Inference in first-order logic

Chapter 9

Outline

- ♢ Reducing first-order inference to propositional inference
- ♢ Unification
- ♢ Generalized Modus Ponens
- ♢ Forward and backward chaining
- ♢ Resolution

Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall\, v \;\; \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g

E.g., $\forall\, x \;\; King(x) \land Greedy(x) \;\Rightarrow\; Evil(x)$ yields

$$King(John) \land Greedy(John) \;\Rightarrow\; Evil(John)$$
$$King(Richard) \land Greedy(Richard) \;\Rightarrow\; Evil(Richard)$$
$$King(Father(John)) \land Greedy(Father(John)) \;\Rightarrow\; Evil(Father(John))$$
$$\vdots$$

Existential instantiation (EI)

For any sentence α, variable v, and constant symbol k
that does not appear elsewhere in the knowledge base:

$$\frac{\exists\, v \;\; \alpha}{\textsc{Subst}(\{v/k\}, \alpha)}$$

E.g., $\exists\, x \;\; Crown(x) \land OnHead(x, John)$ yields

$$Crown(C_1) \land OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a Skolem constant

Another example: from $\exists\, x \;\; d(x^y)/dy = x^y$ we obtain

$$d(e^y)/dy = e^y$$

provided e is a new constant symbol

Existential instantiation contd.

UI can be applied several times to **add** new sentences;
the new KB is logically equivalent to the old

EI can be applied once to **replace** the existential sentence;
the new KB is **not** equivalent to the old,
but is satisfiable iff the old KB was satisfiable

Reduction to propositional inference

Suppose the KB contains just the following:

$$\forall\, x \;\; King(x) \land Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

Instantiating the universal sentence in **all possible** ways, we have

$$King(John) \land Greedy(John) \;\Rightarrow\; Evil(John)$$
$$King(Richard) \land Greedy(Richard) \;\Rightarrow\; Evil(Richard)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

The new KB is propositionalized: proposition symbols are

$King(John),\; Greedy(John),\; Evil(John), King(Richard)$ etc.

Reduction contd.

Claim: a ground sentence* is entailed by new KB iff entailed by original KB

Claim: every FOL KB can be propositionalized so as to preserve entailment

Idea: propositionalize KB and query, apply resolution, return result

Problem: with function symbols, there are infinitely many ground terms,
e.g., $Father(Father(Father(John)))$

Theorem: Herbrand (1930). If a sentence α is entailed by an FOL KB,
it is entailed by a **finite** subset of the propositional KB

Idea: For $n = 0$ to ∞ do
create a propositional KB by instantiating with depth-n terms
see if α is entailed by this KB

Problem: works if α is entailed, loops if α is not entailed

Theorem: Turing (1936), Church (1936), entailment in FOL is semidecidable

Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences.
E.g., from

$\forall x\ \ King(x) \wedge Greedy(x) \;\Rightarrow\; Evil(x)$
$King(John)$
$\forall y\ \ Greedy(y)$
$Brother(Richard, John)$

it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $Greedy(Richard)$ that are irrelevant

Unification

We can get the inference immediately if we can find a substitution θ
such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, OJ)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

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$Knows(John, x)$	$Knows(x, OJ)$	$fail$

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

Generalized Modus Ponens (GMP)

$$\frac{p_1{}',\ p_2{}',\ \ldots,\ p_n{}',\ (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta} \qquad \text{where } p_i{}'\theta = p_i\theta \text{ for all } i$$

$p_1{}'$ is $King(John)$ p_1 is $King(x)$
$p_2{}'$ is $Greedy(y)$ p_2 is $Greedy(x)$
θ is $\{x/John, y/John\}$ q is $Evil(x)$
$q\theta$ is $Evil(John)$

GMP used with KB of definite clauses (**exactly** one positive literal)
All variables assumed universally quantified

Example knowledge base

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Col. West is a criminal

Example knowledge base contd.

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$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$
Nono . . . has some missiles

Example knowledge base contd.

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$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$
Nono . . . has some missiles, i.e., $\exists\, x\ Owns(Nono, x) \wedge Missile(x)$:
$Owns(Nono, M_1)$ and $Missile(M_1)$
. . . all of its missiles were sold to it by Colonel West

Example knowledge base contd.

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$Owns(Nono, M_1)$ and $Missile(M_1)$
. . . all of its missiles were sold to it by Colonel West
$Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$
Missiles are weapons:

Example knowledge base contd.

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$Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$
Missiles are weapons:
$Missile(x) \Rightarrow Weapon(x)$
An enemy of America counts as "hostile":

Example knowledge base contd.

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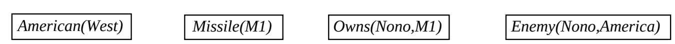
$Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$
Missiles are weapons:
$Missile(x) \Rightarrow Weapon(x)$
An enemy of America counts as "hostile":
$Enemy(x, America) \Rightarrow Hostile(x)$
West, who is American . . .
$American(West)$
The country Nono, an enemy of America . . .
$Enemy(Nono, America)$

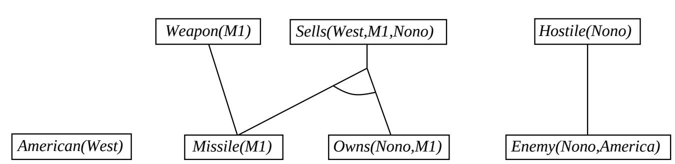
Forward chaining algorithm

```
function FOL-FC-Ask(KB, α) returns a substitution or false
   repeat until new is empty
      new ← { }
      for each sentence r in KB do
         ( p1 ∧ ... ∧ pn ⇒ q) ← Standardize-Apart(r)
         for each θ such that (p1 ∧ ... ∧ pn)θ = (p'1 ∧ ... ∧ p'n)θ
                       for some p'1, ..., p'n in KB
            q' ← Subst(θ, q)
            if q' is not a renaming of a sentence already in KB or new then do
                  add q' to new
                  φ ← Unify(q', α)
                  if φ is not fail then return φ
      add new to KB
   return false
```


Forward chaining proof



Forward chaining proof



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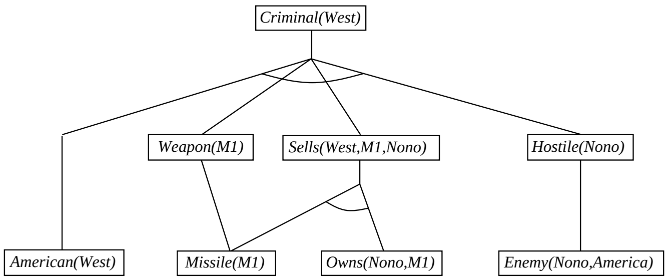


Properties of forward chaining

Sound and complete for first-order definite clauses
(proof similar to propositional proof)

Datalog = first-order definite clauses + **no functions** (e.g., crime KB)
FC terminates for Datalog in poly iterations: at most $p \cdot n^k$ literals

May not terminate in general (with functions) if α is not entailed

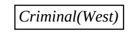
This is unavoidable: entailment with definite clauses is semidecidable

Backward chaining algorithm

```
function FOL-BC-Ask(KB, goals, θ) returns a set of substitutions
   inputs: KB, a knowledge base
           goals, a list of conjuncts forming a query (θ already applied)
           θ, the current substitution, initially the empty substitution { }
   local variables: answers, a set of substitutions, initially empty

   if goals is empty then return {θ}
   q′ ← Subst(θ, First(goals))
   for each sentence r in KB
           where Standardize-Apart(r) = ( p1 ∧ ... ∧ pn ⇒ q)
           and θ′ ← Unify(q, q′) succeeds
       new_goals ← [ p1, ..., pn|Rest(goals)]
       answers ← FOL-BC-Ask(KB, new_goals, Compose(θ′, θ)) ∪ answers
   return answers
```


Backward chaining example



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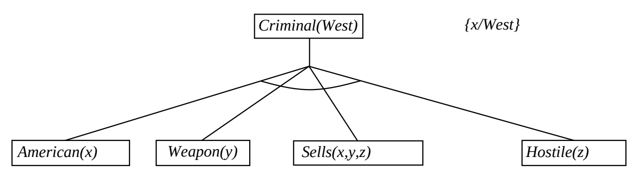

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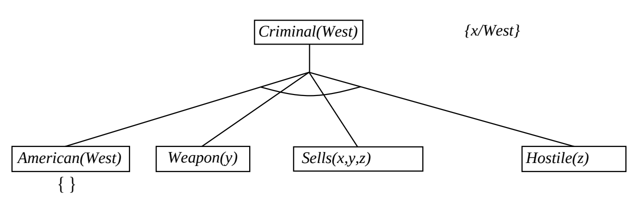

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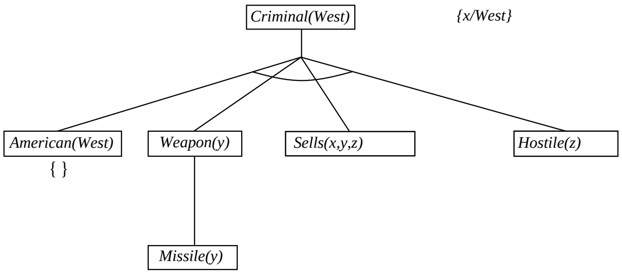

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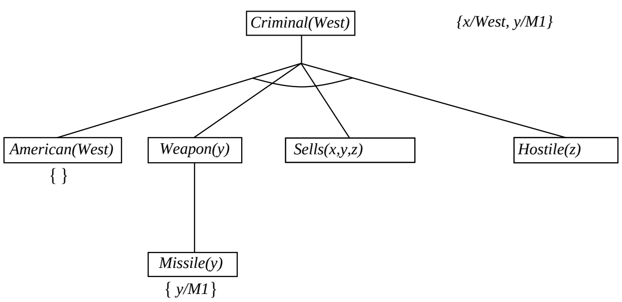

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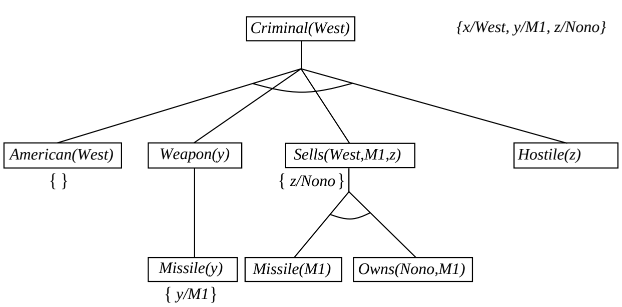

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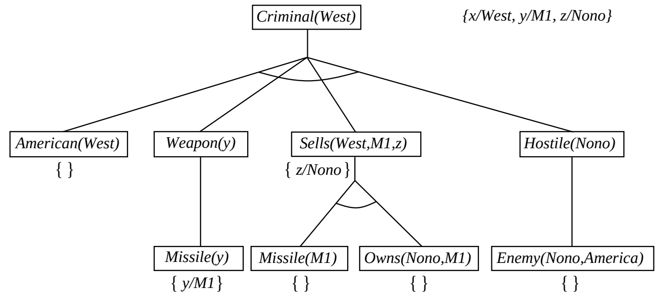

Properties of backward chaining

Depth-first recursive proof search: space is linear in size of proof

Incomplete due to infinite loops
$\Rightarrow$ fix by checking current goal against every goal on stack

Inefficient due to repeated subgoals (both success and failure)
$\Rightarrow$ fix using caching of previous results (extra space!)

Widely used (without improvements!) for logic programming

Logic programming

Sound bite: computation as inference on logical KBs

	Logic programming	Ordinary programming
1.	Identify problem	Identify problem
2.	Assemble information	Assemble information
3.	Tea break	Figure out solution
4.	Encode information in KB	Program solution
5.	Encode problem instance as facts	Encode problem instance as data
6.	Ask queries	Apply program to data
7.	Find false facts	Debug procedural errors

Should be easier to debug $Capital(NewYork, US)$ than $x := x + 2$!

Prolog systems

Basis: backward chaining with Horn clauses + bells & whistles
Widely used in Europe, Japan (basis of 5th Generation project)
Compilation techniques ⇒ approaching a billion LIPS

Program = set of clauses = `head :- literal`$_1$`, ... literal`$_n$.

```
criminal(X) :- american(X), weapon(Y), sells(X,Y,Z), hostile(Z).
```

Depth-first, left-to-right backward chaining
Built-in predicates for arithmetic etc., e.g., `X is Y*Z+3`
Closed-world assumption ("negation as failure")
e.g., given `alive(X) :- not dead(X).`
`alive(joe)` succeeds if `dead(joe)` fails

Prolog examples

Depth-first search from a start state `X`:

```
dfs(X) :- goal(X).
dfs(X) :- successor(X,S),dfs(S).
```

No need to loop over `S`: `successor` succeeds for each

Appending two lists to produce a third:

```
append([],Y,Y).
append([X|L],Y,[X|Z]) :- append(L,Y,Z).

query:   append(A,B,[1,2]) ?
answers: A=[]    B=[1,2]
         A=[1]   B=[2]
         A=[1,2] B=[]
```


Resolution: brief summary

Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{(\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)\theta}$$

where $\textsc{Unify}(\ell_i, \neg m_j) = \theta$.

For example,

$$\frac{\begin{array}{l}\neg Rich(x) \vee Unhappy(x)\\ Rich(Ken)\end{array}}{Unhappy(Ken)}$$

with $\theta = \{x/Ken\}$

Apply resolution steps to $CNF(KB \wedge \neg\alpha)$; complete for FOL

Conversion to CNF

Everyone who loves all animals is loved by someone:
$\forall\, x \;\; [\forall\, y \;\; Animal(y) \;\Rightarrow\; Loves(x,y)] \;\Rightarrow\; [\exists\, y \;\; Loves(y,x)]$

1. Eliminate biconditionals and implications

$\forall\, x \;\; [\neg\forall\, y \;\; \neg Animal(y) \vee Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$

2. Move $\neg$ inwards: $\neg\forall\, x, p \;\; \equiv \exists\, x \;\; \neg p$, $\quad \neg\exists\, x, p \;\; \equiv \forall\, x \;\; \neg p$:

$\forall\, x \;\; [\exists\, y \;\; \neg(\neg Animal(y) \vee Loves(x,y))] \vee [\exists\, y \;\; Loves(y,x)]$
$\forall\, x \;\; [\exists\, y \;\; \neg\neg Animal(y) \wedge \neg Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$
$\forall\, x \;\; [\exists\, y \;\; Animal(y) \wedge \neg Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$

Conversion to CNF contd.

3. Standardize variables: each quantifier should use a different one

$\forall\, x \;\; [\exists\, y \;\; Animal(y) \wedge \neg Loves(x,y)] \vee [\exists\, z \;\; Loves(z,x)]$

4. Skolemize: a more general form of existential instantiation.
Each existential variable is replaced by a Skolem function
of the enclosing universally quantified variables:

$\forall\, x \;\; [Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee Loves(G(x),x)$

5. Drop universal quantifiers:

$[Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee Loves(G(x),x)$

6. Distribute $\wedge$ over $\vee$:

$[Animal(F(x)) \vee Loves(G(x),x)] \wedge [\neg Loves(x,F(x)) \vee Loves(G(x),x)]$

Resolution proof: definite clauses