On Conformally Covariant Spinor Field Equations

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Conformally covariant equations for free spinor fields are determined uniquely by carrying out a descent to Minkowski space from the most general first-order rotationally covariant spinor equations in a six-dimensional flat space. It is found that the introduction of the concept of the “conformally invariant mass” is not possible for spinor fields even if the fields are defined not only on the null hyperquadric but over the entire manifold of coordinates in six-dimensional space.

I. Introduction

The covariance of field equations governing massless fields in Minkowski space under the conformal group has been known for a long time. It was first established for Maxwell’s vacuum field equations by Cunningham (1910) and Bateman (1910) and extended by McLennan (1956) and Bludman (1957) to other types of massless fields, such as the free neutrino field governed by the Weyl equations. Recently (see for example, Barut and Haugen 1971) attempts have been made to enlarge the applicability of this symmetry group to massive fields through exploitation of the isomorphism between the conformal group and the group of rotations in a six-dimensional flat space with metric

\[ \delta^{AB} = \text{diag} (-1,-1,-1,+1,-1,+1) \]

and introduction of the concept of the “conformally invariant mass”. This is certainly formally possible for scalar and vector field equations (see also Ingraham 1971) provided the fields are defined over the entire manifold \( \eta^A \) of coordinates in six-dimensional space, without the customary restriction to the hyperquadric \( L = \eta_A \eta^A = 0 \). However, in the case of spinor fields (for spin \( \frac{1}{2} \), i.e. bispinors of the first rank) attempts at introducing a “conformally invariant mass” term into the field equations encounter a characteristic difficulty, first noticed by Dirac (1936) for the case \( L = 0 \). The purpose of the present work is to demonstrate that this difficulty persists even if one defines the spinor field over the entire manifold, and that one cannot construct conformally covariant spinor field equations of the type envisaged by Barut and Haugen.

Section II contains a summary of the formalism, due to Murai (1958), which is used to describe spinor fields \( \chi(\eta) \) in six-dimensional space. This is followed in Sect. III by the construction of the most general Lagrangian density leading to field equations of not higher than first order which are rotationally covariant in six-dimensional space. In Sect. IV the well-known descent procedure of Mack and Salam (1969) is used to show that the resulting field equations in six-dimensional space amount to Weyl equations for massless fields, if one insists that the “physical” components of the spinors have in Minkowski space the physically required scale dimension \( f = -3/2 \).

II. Six-Dimensional Formalism for Spinors

The coordinates of the six-dimensional space are defined in terms of the Minkowski coordinates \( y^A \) by

\[ \eta^A \equiv \kappa y^A, \quad \kappa \equiv \eta^0 - \eta^3, \quad L = \eta^A \eta_A \]

For a field \( \chi(\eta) \equiv \chi(y, \kappa, L) \), the differentiation \( \partial_A = \partial/\partial \eta^A \) may be rewritten in terms of \( y, \kappa \), and \( L \) so that

\[ \eta^A \partial_A \chi = (\kappa \partial_\kappa + 2L \partial_L) \chi' \]

It is found that although \( m_{AB} \equiv i(\eta_A \partial_B - \eta_B \partial_A) \)

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1. Capital and lower case indices take the values 1–6 and 1–4 respectively unless stated otherwise.
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contain \(L\), they do not contain \(\partial_i\). However, this does not necessitate restricting attention to fields defined only on the hyperquadric \(L = 0\). The generators of conformal transformations can be recombined into operators \(J_{AB} = m_{AB} + S_{AB}\) which satisfy the commutation relations of rotations in six-dimensional space. The spinor representation of \(S_{AB}\) admitting reflections in six dimensions is eight dimensional (Brauer and Weyl 1935). The spin operators \(S_{AB} = \sigma_{AB}\) for spinors can be expressed in terms of six matrices satisfying a Clifford algebra (Murai 1958).

\[
\{\beta^A, \beta^B\} = 2\delta^{AB}, \quad \sigma^{AB} = (i/4)[\beta^A, \beta^B]
\]

The matrices \(\beta^A\) transform as a six-vector

\[
[\beta^A, \sigma^{BC}] = i(\delta^{AB}\beta^C - \delta^{AC}\beta^B)
\]

and have the property

\[
(\beta^A)^* = \delta^{A4}\beta^4 \quad \text{(Do not sum over } A)\]

The product matrix is

\[
\beta^7 = i\beta^1\beta^2\beta^3\beta^4\beta^5\beta^6
\]

\[
[\beta^4, \beta^7] = 0, \quad (\beta^7)^2 = 1
\]

In the representation of Kastrup (1966), \(\beta^7 = I \otimes \sigma^4\), where \(\sigma^1, 2, 3\) are the Pauli matrices. The matrix of adjunction is

\[
A = -i\beta^4\beta^6, \quad \bar{\chi} = \chi^A A
\]

so that

\[
A\beta^4 A = - (\beta^4)^*
\]

In the following, a decomposition of \(\bar{\chi}\) will be used

\[
\bar{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}
\]

where \(\chi_1\) and \(\chi_2\) are the upper and lower four components of \(\chi\). In this space \(I \pm \beta^7\) acts as a projection operator which selects the upper and lower components of \(\chi\).

III. Uniqueness of Conformally Invariant Lagrangian for a Spinor Field

Sixty-four linearly independent matrices can be constructed by forming products of the \(\beta\) matrices. These are

\[
I; \quad \beta^A; \quad \sigma^{AB}; \quad \beta^4\beta^B\beta^C, \quad A \neq B \neq C;
\]

\[
\beta^7\sigma^{AB}; \quad \beta^7\beta^A; \quad \beta^7
\]

Calling these matrices \(\Gamma^i, i = 1, 2, \ldots, 64\), the number of different \(\Gamma^i\) of each type is 1, 6, 15, 20, 15, 6, and 1 respectively. The first four types of bilinear form \(\chi(q)\Gamma^i\chi(q)\) constructed from the \(\Gamma^i\) transform under rotations as a scalar, vector, antisymmetric second-rank tensor, and completely antisymmetric third-rank tensor, while \(\beta^7\) transforms as a pseudoscalar.

In order to derive linear rotationally covariant field equations in six-dimensional space, of not higher than first order in the derivatives of the field, the Lagrangian density must be a scalar or pseudoscalar function \(L = \mathcal{L}(\chi, \partial_\alpha \chi)\), which can be an explicit function of \(\eta\) since translational invariance is not demanded. One can form ten linearly independent scalars or pseudoscalars bilinear in \(\chi\) and its first derivatives.

\[
I_1 = \bar{\chi}\chi; \quad I_2 = \eta_{AB}\beta^A\chi; \quad I_3 = \eta^{45}\bar{\chi}\partial_\alpha \chi
\]

\[
I_4 = \bar{\chi}\beta^A\partial_\alpha \chi; \quad I_5 = \eta_{AB}\sigma^{AB}\partial_\alpha \chi
\]

\[
I_6 = \bar{\chi}\beta^7\chi; \quad I_7 = \eta_{AB}\beta^7\beta^A\chi
\]

\[
I_8 = \eta^4\bar{\chi}\beta^7\partial_\alpha \chi
\]

\[
I_9 = \bar{\chi}\beta^7\beta^A\partial_\alpha \chi; \quad I_{10} = \eta_{AB}\beta^7\sigma^{AB}\partial_\alpha \chi
\]

The most general Lagrangian density is the linear combination

\[
\mathcal{L} = \sum_{i=1}^{10} a_i I_i
\]

where the arbitrary coefficients \(a_i\) can be functions of \(L\) and still guarantee a rotationally invariant action. In the field equations derived from the corresponding action principle each pseudoscalar term equals a corresponding scalar term multiplied by \(\beta^7\) and a constant. This amounts to being able to set the upper and lower four components separately equal to zero in the field equations derived from just the scalar terms of the Lagrangian, because multiplying the entire field equation by \(\beta^7\) and adding or subtracting leaves the factor \((\eta \pm \beta^7)\) outside the scalar equation. Therefore \(I_6, I_7, I_8, I_{10}\) do not yield any terms in the field equations that are not already contained in the field equations derived from \(I_1, I_2, I_3, I_5, I_{10}\).

It will now be shown that of the remaining scalar terms only two do not differ by a divergence. From the definition of adjunction it follows that \(I_3\) and \(I_4\) have the reality properties

\[
(\eta^4\bar{\chi}\partial_\alpha \chi)^* = \eta^4\bar{\chi}\partial_\alpha \chi
\]

\[
(\bar{\chi}\beta^A\partial_\alpha \chi)^* = -\bar{\chi}\beta^A\partial_\alpha \chi
\]
Consider the divergence

\[ \partial_A (\eta^A \vec{\chi}) = \eta^A \partial_A \vec{\chi} + \eta^A \partial_A \vec{\chi} + 6 \vec{\chi} \]

But

\[ (\eta^A \vec{\chi}) = \partial_A \eta^A \vec{\chi} \]

so that by eq. 11.1,

\[ \partial_A (\eta^A \vec{\chi}) = 2I_3 + 6L \]

and \( I_d \) differs from \( I_1 \) by a divergence only and may therefore be omitted. Considering the divergence

\[ \partial_A (\eta^A \vec{\chi}) = 2\eta \partial_A \vec{\chi} + L \partial_A (\vec{\chi}^A \chi) \]

we use

\[ (\vec{\chi} \vec{\chi} \partial_A \chi) = \partial (\vec{\chi} \vec{\chi} \partial_A \chi) \]

so that by eq. 11.2 \( I_4 \) is shown to be itself a divergence

\[ \partial_A (\vec{\chi} \vec{\chi} \partial_A \chi) = 2L \]

allowing it to be omitted.\(^2\) One also finds from eq. 15

\[ \partial_A (L \vec{\chi} \vec{\chi} \partial_A \chi) = 2L_2 \]

Since the coefficients \( \alpha \) are allowed to be functions of \( L, I_2 \) can also be omitted from the Lagrangian. One is then left with \( I_1 \) and \( I_3 \), and the resulting field equations can be written in the form

\[ \sigma^{-1} \eta \alpha \partial \vec{\chi} + i \lambda \vec{\chi} = 0 \]

where \( \lambda \) is a real number; this is the equation given by Hepner (1962) and Dirac (1936). The equation can also be written as

\[ \beta \lambda \beta \alpha \sigma^{-1} \eta \alpha \partial \vec{\chi} + 4i \lambda \vec{\chi} = 0 \]

The adjoint equation is

\[ \eta \alpha \partial \vec{\chi} + i \lambda \vec{\chi} = 0 \]

and using eq. 5, this is just

\[ \eta \alpha \partial \vec{\chi} + i \lambda \vec{\chi} = 0 \]

\[ \eta \alpha \partial \vec{\chi} \sigma \alpha = \pm \lambda \vec{\chi} \]

**IV. Conformally Covariant Spinor Equations in Minkowski Space**

The descent to Minkowski space is carried out by application of a unitary operator \( U \) which eliminates the intrinsic part \( \pi^k = S^a - S^a \) of

\[ \text{the momentum operator} \quad P^k, \]

\[ U = \exp (-iy \gamma^k) \]

Then the scale dimension \( l \) can be identified as

\[ l = \kappa \partial \chi + iS_{65} \]

One may require the field \( \chi(\eta) \) to be homogeneous of degree \( n \) on the hyperquadric \( L = 0 \). Then the field \( \kappa^{-\gamma} \chi(y, \kappa, 0) \) depends only on \( y \). In addition, Mack and Salam (1969) apply a projection operator \( E \) to \( U \chi \) to select the four "physical" components belonging to the highest eigenvalue of \( S_{65} \), and the resulting equations in Minkowski space are those for the field \( \kappa^{-\gamma} EU \chi \). Following a line of reasoning of Kaempfier (1972), as a second possibility one is also free to consider as "physical" those components selected by the projector \( E \equiv I - L \), and the equations satisfied by the field \( \kappa^{-\gamma} EU \chi \), provided that the resulting spinors in Minkowski space have in this case as well the required scale dimension \( l = -3/2 \).

For eight-spinors \( S_{65} = \frac{1}{2} \text{diag} (1,1,-1,-1,-1,-1,1,1) \); if \( l \) is some constant number for each component of \( \chi(\eta) \), then eq. 23 implies that \( \kappa \partial \chi \) is a constant. For each component of \( \chi(\eta) \), then eq. 23 implies that \( \kappa \partial \chi \) is a constant. Hence, for all \( \eta \),

\[ l = n + iS_{65} \]

Thus it can be seen that it is in general not possible to identify the scale dimension \( l \) with the degree of homogeneity \( n \), as has been proposed by Barut and Haugen (1971).

Equation 19 can now be written as a set of four equations for two-component spinors \( \phi_{1,2} \) and \( \Phi_{1,2} \) in Minkowski space, consistent with restrictions placed on \( l \) by the requirement of translation invariance (see the Appendix). In terms of the upper and lower four components \( \psi_1 \) and \( \psi_2 \) of the field \( \psi \equiv \kappa^{-\gamma} EU \chi \) the two-spinors

\[ \phi_{1,2} = E \psi_{1,2}; \quad \Phi_{1,2} = E \psi_{1,2} \]

with scale dimension \( l \) and \( l \) respectively, satisfy the Weyl equation

\[ \gamma^l \partial \phi_{1,2} = 0 \]

\[ L \gamma^l \partial \Phi_{1,2} = 0 \]

for any \( L \neq 0 \), as well as the two auxiliary conditions

\[ (-n + 2\lambda) \phi_{1,2} = 0 \]

\[ (n + 2\lambda + 4) \Phi_{1,2} = 0 \]
Combining eqs. 24 and 27 we obtain

\[ \ell \varphi = (2A + \frac{1}{2}) \varphi \]

\[ \ell \hat{\varphi} = (-2\lambda - 4) \hat{\varphi} \]

If \( \varphi \) is identified with a "physical" field, \( \lambda \) and \( n \) are fixed through eqs. 27.1 and 28.1 by the requirement that \( \ell \) have the "physical value" \(-3/2\)

\[ \lambda = -1, \quad n = -2 \]

giving the field \( \hat{\varphi} \) the unphysical scale dimension \( l = -5/2 \). Substituting \( \lambda \) and \( n \), eq. 27.2 is then identically satisfied. If, alternatively, \( \hat{\varphi} \) is identified with a "physical" field, then the requirement \( \ell = -3/2 \) yields through eqs. 27.2 and 28.2 the values

\[ \lambda = -3/2, \quad n = -1 \]

This gives \( \varphi \) the unphysical scale dimension \( l = -5/2 \), but eq. 27.1 cannot be satisfied unless \( \varphi = 0 \).

Hence, in either case a "conformally invariant mass" term cannot be introduced into free field equations for spinors.

Appendix: Descent Using Translational Invariance

An explicit construction of \( S_{AB} \) will be made using the Weyl representation of the Dirac matrices in which \( \gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \text{diag} (+, +, +, -) \);

\[ \beta^k = \gamma^k \otimes \sigma^1; \quad \beta^5 = \gamma^5 \otimes \sigma^1; \quad \beta^6 = I \otimes \sigma^2 \]

Rewriting eq. 19 in terms of \( \chi' \),

\[ \sigma^k y_j \partial_j \chi' + \sigma^5 \left[ \frac{1}{2} \left( -\frac{L}{\kappa^2} + 1 + y^2 \right) \partial_j - y_j \partial_k + y_k \partial_j \right] \chi' + \sigma^6 \left[ \frac{1}{2} \left( \frac{L}{\kappa^2} + 1 - y^2 \right) \partial_j + y_j \partial_k - y_k \partial_j \right] \chi' + \sigma^6 \gamma^k \partial_k \chi' + i \kappa' \chi' = 0 \]

Because

\[ (\sigma^6 - \sigma^5)(\sigma^6 - \sigma^5) = 0 \]

\( \pi' \) is nilpotent to the power two. Using

\[ \partial_i (U^{-1}) = i \pi' \delta_{jk} \]

the field equations can be written in terms of \( U \gamma' \). Since the equations in Minkowski space are to be conformally covariant, translational invariance can be exploited to evaluate the equations at \( y = 0 \). Using eq. 33, the equation for the function \( \psi \) when evaluated at \( y = 0 \) is found to be

\[ \frac{1}{2} (\sigma^6 + \sigma^5) \partial_j \psi + \frac{1}{2} (\sigma^6 - \sigma^5) (L/\kappa^2) \partial_j \psi - \sigma^6 \gamma^k \partial_k \psi + (i/2) (\sigma^6 + \sigma^5) (\sigma^6 - \sigma^5) \delta_k \gamma^k \psi + i \kappa' \partial_j \psi = 0 \]

Explicitly,

\[ \sigma^6 - \sigma^5 = -\frac{i}{2} \begin{pmatrix} (y^5 + i) & 0 \\ 0 & (y^5 - i) \end{pmatrix}, \quad \sigma^6 + \sigma^5 = \frac{i}{2} \begin{pmatrix} (y^5 - i) & 0 \\ 0 & (y^5 + i) \end{pmatrix} \]

\[ \sigma^6 = \frac{1}{2} \begin{pmatrix} y^5 & 0 \\ 0 & -y^5 \end{pmatrix} \]

It is seen that eq. 35 reduces to two equations for \( \psi_1 \) and \( \psi_2 \):

\[ [(i/2) (y^5 - i) \gamma^j \partial_j - (iL/\kappa^2) (y^5 + i) \gamma^j \partial_j + (\gamma^5 + 2i\kappa) - 2(y^5 - i)] \psi_1 = 0 \]

\[ [(i/2) (y^5 + i) \gamma^j \partial_j - (iL/\kappa^2) (y^5 - i) \gamma^j \partial_j + (\gamma^5 + 2i\kappa) + 2(y^5 + i)] \psi_2 = 0 \]

Multiplying these equations with the projection operators \((y^5 \pm i)\), they decompose further into the following equations in spinors having only two nonzero components:

\[ i(-n + 2\lambda) (y^5 + i) \psi_1 - (L/\kappa^2) \gamma^j \partial_j (y^5 - i) \psi_1 = 0 \]
\[ 38.2 \]
\[ i(-n + 2\lambda)(\gamma^5 - i)\psi_2 + (L/\kappa^2)\gamma^i \partial \partial_i [(\gamma^5 + i)\psi_2] = 0 \]

\[ 38.3 \]
\[ i(n + 2\lambda + 4)(\gamma^5 - i)\psi_1 - \gamma^i \partial \partial_i [(\gamma^5 + i)\psi_1] = 0 \]

\[ 38.4 \]
\[ i(n + 2\lambda + 4)(\gamma^5 + i)\psi_2 + \gamma^j \partial \partial_j [(\gamma^5 - i)\psi_2] = 0 \]

Introducing the projection operators

\[ E = -\frac{i}{2} \begin{pmatrix} (\gamma^5 + i) & 0 \\ 0 & -(\gamma^5 - i) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \]

\[ E = I - E = \frac{i}{2} \begin{pmatrix} (\gamma^5 - i) & 0 \\ 0 & -(\gamma^5 + i) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \]

eqs. 26 and 27 follow immediately.

**Note added in proof:** The author thanks Mr. R. J. Esch for pointing out the additional invariant \( \frac{1}{2} \eta^{\alpha\beta} \eta^{\gamma\delta} \partial_\alpha \partial_\beta \) which differs from \( I_2 \) only by a divergence.

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