy that is weakly admissible at I_i. We claim that s_i' is sequentially weakly admissible in T'; let I_i be any information set in T' belonging to i.

Case 1: I_i contains a node x in T. Since T is a restriction of T', I_i contains all nodes in $I_T(x)$, where $I_T(x)$ is the information set in T containing x. So if s_i is strictly dominated in T' at I_i, then s_i is strictly dominated in T at $I_T(x)$, contrary to the supposition that s_i is admissible at $I_T(x)$

Case 2: I_i contains no node x in T. By construction, s_i is weakly admissible at I_i.□

PROPOSITION 3 *Let T be a finite game tree. If a play sequence is consistent with common reasoning about sequential admissibility in T, then the play sequence is consistent with common reasoning about sequential weak admissibility. That is, $\{play(s) : s \in CR_{Seq}(T)\} \subseteq \{play(s) : s \in CR_{WA}(T)\}$.*

Proof. We prove by induction on $j \geq 0$ that for each $j, T|Seq^j(T)$ is a restriction of $T|WA^j(T)$.

Base Case, $j = 0$. Then $Seq^0(T) = WA^0(T)$, so the claim is immediate.

Inductive Step: Assume that $T|Seq^j(T)$ is a restriction of $T|WA^j(T)$, and consider $j + 1$. Choose any strategy profile s in $Seq^{j+1}(T)$. By lemma 2, extend each $s[i]$ in s to a strategy $s'[i]$ that agrees with $s[i]$ on information sets that have members both in $T|Seq^j(T)$ and $T|WA^j(T)$, and is sequentially weakly admissible in $T|WA^j(T)$. Call the resulting strategy profile s'; s' is in $WA^{j+1}(T)$. Clearly s and s' result in the same play sequence, i.e. $play(s') = play(s)$, because the same actions are taken at each information set. So all nodes that are consistent with $Seq^{j+1}(T)$ are consistent with $WA^{j+1}(T)$, which means that $T|Seq^{j+1}(T)$ is a restriction of $T|WA^{j+1}(T)$. This completes the proof by induction. □

LEMMA 4 *Let T be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is, $\{play(s) : s$ is sequentially admissible in $T\} = \{play(s) : s$ is sequentially properly admissible in $T\}$.*

Proof. ($\supseteq$) Let s be a sequentially properly admissible strategy profile in T, and let x be any node reached in $play(s)$ such that $I(x)$ belongs to player i. Then $s[i]$ is admissible at $I(x)$ since $I(x)$ is consistent with $s[i]$. Now we may modify s to obtain a strategy profile s^*, in which each player i follows $s[i]$ at any information set containing a node in $play(s)$, and follows an admissible strategy at every other information set. Then s^* is sequentially admissible, and $play(s^*) = play(s)$.

($\subseteq$) This is immediate because all sequentially admissible strategies are sequentially properly admissible.□

PROPOSITION 5 *Let T be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is, $\{play(s) : s \in CR_{Seq}(T)\} = \{play(s) : s \in CR_{PSeq}(T)\}$.*

Proof. We prove by induction on j that for each $j \geq 0, T|Seq^j(T) = T|PSeq^j(T)$.

Base Case, $j = 0$. The claim is immediate since $Seq^0(T) = PSeq^0(T) = S(T)$.

Inductive Case: Assume that $T|Seq^j(T) = T|PSeq^j(T)$, and consider $j + 1$. The claim follows immediately from lemma 4.□

PROPOSITION 6 *Let T be a finite game tree with perfect recall. Then a strategy s_i for player i is admissible in $S(T)$ if and only if s_i is sequentially properly admissible in T.*

Proof. Suppose that a strategy s_i in $S(T)$ for player i is weakly dominated in $S(T)$. Then there is a strategy s_i' consistent with $S(T)$ such that

1. for all strategy profiles s_{-i} consistent with $S(T), u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$, and
2. for some strategy profile s_{-i}^* consistent with $S(T), u_i(s_i, s_{-i}^*) < u_i(s_i', s_{-i}^*)$.

Let x be the first node that appears along both the plays of s_i against s_{-i}^* and s_i against s_{-i}^* at which s_i deviates from s_i', so that $x \in range(play(s_i, s_{-i}^*)) \cap range(play(s_i', s_{-i}^*))$ and $s_i(I_i(x)) \neq s_i'(I_i(x))$. Then x is consistent with s_i and s_i' in T. Let y be any node at $I_i(x)$ consistent with s_i and s_i', and let s_{-i} be any strategy profile of i's opponents. Then $u_i(s_i, s_{-i}, y) \leq u_i(s_i', s_{-i}, y)$; for otherwise, by perfect recall, let s_{-i}^* be a strategy profile of i's opponents such that both $play(s_i, s_{-i}^*)$ and $play(s_i', s_{-i}^*)$ reach y, and such that $s_{-i}^*|T_y = s_{-i}|T_y$. Then $u_i(s_i, s_{-i}^*) > u_i(s_i', s_{-i}^*)$, contrary to the hypothesis that s_i' weakly dominates s_i in $S(T)$. Since we also have that $u_i(s_i, s_{-i}^*, x) < u_i(s_i', s_{-i}^*, x)$, it follows that s_i' weakly dominates s_i at $I_i(x)$ so that s_i is not sequentially properly admissible.

Suppose that a strategy s_i is properly weakly dominated at an information set I_i in T by strategy s_i'. Then there must be a node x in I_i consistent with s_i and a strategy profile s_{-i}' in T such that s_i' yields a higher payoff at x against s_{-i}' than s_i does, i.e. $u_i(s_i, s_{-i}', x) < u_i(s_i', s_{-i}', x)$. Assume without loss of generality that x is reached by the play sequence of s_i against s_{-i}', i.e. $x \in range(play(s_i, s_{-i}'))$. Now we define a strategy s_i^* that weakly dominates s_i in T as follows.

1. At an information set I_i' that does not contain x or any successor of x, $s_i^*(I_i') = s_i(I_i')$.
2. At an information set I_i' that contains x or a successor of $x, s_i^*(I_i') = s_i'(I_i')$.

We show that s_i^* weakly dominates s_i in $S(T)$. Since $play(s_i, s_{-i})$ reaches $x, play(s_i^*, s_{-i})$ also reaches x, and so $u_i(s_i^*, s_{-i}) = u_i(s_i^*, s_{-i}, x) = u_i(s_i', s_{-i}, x) > u_i(s_i, s_{-i}, x) = u_i(s_i, s_{-i})$. Thus s_i^* weakly dominates s_i in $S(T)$ if for no s_{-i} in $T, u_i(s_i, s_{-i}) > u_i(s_i^*, s_{-i})$, which we establish now. Let a strategy profile s_{-i} in T be given.

Case 1: the play sequence of (s_i^*, s_{-i}) does not reach $I_i(x)$. Then $play(s_i^*, s_{-i}) = play(s_i, s_{-i})$, and the claim follows immediately.

Case 2: the play sequence of (s_i^*, s_{-i}) goes through some node y in $I_i(x)$. Since x is consistent with s_i and T is a game with perfect recall, y is consistent with s_i, and so $play(s_i, s_{-i})$ reaches y. As before, we have that (a) $u_i(s_i, s_{-i}, y) = u_i(s_i, s_{-i})$. Also, s_i^* coincides with s_i' after node y, and so (b) $u_i(s_i^*, s_{-i}) = u_i(s_i', s_{-i}, y)$. Since s_i' weakly dominates s_i at $I_i(x)$, and y is in $I_i(x)$, it follows that (c) $u_i(s_i', s_{-i}, y) \geq u_i(s_i, s_{-i}, y)$. Combining (a), (b) and (c) it follows that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$. This establishes that s_i is weakly dominated given $S(T)$.$\Box$

THEOREM 7 *Let T be a finite game tree with perfect recall. A strategy profile s is consistent with common reasoning about sequential proper admissibility if and only if s is consistent with common reasoning about admissibility in the strategic form of T. That is, $CR_{PSeq}(T) = CR_{Ad}(S(T))$.*

Proof. We prove by induction on j that for all $j \geq 0, PSeq^j(T) = Ad^j(S(T))$.

Base Case, $j = 0$. Then by definition, $PSeq^0(T) = S(T) = Ad^0(S(T))$.

Inductive Step: Assume that $PSeq^j(T) = Ad^j(S(T))$ and consider $j+1$. By inductive hypothesis, $T|PSeq^j(T) = T|Ad^j(S(T))$. Now a strategy s_i is in $PSeq_i^{j+1}(T) \iff s_i$ is in $PSeq_i^j(T)$ and s_i is sequentially properly admissible in $T|PSeq^j(T)$. By inductive hypothesis, the first condition implies that s_i is in $Ad^j(S(T))$. By proposition 6 and the facts that $T|PSeq^j(T) = T|Ad^j(S(T))$ and that all restrictions of T are games with perfect recall, the second condition implies that s_i is admissible in $S(T|Ad^j(S(T))) = Ad^j(S(T))$. So s_i is in $Ad^{j+1}(S(T))$. Conversely, a strategy s_i is in $Ad^{j+1}(S(T)) \iff s_i$ is in $Ad^j(S(T))$ and s_i is admissible in $Ad^j(S(T))$. By inductive hypothesis, the first condition implies that s_i is in $PSeq^j(T)$, and the second condition may be restated to say that s_i is admissible in $S(T|Ad^j(S(T)))$. By proposition 6, the second condition then implies that s_i is sequentially properly admissible in $T|Ad^j(S(T)) = T|PSeq^j(T)$. Hence s_i is in $PSeq_i^{j+1}(T)$. This shows that $PSeq^{j+1}(T) = Ad^{j+1}(S(T))$, and completes the proof by induction.□

PROPOSITION 8 *For all finite games G with pure strategy profiles* $S, CR_{Ad}(S) \neq \emptyset$.

Proof. The admissible elements in S_i^j survive at each iteration j , for each player i, and there always is a admissible element in each S_i^j since each S_i^j is finite. Hence $S^j \neq \emptyset$ for any j, and so $S^{\sum_{i \in N} |S_i - 1|} = CR_{Ad}(S) \neq \emptyset$.□

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