Transformation From Minkowski to Euclidean Yang-Mills Solutions.

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Some years ago, a correspondence was established by Bernreuther (1) between certain classical solutions of the sourceless Yang-Mills (YM) equations in Euclidean 4-space and solutions in Minkowski space. Aimed at finding new Minkowski-space finite-action solutions from known Euclidean solutions, this method allows one to construct a Minkowski-space YM potential $A_B(x)$ ($B = 0, 1, 2, 3, g_{AB} = \text{diag} (-, +, +, +)$)

$$A_B = e A^a_B t^a / 2i ,$$

$$F_{AB} = e F^{a}_{AB} t^a / 2i ,$$

$$\partial_B F^{BC} + [A_B, F^{BC}] = 0$$

from a Euclidean-space one $a_\mu(y)$ ($\mu = 1, 2, 3, 4, g_{\mu\nu} = \delta_{\mu\nu}$) provided $a_\mu$ is transverse with respect to Euclidean-space co-ordinates $y^\mu$, $y^\mu a_\mu = 0$, and homogeneous of degree $-1$. Subsequently, it was shown (2) that the conditions on $a_\mu$ can be relaxed to the single requirement

$$y^\mu f_{\mu\nu} = 0$$

for field strengths $f_{\mu\nu}$ constructed from the $a_\mu$. However, it was recently shown (3) that if the Euclidean field $a_\mu$ is both transverse and homogeneous of degree $-1$, an important subclass of (2), then any such field which also gives rise to self-dual field strengths is by necessity a pure gauge field, and so is the Minkowski-space field $A_B$ derived from it. This constraint would seem to restrict the applicability of Bernreuther.

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ther's construction, since it is precisely these self-dual solutions which are most easily determined.

For Minkowski-space YM fields, the above theorem does not apply; in any event imposing a form of self-duality (*) yields complex fields only. In this paper we are interested in going from Minkowski space solutions of the YM equations to Euclidean ones, since one main interest in classical YM solutions stems from their contribution to a path integral over actions in Euclidean space.

Although Bernreuther's method established a relationship between Minkowski and Euclidean fields, it is not convenient here, since the restrictions on Minkowski-space fields implied by restrictions on Euclidean ones are complicated. Indeed, one can construct a correspondence inverse to Bernreuther's method by considering the geometry of the mapping used.

Bernreuther's method exploits the conformal covariance of the YM equations, and the isomorphism between the conformal group on Minkowski space and SO_{4,1}. Minkowski-space fields can be written (5) in terms of ones in flat 6-dimensional space so that the conformal transformations are linearized (*). The method of ref. (1) consists, in essence, of considering Euclidean 4-space to be a hypersurface in the (4 + 2)-dimensional space, with both timelike co-ordinates set to zero.

Exactly the same type of procedure can be used to go in the opposite direction. The Euclidean YM equations are covariant under a Euclidean version of the 15-parameter conformal group—this group is isomorphic to SO_{6,1}. Then, taking a section of the (5 + 1)-dimensional space with two spacelike co-ordinates set to zero, one arrives at fields and equations written in flat (3 + 1)-dimensional space. Restrictions on the Aμ in this Minkowski space then lead to YM solutions in Euclidean space.

To transform co-ordinates and fields from Minkowski space xμ to Euclidean space yμ, consider the special case of the embedding formulae in (6) (j = 1, 2, 3):

\[
\begin{align*}
 y_1 &= x_1, \\
y_2^2 &= -(1 + x^2 + 2x^0), \\
x^0 &= -\frac{1}{2}(1 + y^\mu y_\mu),
\end{align*}
\]

\[
\begin{align*}
 a^j(y) &= A^j(x) + x^j A^0(x), \\
a^4(y) &= y^4 A^0(x).
\end{align*}
\]

The field strengths fμν(y) in Euclidean space, constructed from aμ(y) after the fashion of (1) are

\[
\begin{align*}
f_{4j}(y) &= -y_k F^j_{0k}(x), \\
f_{4k}(y) &= F_{4k}(x) + y_k F^0_{0k}(x) - y_{jk} F_{0k}(x).
\end{align*}
\]

The theorem of (2) persists in this inverse construction and one finds that, by requiring the transversality of F_{AB},

\[
x^4 F_{AB} = 0,
\]

(\(5\)) This transformation is formally identical to the embedding of de Sitter group covariant equations in curved 4-space into flat 5-dimensional space (*).
the fields equations (1) reduce to YM equations in Euclidean space, since

\[
\begin{align*}
\partial_\mu f_{\mu k} + [\alpha^k, f_{\mu k}] &= -y_k (\partial^\beta F_{\beta 0} + [A^\beta, F_{\beta 0}]) = 0, \\
\partial_\mu f_{\mu k} + [\alpha^k, f_{\mu k}] &= (\partial^\beta F_{\beta k} + [A^\beta, F_{\beta k}]) - y_k (\partial^\beta F_{\beta 0} - [A^\beta, F_{\beta 0}]) = 0.
\end{align*}
\]

The square of the field strength in Euclidean space is given simply by

\[f_{\mu} f_{\nu} = F_{\mu \nu} F_{\mu \nu},\]

whenever \( F_{\mu \nu} \) satisfies (6).

As an example of the method, consider a \( SU_2 \) potential in the ansatz (7)

\[A^I = i\sigma^{BC} \partial_0 (\ln \varphi),\]

where \( \sigma^{BC} \) are Minkowski space \( SU_2 \) matrices \( \sigma^{0k} = \frac{1}{2} \delta^{0k} \delta_{ab}, \sigma^{ij} = \pm (1/2) \sigma^I \); \( A^8 \) satisfies the YM equations if \( \varphi \) obeys \( \Box \varphi + \lambda \varphi^8 = 0 \) in Minkowski space. The simplest nontrivial such \( \varphi \) fulfilling the requirement (6) is \( \varphi = (\lambda x^I) \), yielding

\[A^8 = -i\sigma^{BC} x_C / x^2.\]

Here, the denominator is

\[x^2 = x^2 - x^0 = y^2 - \frac{1}{4}(1 + y^2 + y_4^2)^2 = -\frac{1}{4}(1 - y^2)^2 + y_4^2 (y_4^2 + 2y^2 + 2).\]

Transforming to Euclidean space \( y^I \) via (3) and (4), one arrives at a solution \( a^I(y) \):

\[
\begin{align*}
\alpha^I(y) &= \mp \frac{y^I}{2x^2} \sigma^I \cdot y, \\
\alpha^I(y) &= \frac{1}{2x^2} \left\{ -i\epsilon^{ijk} \sigma_a y_b \pm \frac{1}{2}(1 + y^2 + y_4^2) \alpha^I \mp y^I \sigma^I \cdot y \right\}.
\end{align*}
\]

This solution is highly reminiscent of the real version (1,4,8) of the two-meron Euclidean solution (9) of de Alfaro, Fubini and Furlan (DFF), and has an action density

\[L(y) = -\frac{1}{2} \text{Tr} f_{\mu \nu}(y) f_{\mu \nu}(y) = -\frac{3}{2e^2 (x^2)^2}\]

in the same general form.

However \( a^I \) is complex, and has a different singularity structure. In the Euclidean DFF solution, the denominator is proportional to \( (1 - y_4^2) + y^2 (y^2 + 2y_4^2 + 2) \), which vanishes at \( y = 0, y_4 = \pm 1 \). In contrast, in (13) the denominator (11) vanishes on the 2-sphere \( y^2 = 1, y_4 = 0 \). In this sense it resembles the extended object solution of Forgács, Horváth and Palla (10). Unlike the latter solution, the solution (12) does not belong to the Euclidean version of the ansatz (8), although it is derived from a Minkowski potential (10) which does satisfy (9).
