On the Scale Dimension of Conformally Covariant Spinor Fields

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Received December 18, 1972

The spinor equation of Barut and Haugen is shown to admit no solutions having a definite scale dimension in Minkowski space.

On montre que l'équation spinorielle de Barut et Haugen n'admet aucune solution possédant une dimension d'échelle définie dans l'espace de Minkowski.


In an earlier paper (Drew 1972), the isomorphism between the conformal group on Minkowski space and the group of rotations O(4, 2) was used to investigate conformally covariant spinor field equations by considering rotationally invariant action principles in a six-dimensional flat space. As has been pointed out in a helpful criticism by Barut and Haugen (1973), the author used a line of reasoning which was, in part, incorrect, in eliminating certain terms from a Lagrangian density comprising all rotationally invariant expressions bilinear in an eight-component spinor field \( \chi \) and its first derivatives. The field \( \chi \) was defined on the manifold of coordinates \( q^A, A = 1-6 \), with metric \( \delta^{AB} = \text{diag}(-1,-1,-1,1,1,1) \). The purpose of this paper is to demonstrate that the spinor equation proposed by Barut and Haugen (1972a, 1973) responding to two distinct types of two-component fields \( \psi \) in Minkowski space, each governed by the Weyl equation

\[ -ir^j \partial \psi / \partial y^j = 0, \quad j = 1-4 \]

and each having the physically required scale dimension \( l = -3/2 \) (Drew 1972).

The coordinates \( \eta^A \) are defined in terms of the Minkowski coordinates \( y^j \) and two auxiliary variables \( \kappa \) and \( L \) by the relations

\[ \eta^j = \kappa y^j, \quad \kappa = \eta^5 - \eta^6, \quad L = \eta^4 \eta^6 \]

so that \( y^j, \kappa, \) and \( L \) form a set of six independent variables. As is clear from the work of Mack and Salam (1969), on the hyperquadric \( L = 0 \) the scale dimension operator \( l \) can be identified as

\[ l = (\kappa \partial_\kappa + i\sigma_{65}) \hat{\chi}, \quad \partial_\kappa = \partial / \partial \kappa \]

where

\[ \hat{\chi} = U \chi, \quad U = \exp \left(-iy^i(\sigma_{6j} + \sigma_{5j})\right) \]

This result arises when one considers the action on \( \chi \) of the differential generator of rotations \( m_{65} \), where

\[ m_{65} = i(\eta_6 \partial_5 - \eta_5 \partial_6) = i(y^6 \partial_j - \kappa \partial_\kappa) \]

By considering O(4, 2) symmetry with \( L \neq 0 \), one deals with a group of transformations on space-time which properly contains the group of conformal transformations. This group maps “spheres” in four dimensions into other spheres such that the angle between neighboring spheres is preserved; the mappings of null spheres (\( L = 0 \)) constitute the conformal transformation group on Minkowski space (Ingraham 1971). However, because rotations in the 6–5 plane in six dimensions for \( L \neq 0 \) induce the same action on the Minkowski variables \( y^j \) as for the case \( L = 0 \),
L does not appear explicitly in the operator \( \eta_{65} + \sigma_{65} \), which contains both extrinsic and intrinsic parts of the generator of dilations in Minkowski space; the action of the dilation subgroup induced on fields \( \hat{\chi} \) is the same whether or not \( L \) is zero. For this reason, the identification \cite{Drew} holds for all \( L \).

If one works in a representation in which \( \sigma_{65} \) is diagonal, then the requirement that \( \hat{\chi} \) be an eigenfunction of \( \lambda \) amounts, on account of \cite{Drew}, to the requirement that it be an eigenfunction of the operator \( \kappa \partial_x \). Since this operator commutes with \( U \), one can write this requirement in terms of the field \( \chi \) as

\[ \kappa \partial_x \chi = n_0 \chi \]

denoting the eigenvalue to which \( \chi \) corresponds by \( n_0 \). Defining a new variable \( \psi \) by

\[ \psi = \kappa^{-\frac{n}{2}} \hat{\chi} \]

then if \cite{Drew} is satisfied, and if \( n \) is chosen equal to \( n_0 \), the field \( \psi \) does not depend upon \( \kappa \), and for this reason \( \psi \) is identified with the "physical" field in Minkowski space. Care must be taken not to look upon it as an operator identity for \( L \), because when acting on a field \( \psi \) the scale dimension operator \( I \) must be identified as

\[ I \psi = (n + \kappa \partial_x + i\sigma_{65}) \psi \]

This has the meaning that under an infinitesimal scale transformation

\[ y' = (1 + \sigma)y \]

the field \( \psi \) undergoes the classical transformation law

\[ \psi'(y') = [1 + \sigma(n + \kappa \partial_x + i\sigma_{65})] \psi(y) \]

In order that one can assign a definite scale dimension to the components of \( \psi \), one must require equations governing fields \( \chi(n) \) in six-dimensional space to admit solutions satisfying the condition \cite{Drew}. On the null hyperquadric \( L = 0 \), one has

\[ \eta^4 \partial_4 = \kappa \partial_x, \quad L = 0 \]

so that the eigenvalue \( n_0 \) can be identified with the degree of homogeneity of \( \chi \) on \( L = 0 \),

\[ \eta^4 \partial_4 \chi = n_0 \chi, \quad L = 0 \]

Fields \( \chi \), then, must be homogeneous functions when restricted to the null surface \( L = 0 \), in order that they be eigenfunctions of \( I \). For the case \( L \neq 0 \), however, it is not necessary to assume that \( \chi \) is homogeneous in order to have definite values for \( I \), and one may use the identities

\[ \kappa \partial_x = \eta^4 \partial_4 - 2L \partial_L, \quad \partial_L = \partial/\partial L \]

\[ 2 \kappa \partial_L = \partial_5 + \partial_6 \]

to determine whether or not \cite{Drew} can be satisfied.

Consider a solution \( \chi \) of \cite{Drew}. It is permissible to apply the operator \( \eta^4 \partial_4 \) from the left to the field equation, so that one has

\[ 0 = \eta^4 \partial_4 [i \beta^4 \partial_x \chi + M \chi] \]

\[ = i \beta^4 \partial_4 [\eta^4 \partial_4 \chi] + M \eta^8 \partial_8 \chi - i \beta^4 \partial_4 \chi \]

If one insists that \( \chi \) be homogeneous of degree \( n_0 \) on \( L = 0 \), so that \cite{Drew} is satisfied, then on account of \cite{Barut}, \cite{Haugen}, \cite{Drew}, and \cite{Drew}, eq. 19 reduces to

\[ M \chi = 0, \quad \text{for } L = 0 \]

Taking \( M \) to be a real positive constant, the identities \cite{Drew} and \cite{Drew} may be employed to show that

\[ M \chi = 0, \quad \text{for } L \neq 0 \]

provided both eqs. 1 and 10 are satisfied, so that for \( M \neq 0 \) no nonvanishing solutions of \cite{Drew} exist which have a specific scale dimension. Introduction of a conformally invariant mass term into field equations in six-dimensional space is thus seen to preclude the possibility of finding spinor fields in Minkowski space satisfying conformally covariant free field equations and having a unique scale dimension. It is important to realize that for this purpose one must diagonalize both parts of \( I \), and not just the matrix \( i \sigma_{65} \) as Barut and Haugen \cite{Barut} have done. Furthermore, contrary to an earlier statement \cite{Drew}, the author no longer believes it possible to introduce a conformally invariant mass \( M \) into six-dimensional equations for scalars or vectors, where \( M \) is a number, provided one demands that there exist solutions of the field equations having a definite scale dimension. This property of conformally covariant field equations is an essential requirement, in a classical theory of free fields, for giving solutions a physical interpretation in terms of probability currents constructed from the fields. Indeed, the fact that the generator of dilations commutes with the generators of Lorentz transformations means that, as a result of Schur's Lemma, for any representation...
of the conformal group which is also an irreducible representation of the Lorentz group, the finite dimensional part of the generator of dilations must be a constant multiple of the unit matrix.

If one now applies the operator \( \eta^A \partial_A \) from the left to \([3]\), one finds

\[
0 = \eta^c \partial_c \left[ \sigma^{AB} \eta_A \partial_B \chi + i \lambda \chi \right] = \sigma^{AB} \eta_A \partial_B [\eta^c \partial_c \chi] + i \lambda \eta^c \partial_c \chi
\]

Demanding that the condition \([10]\) be satisfied, on \( L = 0 \) the right-hand side of this equation vanishes identically by virtue of the field equation \(3\), showing that \([3]\) admits solutions satisfying \([10]\). For the case \( L \neq 0 \), one may use the identities \([17]\) and \([18]\) to show that

\[
\kappa \partial_c [\sigma^{AB} \eta_A \partial_B \chi + i \lambda \chi] = 0
\]

provided both the field equation \(3\) and the condition \([10]\) are satisfied.

An alternative approach (see, e.g., Fulton, Rohrlich, and Witten 1962) is provided by ascribing a scale dimension \( l_m = -1 \) to the mass \( m \), so that as a result of the dilation \([13]\) in Minkowski space, one has

\[
m' = (1 - \sigma)m
\]

This transformation law must be regarded as a separate postulate. Then considering four-component spinor fields \( \psi \), transforming under dilations with the scale dimension \( l = -3/2 \),

\[
\psi'(y') = (1 + \sigma)\psi(y)
\]

the Dirac equation in Minkowski space is conformally covariant, provided \([24]\) holds. Comparing \([25]\) with \([14]\), one sees that this situation cannot prevail for the four-component projection \( \psi \) of a field derived from \( \chi(\eta) \), satisfying

\[
(-i \gamma^j \partial_j + \kappa M)\psi = 0
\]

with the postulate \([24]\) replaced by the transformation law

\[
M' = M(1 - \sigma) M
\]

since in this case \( \psi \) cannot satisfy both eq. 26 and the condition

\[
(n + \kappa \partial_\kappa + i \sigma_{\kappa \beta})\psi = -3/2 \psi
\]

In conclusion, in the light of the useful criticism of Barut and Haugen, the claim that \([3]\) is the only conformally covariant spinor equation, which was made in the author's original paper, should be restated to read that the Weyl equation is the only conformally covariant spinor equation, derivable from the O(4, 2) formalism, having solutions which correspond to fields in Minkowski space having a definite scale dimension, whereas no solution of the equation of Barut and Haugen has this property.

**Acknowledgments**

The author is grateful to Dr. A. O. Barut and Dr. R. B. Haugen for their courtesy in providing him with a copy of their paper prior to its publication. The author is indebted to Professor F. A. Kaempf for advice and criticism, and to Mr. R. J. Esch and Dr. P. M. McCarthy for helpful conversations. The financial assistance of the National Research Council of Canada in the form of a Postgraduate Scholarship is gratefully acknowledged.


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