First-order logic

Chapter 8

Outline

◇ Why FOL?

◇ Syntax and semantics of FOL

◇ Fun with sentences

◇ Wumpus world in FOL

Pros and cons of propositional logic

Propositional logic is **declarative**: pieces of syntax correspond to facts

Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

Propositional logic is **compositional**:
meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

Propositional logic has very limited expressive power
(unlike natural language)
E.g., cannot say "pits cause breezes in adjacent squares"
except by writing one sentence for each square

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . .,
brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants	$KingJohn,\ 2,\ SFU, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

FOL Sentences

Sentence = $AtomicSentence$
or $(Sentence\ Connective\ Sentence)$
or $Quantifier\ Variable, \ldots\ Sentence$
or $\neg Sentence$

Atomic sentences

Atomic sentence = $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term = $function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$

$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg{>}(1,2)$

Truth in first-order logic

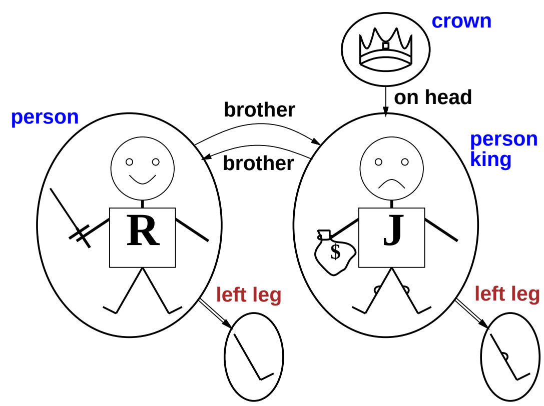
Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for
constant symbols $\rightarrow$ objects
predicate symbols $\rightarrow$ relations
function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example



Truth example

Consider the interpretation in which
$Richard \rightarrow$ Richard the Lionheart (man with R on chest)
$John \rightarrow$ the evil King John (man with J on chest)
$Brother \rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true
just in the case that Richard the Lionheart and the evil King John
are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
  For each k-ary predicate P_k in the vocabulary
    For each possible k-ary relation on n objects
      For each constant symbol C in the vocabulary
        For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at SFU is smart:
$\forall x \ \ At(x, SFU) \Rightarrow Smart(x)$

$\forall x \ \ P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, SFU) \Rightarrow Smart(KingJohn)) \\ \wedge\ & (At(Richard, SFU) \Rightarrow Smart(Richard)) \\ \wedge\ & (At(SFU, SFU) \Rightarrow Smart(SFU)) \\ \wedge\ & \ldots \end{aligned}$$

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$\forall x \ \ At(x, SFU) \wedge Smart(x)$

means "Everyone is at SFU and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \ \langle sentence \rangle$

Someone at UBC is smart:
$\exists x \ \ At(x, UBC) \wedge Smart(x)$

$\exists x \ \ P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, UBC) \wedge Smart(KingJohn)) \\ \vee\ & (At(Richard, UBC) \wedge Smart(Richard)) \\ \vee\ & (At(UBC, UBC) \wedge Smart(UBC)) \\ \vee\ & \ldots \end{aligned}$$

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$\exists x \ \ At(x, UBC) \Rightarrow Smart(x)$

is true if there is anyone who is not at UBC!

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$ (why??)

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$ (why??)

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ \ Loves(x, y)$
$\forall y \ \exists x \ \ Loves(x, y)$

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$\exists x\ \ \forall y\ \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y\ \ \exists x\ \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x\ \ Likes(x, IceCream) \qquad \neg\exists x\ \ \neg Likes(x, IceCream)$

$\exists x\ \ Likes(x, Broccoli) \qquad \neg\forall x\ \ \neg Likes(x, Broccoli)$

Fun with sentences

Brothers are siblings

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$\forall x, y\ \ Brother(x, y) \Rightarrow Sibling(x, y)$.

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"Sibling" is symmetric

$\forall x, y\ \ Sibling(x, y) \Leftrightarrow Sibling(y, x)$.

One's mother is one's female parent

Fun with sentences

Brothers are siblings

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"Sibling" is symmetric

$\forall x, y\ \ Sibling(x, y) \Leftrightarrow Sibling(y, x)$.

One's mother is one's female parent

$\forall x, y\ \ Mother(x, y) \Leftrightarrow (Female(x) \land Parent(x, y))$.

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$\forall x, y\ \ Brother(x, y) \Rightarrow Sibling(x, y)$.

"Sibling" is symmetric

$\forall x, y\ \ Sibling(x, y) \Leftrightarrow Sibling(y, x)$.

One's mother is one's female parent

$\forall x, y\ \ Mother(x, y) \Leftrightarrow (Female(x) \land Parent(x, y))$.

A first cousin is a child of a parent's sibling

$\forall x, y\ \ FirstCousin(x, y) \Leftrightarrow \exists p, ps\ \ Parent(p, x) \land Sibling(ps, p) \land Parent(ps, y)$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \land \exists m, f \;\; \neg(m = f) \land Parent(m, x) \land Parent(f, x) \land Parent(m, y) \land Parent(f, y)]$$

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists a \;\; Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes, \; \{a/Shoot\}$ $\leftarrow$ substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"
$\forall b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$
$\forall s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall t \;\; AtGold(t) \land \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed
$\Rightarrow$ keeping track of change is essential

Deducing hidden properties

Properties of locations:
$\forall x, t \;\; At(Agent, x, t) \land Smelt(t) \;\Rightarrow\; Smelly(x)$
$\forall x, t \;\; At(Agent, x, t) \land Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect
$\forall y \;\; Breezy(y) \;\Rightarrow\; \exists x \;\; Pit(x) \land Adjacent(x, y)$

Causal rule—infer effect from cause
$\forall x, y \;\; Pit(x) \land Adjacent(x, y) \;\Rightarrow\; Breezy(y)$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

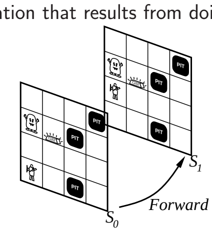
Definition for the $Breezy$ predicate:
$\forall y \;\; Breezy(y) \;\Leftrightarrow\; [\exists x \;\; Pit(x) \land Adjacent(x, y)]$

Keeping track of change

Facts hold in situations, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:
Adds a situation argument to each non-eternal predicate
E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s



Describing actions I

"Effect" axiom—describe changes due to action
$\forall s \;\; AtGold(s) \;\Rightarrow\; Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe **non-changes** due to action
$\forall s \;\; HaveArrow(s) \;\Rightarrow\; HaveArrow(Result(Grab, s))$

Frame problem: find an elegant way to handle non-change
(a) representation—avoid frame axioms
(b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

P true afterwards ⇔ [an action made P true
∨ P true already and no action made P false]

For holding the gold:
??

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

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For holding the gold:

$\forall\, a, s\ \ Holding(Gold, Result(a, s)) \ \Leftrightarrow$
$[(a = Grab \wedge AtGold(s))$
$\vee\, (Holding(Gold, s) \wedge a \neq Release)]$

Making plans

Initial condition in KB:
$At(Agent, [1, 1], S_0)$
$At(Gold, [1, 2], S_0)$

Query: $Ask(KB, \exists\, s\ \ Holding(Gold, s))$
i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists\, p\ \ Holding(Gold, PlanResult(p, S_0)))$ has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:
$\forall\, s\ \ PlanResult([\,], s) = s$
$\forall\, a, p, s\ \ PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

First-order logic:
– objects and relations are semantic primitives
– syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
– conventions for describing actions and change in FOL
– can formulate planning as inference on a situation calculus KB