Conservation Laws in de Sitter Space from
Action Principles in Five-Space

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By recognizing the resemblance of the de Sitter group algebra to that of the conformal group, the method by which manifestly conformally covariant field equations in six-dimensional space are rewritten in Minkowski space is adapted to fields in flat five-dimensional space, the embedding space of de Sitter space. A quantum action principle based solely on rotational invariance in five-dimensional space is devised, and the resulting commutation relations are shown to correspond to the correct ones in curved four-space. As well as recovering the ten conservation laws associated with de Sitter group invariance, the five extra conservation laws present whenever conformal symmetry holds are determined directly in five-space. The derivation is found to be complicated by a new feature—the Lagrangian density does not transform as a field either for special conformal transformations or for dilations; this is true only for the former transformations in flat space.

INTRODUCTION

One of the simplest curved spaces one can consider is a space of constant curvature, i.e., a de Sitter space [1]. Of these spaces, there are two fundamentally different kinds—those with either closed or open timelike geodesics. The latter will be treated here, and use will be made of the fact that this type of space can be embedded in a (4 + 1)-dimensional flat space.

Despite the fact that this embedding has been well known for a long time [2], an action principle and a resulting prescription for field quantization have never been devised in the five-dimensional space, although there have been tentative efforts in this direction [3], inspired by the pioneering work of Dirac [4]. This lack seems curious in light of the fact that relatively few spaces afford the opportunity for constructing quantum fields in a curved space background by using a flat-space formalism with dimensionality lower than ten [5].

Quantization of fields directly in the five-space has an interest of its own regardless of the fact that quantization in the de Sitter background can be carried out in the framework of the four-dimensional Riemannian spacetime [6–9]. Not only are calculations of symmetry operators greatly simplified by using a formalism that is manifestly invariant under the de Sitter group, but also, presumably, one should be
able to carry out scattering calculations in the flat five-space after the fashion of Adler's work on massless quantum electrodynamics in Euclidean space [10].

In this investigation, action principles and commutation relations for free fields are formulated directly in the five-dimensional space. The method of descent to the curved four-space that is employed here, by which fields in five dimensions are written in terms of de Sitter space variables, is the extension off the null hyperquadric in higher-dimensional space of the procedure of Dirac [11] and Mack and Salam [12] for writing conformally covariant fields in Minkowski space as rotationally covariant fields in a flat (4 + 2)-dimensional space. By tailoring their method to the five-dimensional situation, this procedure allows one to recover, after descent to the de Sitter space, the ten quantum field theory generators associated with the de Sitter group as well as the five additional quantities that are conserved in the case of vanishing mass. These extra conservation laws result from the generalization to curved spacetime of the conformal group of transformations on Minkowski space. It is found that, just as in the case of massless fields in Minkowski space, the variation of the Lagrangian density is not necessarily that of a field variable for special conformal transformations (see [13]). However, because dilations and special conformal transformations are unified here, this feature is carried over to dilations as well, in contrast to the situation in Minkowski space. This makes constructing the generators of these transformations for arbitrary spin a more difficult matter than in the flat space limit, although it is still possible to determine the conserved currents for each free field separately.

In Section 1 the various coordinate systems needed to study the de Sitter space are outlined. In terms of the parameter $R$, Minkowski space results from passage to the limit $R \to \infty$. In Section 2, the descent to four-dimensional space is treated. By recognizing that the de Sitter group algebra can be rearranged so as to resemble that of the conformal group in Minkowski space, it is possible to find a transformation $T$ in the index space of a field $\chi$ in five dimensions such that the representation of the de Sitter group acting on $T\chi$ contains only Lorentz group spin matrices. Field equations in five-space can thus be compared directly with equations derived from those in Minkowski space by replacement of ordinary derivatives by covariant ones.

It is found in Section 3 that in order to be able to construct a real Lagrangian density for a spinor field without introducing a coordinate-dependent definition of an adjoint spinor in five-dimensional space, the original form of Dirac's equation [4] in five-space must be modified. Minimal coupling in five-dimensional space introduces a five-vector field; the form of the interaction in curved four-space is shown not to contain the fifth component of the vector potential, notwithstanding the fact that here this component is not automatically eliminated by the imposition of a transversality condition in five-space. The five-space vector equations analogous to the equation for scalars are shown to reduce to the covariant version of the Proca equations in the Lorentz gauge, when these equations are written in terms of the components of the vector with respect to the local orthonormal tetrad.

In Section 4, an action principle in five-dimensional space is devised which is appropriate to situations in which rotational invariance is the sole guiding physical
principle. This is accomplished by means of replacing Gauss' theorem by Stokes' theorem in five-space. By interpreting the conserved quantities derived by varying the boundary of the action integral as quantum field theory generators, the (anti-)commutation relations satisfied by fields in five-space are established as consistency conditions. These reduce to the correct (anti-)commutation relations for equal timelike variables when the descent to curved four-space is carried out. The ten conservation laws derived from rotational invariance in five-space become the conservation laws associated with invariance under the de Sitter group in four-space.

The procedure is extended in Section 5 to include conformal transformations in de Sitter space. It is demonstrated that the additional conservation laws due to conformal symmetry can be grasped in a unified manner by considering the transformation behavior of fields in five-space. The complications associated with formulating conservation laws for fields of arbitrary spin for the special conformal transformations in flat space are shown to persist in curved space, and as well these difficulties are shown to extend also to the analog of dilations in de Sitter space.

1. COORDINATE SYSTEMS

In terms of "conformal" coordinates $x^\mu (\mu = 1, 2, 3, 4)$, the de Sitter line element takes the form [14]

$$ds^2 = \varphi^2 \delta_{\mu\nu} dx^\mu dx^\nu,$$

(1.1)

where $\delta_{\mu\nu}$ is the Minkowski metric $\delta_{\mu\nu} = \text{diag}(-1, -1, -1, +1)$ and

$$\varphi = 1 - x^2/4R^2, \quad x^2 = \delta_{\mu\nu} x^\mu x^\nu,$$

$R$ being the radius of the universe. This form of the metric shows clearly that the de Sitter space is conformally flat. The Minkowski space limit is given by $R \to \infty$. The five-space formulation is achieved by embedding the above space in a pseudo-Euclidean space endowed with the metric $\delta_{AB} = \text{diag}(-1, -1, -1, +1, -1)$ in terms of Cartesian coordinates $\eta^A (A = 1, 2, 3, 4, 5)$, by means of the mapping

$$\eta^\mu = \varphi x^\mu, \quad \eta^5 = R(1 - 2\varphi).$$

(1.2a)

This takes the five variables $(x^\mu, R)$ into the single-sheet hyperboloid of revolution

$$\eta^A \eta_A = -R^2.$$

(1.2b)

Then for constant $R$ the line element has the form

$$ds^2 = \delta_{AB} d\eta^A d\eta^B.$$

(1.3)
An alternative coordinate system with three spacelike variables \( y^k \) \( (k = 1, 2, 3) \) and a timelike variable \( \lambda \) is defined in terms of the \( \eta^A \) by \[6\]

\[
\begin{align*}
\eta^k &= -\lambda^{-1} y^k, \\
\eta^4 &= R/\lambda, \\
\eta^4 - \eta^5 &= (R^2\lambda^2 - y^2)/R\lambda,
\end{align*}
\]

with the metric

\[
\begin{align*}
g_{\mu\nu} &= \text{diag}(-\lambda^{-2}, -\lambda^{-2}, -\lambda^{-2}, R^2\lambda^{-2}), \\
(-g)^{1/2} &= R\lambda^{-4},
\end{align*}
\]

By inserting the definitions of the \( x^\mu \) in terms of the \( \eta^A \), one finds \[8\] that in the Minkowski space limit \( R \to \infty \), one has \( \lambda \to 1 \), \( y \to x \), and \( \tau = R(\lambda - 1) \to x^4 \). The property of conformal flatness can be displayed by writing (1.3) as

\[
ds^2 = (1 + \tau/R)^2 (d\tau^2 - dy^2).
\]

Some care must be used in taking the limit \( R \to \infty \), since one also finds

\[
R(\lambda^2 - 1) \to 2x^4, \quad R \to \infty. \tag{1.6}
\]

## 2. De Sitter Group Generators

The group of isometries on the surface (1.2b) is the pseudorotation group \( O(4, 1) \). The infinitesimal rotations are

\[
\delta\eta^A = E^A_B \delta\eta^B, \quad E^{AB} = -E^{BA}, \tag{2.1}
\]

with ten parameters \( E_{AB} \). In any representation, with generators \( M_{AB} \), the rotation group commutation relations are satisfied,

\[
[M_{AB}, M_{CD}] = i(\delta_{AD}M_{BC} + \delta_{HC}M_{AD} - \delta_{AC}M_{BD} - \delta_{BD}M_{AC}). \tag{2.2}
\]

Then defining \( P_\mu = M_{\mu\nu}/R \), the algebra of the generators \( M_{AB} \) can be written in a form which becomes that of the generators of the Poincaré group when \( R \to \infty \),

\[
[M_{\mu\nu}, P_\rho] = i(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu), \quad [P_\mu, P_\rho] = iM_{\mu\rho}/R^2. \tag{2.3}
\]

However, there is also the possibility of interpreting the algebra (2.2) in another way, by defining

\[
M'_{jk} = M_{jk}, \quad P'_j = (M_{5j} - M_{4j})/R, \quad \Phi' = M_{54}, \quad K'_j = -R(M_{5j} + M_{4j}). \tag{2.4}
\]
Then in terms of $M'_{jk}$, $P'_j$, $\Phi'$, $K'_j$ (2.2) can be written

$$\left[ M'_{jk}, M'_{im} \right] = i(\delta_{jm} M'_{ik} + \delta_{km} M'_{ij} - \delta_{im} M'_{kj} - \delta_{jm} M'_{kl});$$

$$\left[ M'_{jk}, P'_m \right] = i(\delta_{km} P'_j - \delta_{jm} P'_k), \quad \left[ M'_{jk}, \Phi' \right] = 0, \quad \left[ P'_j, P'_m \right] = 0;$$

$$\left[ M'_{jk}, K'_m \right] = i(\delta_{km} K'_j - \delta_{jm} K'_k), \quad \left[ P'_j, \Phi' \right] = iP'_j, \quad \left[ K'_j, K'_m \right] = 0;$$

$$\left[ P'_j, K'_m \right] = 2i(\delta_{jm} \Phi' + M'_{jk}),$$

with $\delta_{jk} = \text{diag}(-1, -1, -1)$. Now the algebra (2.5) is that of the restriction to three-dimensional Euclidean space of the conformal group on Minkowski space. In fact, a similar rearrangement of generators is possible for any group $SO(p, q)$ [15].

The differential generators of the rotations (2.1) are given by

$$m_{AB} = i(\eta_A \partial/\partial \eta^B - \eta_B \partial/\partial \eta^A),$$

and spelled out in the $(y^k, \lambda)$ coordinates they become [8]

$$m_{jk} = -i(y^i \partial_k - y^k \partial_j), \quad \partial_j = \partial/\partial y^j;$$

$$m_{\lambda A} = i(y^k \partial_\lambda + \lambda \partial_k), \quad m_{\lambda j} - m_{\lambda j} = -iR \partial_j;$$

$$m_{\lambda j} + m_{\lambda j} = i[(R \lambda^2 - y^2/R) \partial_j + 2R^{-1}y^j y^k \partial_k + 2R^{-1} \lambda y^j \partial_\lambda], \quad \partial_\lambda = \partial/\partial \lambda. \quad (2.7)$$

Therefore, in terms of the $(y^k, \lambda)$ coordinates, the operator $p_\lambda = m_{\lambda A}/R$ is the generator of dilations. Nevertheless, in the limit $R \to \infty$, the operators $m_{\alpha \mu}$ and $m_{\alpha \mu}/R$ become the differential generators of the Poincaré group in Minkowski space.

Suppose now that $\chi(\eta)$ is a scalar field in five-space that transforms under the representation of the rotations with generators (2.6); i.e., the infinitesimal transformation (2.1) induces on $\chi$ the variation

$$\delta \chi = \chi'(\eta) - \chi(\eta) = (-i/2) E^{AB} m_{AB} \chi(\eta). \quad (2.8)$$

This definition, and identification of the parameters $E^{AB}$ with the Lorentz transformation parameters $\epsilon^{\mu \nu} = E^{\mu \nu}$ and the parameters of generalized translations $\alpha^\mu = R \epsilon^{\mu \nu}$, yield the transformation behavior of a scalar field in $(y^k, \lambda)$ space. In the limit $R \to \infty$, this reduces to the transformation of a scalar field under the Poincaré group.

The problem occurring at this point is how to generalize (2.8) to fields with spin. Börner and Dürr [8] have found a representation by assuming that under ordinary three-dimensional rotations, fields with spin transform with the same spin matrices $s_{jk}$ as in Lorentz invariant field theory; they then make an ansatz for the complete representation involving only Lorentz spin matrices $s_{\mu \nu}$ and narrow down the representation uniquely by using the group commutation relations.

However, by this method there is lost the connection between fields and field equations in five-dimensional space and their counterparts in four-dimensional space which is present in the case of scalars. This defect can be repaired if one keeps in mind
the fact that, according to (2.5), fields transforming according to a representation of the rotations (2.1) also transform according to a representation of a group isomorphic to the restriction of conformal transformations in Minkowski space to three-dimensional Euclidean space. Because this is the case, it is possible to construct a "descent" operator to take fields with spin in five-space into their four-space versions, after the fashion of the procedure of Mack and Salam for writing fields in flat six-space in terms of Minkowski space variables.

To do this one assumes that if \( \chi(\eta) \) is a field with spin in five-space, the transformation (2.1) induces on \( \chi(\eta) \) the variation

\[
\delta \chi = \chi'(\eta) - \chi(\eta) = (-i/2) E^{AB} j_{AB} \chi
\]

with

\[
j_{AB} = m_{AB} + s_{AB} ,
\]

where \( s_{AB} = -s_{BA} \) are ten spin matrices in five-space. Then the representation on \( \chi \) of \( P'_j \), defined by (2.4), has the form

\[
p'_j = i \partial_j + (s_{5j} - s_{4j})/R .
\]

Now in the similar problem of fields on the null surface in \((4 + 2)\)-dimensional space [11, 12], one eliminates the spin part in the operator analogous to (2.10) by means of a similarity transformation with a space-time-dependent matrix \( U \), because then the resulting operator can be identified with the momentum operator in Minkowski space. The same idea can be put to use here, since by applying \( U \) to wave equations in five-space, the resulting equations in four-space will be manifestly translationally invariant in \( y^k \) on account of the conformal invariance implied by (2.5).

The solution of the problem

\[
U p'_j U^{-1} = i \partial_j
\]

is given by the analog of the usual operator \( U \) [12] for five-dimensional space, where one does not work on the null surface:

\[
U = \exp[-iy^j(s_{5j} - s_{4j})/R].
\]

Since the operators \( (s_{5j} - s_{4j}) \) commute among themselves, the inverse is

\[
U^{-1} = \exp[iy^j(s_{5j} - s_{4j})/R].
\]
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It is a tedious but straightforward task to compute the effects of $U$ on the operators $j_{AB}$, with the results

$$Uj_{kn}U^{-1} = m_{kn} + s_{kn}, \quad Uj_{54}U^{-1} = m_{54} + s_{54},$$

$$U(j_{sk} - j_{4k}) U^{-1} = m_{sk} - m_{4k};$$

$$U(j_{sk} + j_{4k}) U^{-1} = m_{sk} + m_{4k} + 2y' s_{jk}/R + 2y' s_{54}/R - \lambda(s_{5k} - s_{4k}) + (s_{5k} + s_{4k}).$$

At this point one can introduce a second similarity transformation in the index space of $\chi$, which has the effect of preserving (2.11) but eliminating all the matrices $s_{54}$ from (2.13). The reason for doing this is to be able to compare field equations in four-space with field equations constructed from Minkowski space equations by replacing ordinary derivatives with covariant ones. Such equations contain only the Lorentz matrices $s_{\mu\nu}$. Accordingly, consider the matrix

$$W = \exp[-i \ln \Lambda s_{54}], \quad W^{-1} = \exp[i \ln \Lambda s_{54}],$$

where either $\Lambda$ is restricted to $\Lambda > 0$, or one uses the definition of $\ln z$ for complex $z$ to define $\ln \Lambda$ for $\Lambda = -|\Lambda|$ by $d(\ln \Lambda)/d\Lambda = -|\Lambda|^{-1}$, $\exp(\ln \Lambda) = -(|\Lambda|$. Denoting by $\bar{m}_{AB}$ the results of applying the operator $T = WU \neq UW$ to the operators $j_{AB}$,

$$\bar{m}_{AB} = Tj_{AB}T^{-1} = WUj_{AB}U^{-1}W^{-1},$$

one finds that these operators are given by

$$\bar{m}_{kn} = j_{kn} = m_{kn} + s_{kn}, \quad \bar{m}_{54} = m_{54}, \quad \bar{m}_{5k} - \bar{m}_{4k} = m_{5k} - m_{4k};$$

$$\bar{m}_{sk} + \bar{m}_{4k} = m_{sk} + m_{4k} + 2y's_{jk}/R + 2y's_{54}/R - 2s_{5k}. \quad (2.16)$$

This is precisely the form of the representation found by Bärner and Dürr [8]. Therefore the operator $T = WU$ provides the desired connection between fields in five-dimensional space and their counterparts in curved four-space. The operators $\bar{m}_{\mu\nu}$ become, in the limit $R \to \infty$, the usual representation of Lorentz transformations for particles with spin, while $\bar{m}_{5\mu}/R$ become the momentum operators $i\partial/\partial x^\mu$.

3. FIELD EQUATIONS IN FIVE-DIMENSIONAL SPACE

3.1. Scalar Equation

If the wavefunction $\chi(\eta)$ for a spin-0 boson in five-dimensional space is a scalar, then the spin-0 equation can be taken to be [8]

$$[(1/2R^2) m_{AB} m^{AB} - I_4] \chi = 0,$$

with $I_4$ a constant and natural units, $\hbar = c = 1$. This form derives from the fact that
the second-order differential operator in (3.1) is the first Casimir operator of the pseudorotation group in the representation (2.6). To rewrite the field equation in terms of $y^k, \lambda, R$ the operators $m_{AB}$ are replaced by their equivalents in these coordinates, given by (2.7), with the result

$$(1/2R^2) m_{AB} m^{AB} = R^{-2}\lambda^2 \partial_\lambda \partial_\lambda - 2R^{-2}\lambda \partial_\lambda - \lambda^2 \partial_j \partial_j. $$

(3.2a)

In the limit $R \to \infty$, the above operator becomes the D’Alembertian, so that as far as the flat-space limit is concerned, it is sufficient to take $I_1 = -m^2$,

$$I_1 = -m^2;$$

(3.2b)

where $m$ is the mass. However, this value of $I_1$ is not necessary as well, since any constant having $R^2$ in the denominator vanishes for $R \to \infty$. Nevertheless, for definiteness let the scalar wave equation be (3.1) with (3.2b), since the operator (3.2a) is in fact the covariant generalization of the D’Alembertian to de Sitter space, in the coordinates $(y^k, \lambda) \equiv y^\mu$:

$$\Box_e = (-g)^{-1/2} (\partial/\partial y^\mu)[(-g)^{-1/2} g^{\mu\nu}(\partial/\partial y^\nu)] = (1/2R^2) m_{AB} m^{AB}. $$

(3.3)

3.2. Spinor Equation

One need not go to a higher-dimensional matrix representation than the usual Dirac matrices to accommodate the additional spin matrices in five-space. With $\gamma^A = (\gamma^\mu, \gamma^5)$, the simplest representation of the $s^{AB}$ is constructed in analogy with the Minkowski space matrices,

$$\sigma_{AB} = (i/4)[\gamma_A, \gamma_B].$$

(3.4a)

The representation used here is that with

$$(\gamma^k)^+ = -\gamma^k, \quad (\gamma^A)^+ = \gamma^A, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = -(\gamma^5)^+. $$

(3.4b)

Dirac [4] postulated a wave equation for a spinor $\chi(\eta)$ in the form

$$(1/2R) \gamma^A \gamma^B m_{AB} \chi(\eta) = \kappa \chi(\eta).$$

(3.5)

As shown clearly by Gürsey [14], this equation can be interpreted as a Dirac equation in de Sitter space provided two separate procedures are followed. First, the parameter $\kappa$ must be identified with the complex number $m + 2i/R$, where $m$ is the mass; and second, one must define a new set of Dirac matrices $\gamma^\mu = \gamma^\mu, \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = -(\gamma^5)^+$, so that in the limit $R \to \infty$, the equation reads $i\gamma^\mu(\partial/\partial x^\mu) \chi - m \chi = 0$.

To develop conservation laws for spinors, it will be shown that it is necessary to find a Lagrangian density $\mathcal{L}$ in five-space which is real up to a divergence $m_{AB} F^{AB}$. Such an $\mathcal{L}$ for spinors is formed by multiplying the field equation matrix-differential
operator by $\tilde{x} = \chi^+ A$, where $A$ defines the Dirac adjoint. The matrix $A$ can be taken equal to $\gamma^a$ [3], and the $\mathcal{L}$ for (3.5) is

$$\mathcal{L}_D = \left(1/2R\right) \bar{\chi} \gamma^a \gamma^b m_{AB} \chi - m \bar{\chi} \chi - (2i/R) \bar{\chi} \chi. \quad (3.6)$$

A straightforward calculation yields for the hermitian adjoint of the first expression

$$\left(1/2R\right)(\bar{\chi} \gamma^a \gamma^b m_{AB} \chi)^\dagger = m_{AB}(\bar{\chi} \gamma^a \gamma^b \chi/2R) - \left(1/2R\right) \bar{\chi} \gamma^a \gamma^b m_{AB} \chi, \quad (3.7)$$

whereas the second and third terms are real and imaginary, respectively. The first term on the right-hand side of (3.7) is in fact a divergence of the form $m_{AB} E^{AB} = i\partial(\eta_A E^{AB})/\partial \eta^B$, so that the first expression in (3.6) is imaginary up to a divergence. Thus, one has the result that (3.5) cannot be derived from a real Lagrangian density.

There are two possible avenues open to remedy this situation—one can modify either the field equation or the matrix $A$. The latter approach amounts to defining a new $A$,

$$A = -\gamma^a \eta_A \gamma^a/R, \quad A^{-1} = \eta_A \gamma^a \gamma^a/R, \quad (3.8)$$

which does define an adjoint by virtue of leaving $\chi^+ A \chi$ invariant under the replacement $\chi'(\eta) = \chi(\eta) - (i/2) E^{AB} \sigma_{AB} \chi(\eta)$, provided $\gamma^a$ is also replaced by $\gamma'^a = \gamma^a + E^a r^b$ in (3.8). With this $A$, $\mathcal{L}_D$ is real up to a divergence.

However, modifying the field equation and retaining $A = \gamma^a$ yields the same results in four-space without the necessity of replacing $\gamma^a$ by $\gamma'^a$. This modification reads

$$[-(\eta_C \gamma^C / R)(1/2R) \gamma^a \gamma^b m_{AB} + 2i(\eta_C \gamma^C / R^2) - m] \chi = 0, \quad (3.9)$$

with $\mathcal{L}$ real up to a divergence

$$\mathcal{L} = \bar{\chi}[-(\eta_C \gamma^C / R)(1/2R) \gamma^a \gamma^b m_{AB} + 2i(\eta_C \gamma^C / R^2) - m] \chi, \quad \bar{\chi} = \chi^+ \gamma^a. \quad (3.10)$$

Equation (3.9) is the free-field equation for spinors adopted here.

For spinors, the matrices $\sigma_{5j} - \sigma_{4j}$ are found to be nilpotent to the power two, so that the matrix (2.12) is given by

$$U = I - iy^l(\sigma_{5j} - \sigma_{4j})/R, \quad U^{-1} = I + iy^l(\sigma_{5j} - \sigma_{4j})/R. \quad (3.11a)$$

The matrix $\sigma^{54}$ satisfies $(\sigma^{54})^2 = -\frac{1}{4}$, so that the matrix (2.14) is given by

$$W = \cosh(\frac{1}{2} \ln \lambda) + 2i \sinh(\frac{1}{2} \ln \lambda) \sigma^{54}. \quad (3.11b)$$

It is straightforward to carry out the similarity transformation $T(\ ) T^{-1}$ on (3.9),
with $T = WU$, but the calculation is long. The task can be simplified by recalling that translational invariance in $y^k$ is guaranteed. Thus one finds

$$T(\eta C_\gamma^C / R) T^{-1} = \gamma$$

(3.12a)

and

$$T[(-\eta C_\gamma^C / 2R^2) \gamma^A \gamma^B m_{AB}] T^{-1} = i[\gamma^A(\lambda / R) \partial_\lambda + \gamma^I \partial_I - (3/2R) \gamma^I]$$

$$- (2i/R) \gamma^I.$$  

(3.12b)

Hence, defining derivation operations (cf. [6])

$$\nabla_j = \lambda \partial_j - (1/2R) \gamma^I \partial_I,$$

$$\nabla_4 = (\lambda / R) \partial_\lambda,$$

(3.13)

the field equation for the four-space spinor field $\psi = T\chi$ takes the form

$$(iy^\mu \nabla_\mu - m) \psi = 0.$$  

(3.14)

This generalization of the Dirac equation to de Sitter space is in agreement with that found by Nachtmann [6] by the method of induced representations in the curved four-space, using the $(y^k, \lambda)$ coordinates. Börner and Dürr [8] proposed a field equation for spinors, which was essentially the iterated version of this equation, by equating to a constant the first Casimir operator for the spinor representation. However, they used the representation (2.16) as opposed to the representation in five-space (2.9b), and so did not establish a connection between $\chi$ and $\psi$. Gürsey and Lee [3, 14] introduced an operator comparable to $U$ written in terms of the coordinates $x^u$, which are defined by (1.1). But one of the principal advantages of the operator $T$ derives from the fact that it yields directly fields in four-space defined over the coordinates $(y^k, \lambda)$ for any spin, and not just for spinors; it is in terms of $(y^k, \lambda)$ that solutions of field equations have been found [6, 8] and in terms of these coordinates that calculations can be simplified by exploitation of the translational invariance with respect to $y^k$.

3.3. Minimal Coupling; Vector Equation

The Lagrangian density (3.10) can be made invariant under coordinate-dependent phase transformations

$$\chi \rightarrow \chi \exp[iq \Lambda(\eta)]$$

(3.15a)

by replacing $m_{AB}$ by

$$m_{BC} \rightarrow m_{BC} + qH_{BC},$$

where $H_{BC} = -H_{CB}$ is a five-tensor compensating field that undergoes a gauge transformation

$$H_{BC} \rightarrow H_{BC} - im_{BC} \Lambda$$

(3.15b)
whenever $\chi$ transforms as in (3.15a). The simplest such $H_{BC}$ is

$$H_{BC} = \gamma_B A_C - \gamma_C A_B,$$

with $A_B$ a five-vector having the gauge transformation

$$A_B \rightarrow A_B + \partial A/\partial \eta^B.$$  

(3.15c)

To determine the form of the minimal coupling interaction in four-space, it is necessary to find the operator $T$ for vectors, and define the four-space field $a_B$ by

$$a_B = T_B^C A_C.$$

The vector representation of the matrices $s_{AB}$ is given by

$$(s_{AB})_C^D = i(\delta_{AC}\delta_B^D - \delta_A^D\delta_{BC}).$$

(3.16)

The matrices $(s^5, s^{4j})$ occurring in (2.12) are here found to be nilpotent to the power three, and the matrix $s^{54}$ in (2.14) satisfies $(s^{54})^3 = -s^{54}$. Accordingly, one can determine the matrix $T = WU$ in closed form, with the result

$$T_B^C = \delta_B^k \delta_k^C + (y^h/R)(\delta_{B^4}^C + \delta_{B^5}^C) + \frac{1}{2}\lambda(\delta_{B^4}^4 - \delta_{B^5}^5)(\delta_{C^4}^4 - \delta_{C^5}^5)$$

$$+ y^h/R)(\delta_{B^4}^4 - \delta_{B^5}^5) - (y^k/R)(\delta_{B^4}^k + \delta_{B^5}^k) \delta_{k^C}$$

$$+ (y^2/2R^2)(\delta_{B^4}^4 + \delta_{B^5}^5)(\delta_{C^4}^4 - \delta_{C^5}^5) = (T^{-1})_B^C.$$

(3.17)

Written out in components, this reads

$$a_k = A_k - (y^h/R)(A_4 - A_5), \quad a_4 - a_5 = \lambda(A_4 - A_5);$$

$$a_4 + a_5 = \lambda^{-1}[A_4 + A_5 - (2/R) y^k A_k + (y^2/R^2)(A_4 - A_5)].$$

From these relations, it is seen that the component $a_5$ can be eliminated by requiring transversality of $A_B$ in five-space, since one has

$$a_5 = (1/R) \eta^B A_B.$$

However, $\eta^B A_B = 0$ will not be insisted upon here, although this is usually assumed in similar formulations (cf. [10]), since it will be shown that $a_5$ does not enter into the interaction in four-space even if it is not assumed to vanish. The reason is that since the representation (2.16) contains no matrices $s_{5k}$, the component $a_5$ is completely decoupled from $a_4$, so that the variation

$$\delta A_B = (-i/2) E^{CD}(j_{CD})_B^F A_F$$


has the effects
\[ \delta a_\mu = (-i/2) E^{CD}(\bar{m}_{CD})_\mu a_\tau, \quad \delta a_5 = (-i/2) E^{CD}m_{CD}a_5. \]

Thus one is left with a decoupled vector and scalar in four-space.

Using (3.17), the extra term in the spinor equation due to minimal coupling can now be evaluated. The equation for \( \chi \) can be written in the form
\[ \left[ -\left( \eta_{44}/2R^2 \right) \sigma^{BC} \eta_B(\partial/\partial \eta^C - i q A_C) + 2i \eta_{44}/R - m \right] \chi = 0. \quad (3.18) \]

One finds that carrying out the transformation with the operator \( T \) for spinors, the expression involving \( A_\tau \) becomes
\[ T(\sigma^{BC} \eta_B A_C) T^{-1} = R(\sigma^5 a_\mu), \]

so that \( \psi = T\chi \) satisfies
\[ [i \gamma^\mu(\nabla_\mu - ig a_\mu) - m] \psi = 0. \quad (3.19) \]

From (3.15c), one finds that the gauge transformation for the field \( a_\mu \) is given by
\[ a_j \to a_j + \lambda \partial_j A, \quad a_4 \to a_4 + (\lambda/R) \partial^4 A, \quad (3.20) \]
i.e., \( a_\mu \) undergoes
\[ a_\mu \to a_\mu + D_\mu A, \quad (3.21) \]
where \( D_\mu \) are the versions of the operations (3.13) which act on scalars,
\[ D_j = \lambda \partial_j, \quad D_4 = (\lambda/R) \partial^4. \quad (3.22) \]

Equation (3.19), together with (3.21), constitutes the form which the minimal coupling interaction takes in de Sitter space. The fact that \( a_5 \) does not appear in the interaction is a feature that persists for the coupling of an intermediate vector boson to the weak current. The details for weak interactions are not presented here.

A possible vector free-field equation suggested by (3.1) and (3.2) is formed by setting the first Casimir operator in the vector representation equal to minus the mass squared,
\[ \left( 1/2R^2 \right)(m_{BC} + s_{BC})_D^E (m^{BC} + s^{BC})_E F A_F = -m^2 A_D. \quad (3.23) \]
Substituting (3.16) this reads
\[ [(1/2R^2) m_{BC}m^{BC} + (4/R^2) + m^2] A_D + (2i/R^2) m_{DF} A_F = 0. \quad (3.24) \]
To determine the gauge group of this equation, one must commute the derivative \( \partial/\partial \eta^x \) through the field equation operator. Keeping in mind that \( R^2 \) appearing in (3.23) is shorthand for \( -\eta^B \eta^B \), one finds that the form (3.23) is preserved when \( A_B \) undergoes the transformation (3.15c) provided the gauge function \( \Lambda \) satisfies
\[
(\partial/\partial \eta^D)[(1/2 R^2) m_{BC} m^{BC} A + m^2 A] - (\eta_D/R^4)(m_{BC} m^{BC} A) = 0
\]  
(3.25a)

Thus \( A_B \) is required to be massless, \( m^2 = 0 \), and \( \Lambda \) is constrained to satisfy the scalar field equation (3.1) with \( \eta^I = 0 \),
\[
(1/2 R^2) m_{CD} m^{CD} A = \Box_c A = 0
\]  
(3.25b)

This indicates that (3.23) reduces in four-space to the equation for a vector field in the covariant generalization of the Lorentz gauge. Equation (3.25b) is the defining relation for this gauge, so the Lorentz gauge reads in five-space
\[
(i/2 R^2) m_{CD}(\eta^C A^D - \eta^D A^C) = 0
\]  
(3.26a)

since this condition is preserved under gauge transformations provided \( \Lambda \) satisfies (3.25b). To write (3.26a) in terms of four-space components \( a_B \), one substitutes for \( A_B \) by means of (3.17) and carries out the indicated derivatives, and again this procedure can be greatly simplified by invoking the necessity of translational invariance in \( y^L \). The result is
\[
0 = -D_j a_j + D_\alpha a_\alpha - (3/R) a_4 = -[D_j \delta_j^x - (i/R)(s_{ij})_j^x] a_\alpha + D_\alpha a_\alpha
\]  
(3.26b)

with \( D_\alpha \) given by (3.22). It is easy to show that (3.26b) is indeed the condition which is maintained under gauge replacements (3.21) for gauge functions satisfying \( \Box_c A = 0 \).

Writing the field equation (3.24) in terms of \( a_B \) amounts to replacing \( m_{BC} + s_{BC} \) in (3.23) by the operators (2.16). Consequently the equations for the first four components \( a_\alpha \) reduce to those considered by Börner and Dür [8] and given in the \((y^k, \lambda)\) coordinates by Börner [16], while the equation for \( a_5 \) is (3.1) with (3.2b). One finds
\[
\Box_c a_k - 2(\lambda/R) \delta_k a_4 + (m^2 + 2/R^2) a_k = 0;
\]
\[
\Box_c a_4 - 2(\lambda/R) \delta_4 a_k + m^2 a_4 = 0,
\]  
(3.27)
\[
\Box_c a_5 + m^2 a_5 = 0,
\]

with \( \Box_c \) given by (3.3). Now the covariant version of the Proca equations, expressed in the \((y^k, \lambda)\) coordinates, can be put in a form very similar to (3.27) by making use of the covariant version of the Lorentz gauge. Defining the field tensor \( F_{\mu \nu} \) by
\[
F_{\mu \nu} = \partial B_\mu / \partial y^\nu - \partial B_\nu / \partial y^\mu
\]
with \( y^\mu = (y^k, \lambda) \), the covariant Lorentz condition sets the covariant derivative of \( B_\mu \) equal to zero:

\[
0 = g^{\mu\nu} B_{\mu,\nu} = (\lambda^2/R^2) \partial_j B_4 - \lambda^2 \partial_j B_4 - 2(\lambda/R^2) B_4,
\]

(3.28)

and the Proca equations

\[
F^{\mu\nu} + m^2 B^\mu = 0
\]

become

\[
[(\lambda^2/R^2) \partial_i \partial_j \partial_i \partial_j + m^2] B_4 - 2(\lambda/R^2) \partial_i B_4 = 0,
\]

(3.29b)

To see the connection between (3.28) and (3.29) and (3.26) and (3.27), one must notice that the differential operators \( D_\mu \) are the components of the gradients \( \partial_j \), \( \partial_\lambda \) written in the local Cartesian basis consisting of four vectors mutually orthogonal with respect to the Minkowski metric. The transformation between components \( B_\mu \) and components \( a_\alpha \) is effected by means of the vierbein components \( e^{\mu}_\alpha \) satisfying

\[
e^{\mu}_\alpha e^{\nu}_\beta \delta_{\alpha\beta} = g^{\mu\nu}, \quad e^{\mu}_\alpha e^{\nu}_\beta \delta_{\alpha\beta} = \delta^{\mu}_\nu, \quad e^{\mu}_\alpha e^{\mu}_\beta = \delta^{\alpha}_\beta.
\]

Then one has

\[
a_\alpha = e^{\mu}_\alpha B_\mu .
\]

From (3.22), one has, by inspection

\[
e^{\mu}_\lambda = \delta^{\mu}_j \lambda, \quad e^{\mu}_4 = \delta^{\mu}_4 (\lambda/R),
\]

so that

\[
a_j = \lambda B_j, \quad a_4 = (\lambda/R) B_4 .
\]

(3.30)

This identification brings into coincidence the covariant version of the vector equations and the equations one recovers from the five-dimensional formalism by descent to curved four-space. The necessity of transforming the components of the vector potential via (3.30) seems not to have been appreciated before (cf. [8]).

4. Action Principles in Five-Dimensional Space

4.1. Conservation Laws on Hyperboloids

In order to develop a method by which field equations can be derived from a Lagrangian density \( \mathcal{L} \) in five-space by means of an action principle, it is necessary to determine the form that conservation laws assume when only rotational and not translational invariance is the required property of the action integral.
To this purpose one must find an analog in five dimensions of Gauss' theorem, which can be applied to situations for which the variations of the coordinates $\eta^a$ take points on the hyperboloid $\eta^a \eta_a = -R^2$ into points on the same hyperboloid. Then by studying the variations of fields on this hyperboloid, one can construct an alternative to the Euler–Lagrange equations that is applicable to situations in which translational invariance does not obtain.

An identity relating a four-dimensional integral over some region $\Omega_4$ of this hyperboloid to a three-dimensional integral taken around the (closed) boundary $\Omega_3$ of $\Omega_4$ is provided by the generalization of Stokes' theorem (see, e.g., [17]) to the case of five dimensions. Written in terms of the antisymmetric tensor density $F^{AB}$, this relation is

$$\int_{\Omega_3} F^{AB} dS_{AB} = 2 \int_{\Omega_4} (\partial F^{AB}/\partial \eta^A) dS_B .$$

A surface element $dS_A$ that brings the integrand of the four-dimensional integral into the form $m_{AB} F^{AB}$ can be found by converting the right-hand side of (4.1) into an integral over a region $\Omega_5$ in five-dimensional space. This can be accomplished by defining the one-parameter family of surfaces $f(\eta) = \text{constant}$, so that in terms of $f(\eta)$ the element of area $dS_A$ on the surface $f = f_0$ is given by (cf. [18])

$$dS_A = d^5 \eta \delta(f - f_0) \frac{\partial f}{\partial \eta^A} .$$

Then on $f = R = (-\eta_A \eta^A)^{1/2}$ equal to a constant $R_0$ the surface element is

$$dS_A = -d^5 \eta \delta(R - R_0) \frac{\partial f}{\partial \eta^A} ,$$

where the integration is to be carried out over $\Omega_5$. One can prove that (4.2b) is correct by parametrizing the surface $R = R_0$ with $(y^k, \lambda)$ and producing the Jacobian $J(\eta^A; y^k, \lambda, R)$ by forming

$$(\partial \eta^A/\partial R) dS_A = d^5 \eta \delta(R - R_0) J(\eta^A; y^k, \lambda, R) \delta(R - R_0) d^5 y \, d\lambda \, dR .$$

Up to the factor setting $R = R_0$, this is the same result one arrives at by explicitly defining $dS_A$ as the contraction of the fourth-rank antisymmetric tensor surface element. One finds from (1.4) that the Jacobian is just $(-g)^{1/2}$ up to a minus sign:

$$J(\eta^A; y^k, \lambda, R) = -R/\lambda^4 .$$

The surface element $dS_A$ contains the unit vector $n_A$, defined by

$$n_A = \partial R/\eta^A = -\eta_A/R , \quad n_A n^A = -1 .$$

The element of area $dS_{AB}$ is found by forming an antisymmetric product with another vector in an orthogonal direction, i.e.,

$$dS_{AB} = d^5 \eta (\partial h/\partial \eta^A)_{h=h_0} \frac{\partial f}{\partial \eta^B} \delta(f - f_0) \delta(h - h_0) .$$
where \( h \) defines another one-parameter family, and the square brackets have the meaning of antisymmetrization,

\[
\eta^{[A} \eta^{B]} = \eta^{A} \eta^{B} - \eta^{B} \eta^{A}.
\]

The obvious choice here is \( h(\eta) = \lambda \), and

\[
\partial \lambda / \partial \eta^{A} = (\lambda / R) n_{A}', \quad n_{A}' = -(\eta_{A}/R) + \lambda(\delta_{A}^{4} + \delta_{A}^{5}), \quad n_{A} n'A = 1.
\]

Hence

\[
dS_{AB} = d^{5} \eta \left[ \lambda / R \right] n_{A} n_{B} \delta(R - R_{0}) \delta(\lambda - \lambda_{0})
\]

\[
= -d^{5} \eta (\lambda / R^{2})(\delta^{A}_{A} + \delta^{B}_{B}) \eta_{B} \cdot \delta(R - R_{0}) \delta(\lambda - \lambda_{0}).
\]

From the foregoing results, one has, finally, that Stokes' theorem (4.1) reads

\[
-i \int m_{AB} F_{AB} R^{-1} \delta(R - R_{0}) d^{5} \eta
\]

\[
- \left\{ \int_{\lambda_{0} - \lambda} - \int_{\lambda_{0} - \lambda} \right\} F_{AB}^{AB}(2 \lambda^{2}/R^{2}) \eta_{A}(\delta_{A}^{4} + \delta_{A}^{5}) \delta(R - R_{0}) \delta(\lambda - \lambda_{0}) d^{5} \eta.
\]

Thus whenever \( F_{AB} \) obeys the divergence condition

\[
m_{AB} F_{AB} = 0,
\]

the boundary integral on the right-hand side of (4.4) vanishes and one has a conservation law. Indeed, assuming that (4.5) holds, it is straightforward to show that

\[
(d/d \lambda) \int \eta_{A}(F_{AA} + F_{AB})(2 \lambda^{2}/R^{2}) \delta(R - R_{0}) \delta(\lambda - \lambda_{0}) d^{5} \eta = 0
\]

by using the relation

\[
\partial_{\lambda} = (i/R) \eta^{C}(m_{c4} - m_{c5}).
\]

4.2. Action Principle

The integrand in an action integral in five-space must be constrained such that \( \eta^{A} \) lies on \( \eta^{2} = -R_{0}^{2} \), and this can be accomplished by defining the action \( I \) as

\[
I = \int_{\Omega_{h}} L^{0} \delta(R - R_{0}) d^{5} \eta.
\]
The form of the Lagrangian density $\mathcal{L}$ suggested by the field equations adopted in Section 3 is.

$$\mathcal{L} = \mathcal{L}(\eta, \chi, m_{AB}\chi),$$  \hfill (4.6b)

including $\eta^4$ among the explicit arguments of $\mathcal{L}$. For an arbitrary transformation of the field and the coordinates,

$$\delta \eta^4 = \eta' - \eta^4, \quad \delta \chi = \chi'(\eta) - \chi(\eta),$$

one can now calculate the variation of $I$. Assuming that $\mathcal{L}$ is form invariant under these replacements, i.e.,

$$\mathcal{L}'[\eta'] = \mathcal{L}[\eta'] = \mathcal{L}[\eta', \chi'(\eta'), m'_{AB}\chi'(\eta')],$$  \hfill (4.7)

the calculation is similar to the usual one (see, e.g., [19]). Restricting attention to only those transformations $\delta \eta^4$ that preserve the value of $\eta^2$,

$$(\eta + \delta \eta)^2 = \eta^2,$$  \hfill (4.8)

the result is

$$I' - I = \int \{ (\partial \mathcal{L}/\partial \eta') \delta \eta^4 + (\partial \mathcal{L}/\partial \chi') \delta \chi + \pi^{CD} m_{CD} \delta \chi \} \delta (R - R_0) d^5\eta, \quad (4.9a)$$

where $\pi^{CD}$ is the conjugate momentum

$$\pi^{CD} = \partial \mathcal{L}/\partial (m_{CD}\chi).$$

Now (4.8) implies transverse variations $\eta_B \delta \eta^B = 0$, allowing one to write the term in (4.9a) containing $\mathcal{L}$ in the form of a divergence $m_{AB} F^{AB}$. Thus the variation of $I$ splits into two parts,

$$I' - I = \int \{ (\partial \mathcal{L}/\partial \chi') - m_{AB}\pi^{AB} \} \delta \chi$$

$$+ m_{AB} [\pi^{AB} \delta \chi + (i/2R^2) \eta^{[A} \mathcal{L} \delta \eta^{B]}] \delta (R - R_0) d^5\eta. \quad (4.9b)$$

The first part yields the form of the Euler–Lagrange equations following from the action (4.6).

$$(\partial \mathcal{L}/\partial \chi') - m_{AB}\pi^{AB} = 0,$$  \hfill (4.10)

and the second part defines a conserved surface integral $F$ via Stokes' theorem,

$$F = i \int \{ R \pi^{CD} \delta \chi + (i/2R) \eta^{[C} \mathcal{L} \delta \eta^{D]} \} dS_{CD}. \quad (4.11)$$
Following Schwinger, one can identify this conserved integral with the quantum field theory generator of rotations when $\delta \eta^t$ and $\delta \chi$ are given by (2.1) and (2.9), respectively. From the definition

$$\chi'(\eta) = R^{-1}\chi R,$$

$R$ is formed by

$$R^{-1} = \exp(iF),$$

so that with infinitesimal rotation parameters $E^{AB}$, one has

$$F = \frac{1}{2}E^{AB}M_{AB},$$

if $M_{AB}$ is the generator of rotations,

$$[\chi, M_{AB}] = j_{AB}\chi, \quad j_{AB} = m_{AB} + s_{AB}.$$  

Thus, writing

$$M_{AB} = \int S_{AB}^{CD} dS_{CD},$$

the angular momentum tensor in five-dimensional space is given by

$$S_{AB}^{CD} = R\pi^{CD}j_{AB\chi} - (1/2R)\mathcal{L}(\eta^C\delta_{[A|\eta_B]} - \eta^D\delta_{[A\eta_B]}).$$  \hspace{1cm} (4.12)

By dropping a term with vanishing divergence, (4.12) can be made more transparent. The simpler form of the angular momentum tensor is

$$S_{AB}^{CD} = R\pi^{CD}j_{AB\chi} - \frac{1}{2}\delta_{[A|\eta_B]}\mathcal{L}'(\eta).$$  \hspace{1cm} (4.13)

For an arbitrary $\delta \chi$ generated by the generic form of $F$ (4.11),

$$\delta \chi = i[F, \chi],$$  \hspace{1cm} (4.14)

accompanied by no coordinate variation, $\delta \eta^a = 0$, one can establish the commutation relations for the field as consistency conditions on (4.14). In this case, spelling out the implicit summation over independent field components $\chi\alpha$, $F$ is

$$F = i \int R\pi^{CD}_\alpha \delta \chi\alpha dS_{CD}.$$  

Thus, for any $\eta$ on the surface defined by $dS_{CD}(\eta')$, one must have

$$\delta \chi\alpha(\eta) = \int R[-\pi^{CD}_\alpha(\eta')[\delta \chi\alpha(\eta'), \chi\alpha(\eta)]] + \ [\chi\alpha(\eta), \pi^{CD}_\alpha(\eta')] \delta \chi\alpha(\eta')] dS_{CD}(\eta').$$  \hspace{1cm} (4.15)
If one assumes, further, that (cf. [20])

\[ [\chi_\alpha(\eta), \delta \chi_\beta(\eta')] = 0 \quad (4.16) \]

for all \( \alpha, \beta \) and all \( \eta, \eta' \) on the surface \( \Omega_3 \), then the simplest solution for (4.15) is found by writing

\[ R[\chi_\alpha(\eta), \pi^C_\alpha(\eta')] dS_{CD}(\eta') = \delta_{\alpha\beta} \delta^{CD}(\eta - \eta') dS_{CD}(\eta'). \quad (4.17) \]

Here the singular function \( \delta^{CD}(\eta - \eta') \) is defined for two points on \( \Omega_3 \) by (cf. the function \( \delta(x - x') \) in [20])

\[ \int_{\Omega_3} g(\eta') \delta^{CD}(\eta - \eta') dS_{CD}(\eta') = g(\eta), \]

and because of (4.3) has the form

\[ \delta^{AB}(\eta - \eta') = (\lambda^3/2) \delta(y - y') n'^{[A} n^{B]}. \]

Thus, substituting (4.3), the relation (4.17) reads

\[ [\chi_\alpha(\eta), \eta C('\pi^C_\alpha(\eta') + \pi^C_\alpha(\eta'))] = -\lambda^2 \delta_{\alpha\beta} \delta(y - y'), \quad \lambda = \lambda', \quad R = R'. \]

or, explicitly,

\[ [\chi_\alpha(\eta), (\pi^C_\alpha(\eta') - (y'^{k}/\sqrt{R})(\pi^4_\alpha(\eta') + \pi^5_\alpha(\eta')))] = -\lambda^2 \delta_{\alpha\beta} \delta(y - y'), \quad \lambda = \lambda', \quad R = R'. \quad (4.18) \]

Also, (4.16) implies that

\[ 0 = \delta[\chi_\alpha(\eta'), \chi_\beta(\eta')] \]

and since the variation is arbitrary,

\[ [\chi_\alpha(\eta'), \chi_\beta(\eta')] = 0 \quad (4.19) \]

for \( \eta, \eta' \) on \( \Omega_3 \). Equations (4.18) and (4.19) are the canonical commutation relations governing fields in five-dimensional space.

### 4.3. Scalar Fields

The Lagrangian density yielding the scalar field equation (3.1) with (3.2b) is

\[ \mathcal{L} = -(1/4R^2) m_{A\beta} m^{A\beta} + \frac{1}{2} m^2 \chi^2, \quad (4.20a) \]
so that the conjugate momentum is given in this case by

\[ \pi^{AB} = -(1/2R^2) m^{AB} \chi. \]  

(4.20b)

Expressing the action integral (4.6) in terms of \((y^k, \lambda, R)\), one arrives at the usual form

\[ I = \int (-g)^{1/2} \mathcal{L}_4 \, d^2 y \, d\lambda \]

with

\[ \mathcal{L}_4 = (\lambda^2/2R^2) \partial_\lambda \partial_\lambda \chi - \frac{1}{2} \lambda^2 \partial_k \partial^k \chi - \frac{1}{2} m^2 \chi^2. \]

Assuming commutation relations for bosons, the quantum conditions on \(\chi\) given by (4.18) and (4.19) read (with \(R = R'\) implicit)

\[ [\chi(y, \lambda), \chi(y', \lambda)] = 0, \quad [\chi(y, \lambda), D_4 \chi(y', \lambda)] = i \lambda^4 \delta(y - y'), \]

(4.21)

where \(D_4 = (\lambda/R) \partial_\lambda\). These are precisely the commutation relations arrived at by Nachtmann [6] by quantization in curved four-space. This result therefore verifies the suitability of the quantum action principle for fields in five-dimensional space developed above.

The conservation laws flowing from de Sitter group invariance can be established in equally short order. The results that follow from (4.13) are

\[ P_j = M_{5j}/R = \int_{\lambda = \lambda_0} (-i D_4 \chi(m_{5j}/R) \chi - (y^j/R) \mathcal{L}_3) \lambda^{-3} \, d^3 y, \]

\[ P_4 = M_{44}/R = \int_{\lambda = \lambda_0} (-i D_4 \chi(m_{44}/R) \chi - \mathcal{L}_3) \lambda^{-3} \, d^3 y, \]

\[ M_{jk} = \int_{\lambda = \lambda_0} (-i D_4 \chi(m_{jk} \chi) \lambda^{-3} \, d^3 y, \]

\[ M_{4i} = \int_{\lambda = \lambda_0} (-i D_4 \chi(m_{4i} \chi - y^i \mathcal{L}_3) \lambda^{-3} \, d^3 y. \]

(4.22)

The terms involving \(\mathcal{L}_4\) above can be recast by writing them in terms of the Killing vectors associated with the de Sitter group transformations. Denoting the components of the Killing vector for the \(P_4\) transformations by

\[ (\xi_4)^i = (1/\alpha^4)(\delta y^k, \delta \lambda) = (y^k/R, \lambda/R), \quad \alpha^4 = RE^{45}, \]

the components \((\xi_4)^i\) are determined by the vierbein components \(e^\alpha_a\), as in (3.30). Since one finds

\[ (\xi_4)^a = (y^k/R\lambda, 1), \]
the pertinent term in \( P_4 \) is \(-\langle \xi_4 \rangle^4 \). Similarly the term in \( P_1 \) is \(-\langle \xi_1 \rangle^4 \). In this form, the expressions (4.22) are in agreement with the conserved quantities obtained by Nachtmann [6] for the case of scalar fields in de Sitter space. In terms of the inner product of solutions of the field equation in five-space, which is

\[
(\varphi, \chi) = \int \frac{1}{2R} (\varphi m^{AB} \chi - m^{AB} \varphi \chi) dS_{AB} = i \int_{\lambda - \lambda_0} (\varphi D_4 \chi - D_4 \varphi \chi) \lambda^{-3} d^3 y,
\]

using the field equation it is easy to show that the quantities (4.22) can be written as expectation values

\[
M_{AB} = \langle \chi, m_{AB} \chi \rangle.
\]

In the limit \( R \to \infty \), as expected these quantities go over to the usual forms expressing conservation of momentum and angular momentum.

### 4.4. Spinor Fields

The field equation (3.9) follows from the Lagrangian density (3.10), so that for the case of spinors the action integral (4.6) reads

\[
I = \int (-g)^{1/2} \mathcal{L}_4 d^3 y d\lambda, \quad \mathcal{L}_4 = \psi (i \gamma^\mu \nabla_\mu + m) \psi,
\]

where \( \psi = T\chi \), as in (3.14). The conjugate momenta are given here by

\[
\pi^{AB} = (-i/R^2) \chi \gamma_5 \gamma^C \sigma^{AB}.
\]

Assuming anticommutation relations for fermions, one arrives at the following conditions for \( \gamma \) by substituting (4.23) into (4.18) and (4.19) and applying \( T(\gamma) T^{-1} \) to the results:

\[
[\psi_\alpha(y, \lambda), \psi_\beta(y', \lambda)]_+ = 0, \quad [\psi_\alpha(y, \lambda), \psi_{\beta+}(y', \lambda)]_+ = \delta_{\alpha\beta} \delta^{3}(y - y').
\]

These relations agree with the results for spinors in four-space derived by Nachtmann [6], with the constant \( c_1 = c_1^* > 0 \) appearing in his work being equal to unity here. In the limit \( R \to \infty \) these become the familiar canonical anticommutation relations in Minkowski space.

In terms of the inner product

\[
(\varphi, \chi) = \int \frac{1}{2R} \bar{\varphi} (\gamma^C R) \gamma^{AB} \chi dS_{AB} = \int_{\lambda = \lambda_0} \bar{\psi} (T \varphi) \chi \psi \psi \psi \lambda^{-3} d^3 y,
\]

the ten conserved quantities associated with de Sitter group invariance can be written

\[
M_{AB} = (\chi, m_{AB} + \sigma_{AB}) \chi = \int_{\lambda = \lambda_0} \bar{\psi} \gamma^A \bar{m}_{AB} \psi \lambda^{-3} d^3 y,
\]
since $\mathcal{L} = 0$ for solutions of the field equations. Here, $\bar{m}_{AB}$ is the version of the representation (2.16) with rotation matrices $\sigma_{AB}$ given by (3.4). Also, the charge operator has the simple form

$$Q = (\chi, \chi) = \int_{\lambda=1} (\bar{\psi} \gamma^A \psi) \lambda^{-3} d^3y. \quad (4.26)$$

Once again, in the limit $R \rightarrow \infty$ one finds that $M_{\mu\nu}$ and $M_{5\nu}/R$ become the usual generators of the Poincaré group in the spinor representation, and $Q$ becomes the customary charge operator.

These results show that by making use of the operator $T$, defined in Section 3, one can quantize fields with spin by means of an action principle in five-dimensional space and arrive at the correct conservation laws in the curved four-space. In Section 5, this method is extended to establish the extra conservation laws that are satisfied whenever conformal symmetry is present in addition to invariance under the de Sitter group.

5. **Conformal Invariance**

5.1. **Conformal Group Generators**

Recently Fronsdal, studying the rotations in $(3 + 2)$-dimensional space, has shown how to complete the algebra of the $SO(4, 2)$ conformal group [21]. In the similar situation here, the additional five generators, denoted $j_{6A}$, are given by

$$j_{6A} = (1/R)(\eta^B j_{AB} - i\ell \eta_A), \quad (5.1)$$

where $j_{AB}$ are the rotation group generators (2.9b) and $\ell$ is the analog of the canonical scale dimension in Minkowski space (see, e.g., [12]). The operators $j_{6A}$ together with the $j_{AB}$ satisfy the $SO(4, 2)$ algebra for any value of the c-number $\ell$, but it will be shown below that in order to actually have symmetry for field equations for massless particles, $\ell$ must assume the same value as does the scale dimension in Minkowski space for free fields, viz, $\ell = -1$ for scalars and $\ell = -3/2$ for spinors. Here the 15-parameter conformal transformation group is interpreted as point transformations extending the de Sitter group, rather than as conformal metric transformations (see, e.g., [22]).

In general, the action of (5.1) on fields in four-space is found by applying to $j_{6A}$ the similarity transformation $T$. Denoting the results obtained by $\bar{m}_{6A}$, one has

$$\bar{m}_{6A} = Tj_{6A}T^{-1} = (1/R)(\eta^B \bar{m}_{AB} - i\ell \eta_A), \quad (5.2)$$

with $\bar{m}_{AB}$ given by (2.16). The analog of the generator of dilations in Minkowski space is $\bar{m}_{65}$, which is given in detail by

$$\bar{m}_{65} = i[\lambda y^k \partial_k + (1/2R^a)(R^a \lambda^2 - R^2 + y^a) \partial_\lambda - \ell] + (y^k/R) s_{4k}. \quad (5.3)$$
Because of (1.6), this operator becomes the generator of dilations \( i(x^\mu \partial_x^\mu - \ell) \) in the flat space limit. The analogs of the generators of special conformal transformations are given here by \( 2R(\bar{m}_6 - \bar{m}_5) \). Explicitly, these are

\[
2R(\bar{m}_6 - \bar{m}_5) = -i[2y^k\partial_j + 2y^\ell(\lambda - 1) \partial_\lambda - y^k \partial_k + (R \lambda - R)^2 \partial_k] + 2y^k s^k_j + 2(R \lambda - R)^2 s_4 + 2iy^k \lambda,
\]

\[
2R(\bar{m}_5 - \bar{m}_5) = i[2(R \lambda - R) y^k \partial_k + (y^2/R) \partial_\lambda + R^{-1}(R \lambda - R)^2 \partial_\lambda] + 2y^k s^k_4 - i\ell(R^2 \lambda^2 - R^2 - y^2)/R \lambda.
\]

In the limit \( R \to \infty \), they go over to the generators of special conformal transformations in Minkowski space (see [12]). The five-dimensional form (5.1) is obviously a good deal simpler than the corresponding generators in de Sitter space. The coordinate transformation corresponding to (5.1) is

\[
\delta \eta^B = -(E^A/R)(\eta_A \eta^C + R^2 \delta_A^C),
\]

with infinitesimal parameters \( E^A \). The corresponding transformations in terms of \((y^k, \lambda, R)\) are quite complicated, but just as for the rotations, the infinitesimal transformations (5.5a) leave \( R \) invariant, since, to first order,

\[
(\eta^A + \delta \eta^A)(\eta_A + \delta \eta_A) = \eta^A \eta_A = -R^2.
\]

Whenever conformal symmetry is present, the algebra of the de Sitter group (2.3) can be rewritten with the aid of the general representation \( M_{6,4} \). For if one has available the five \( M_{6,4} \), then defining the momentum operators by \( P_\mu = (1/2R)(M_{6u} + M_{5u}) \), the algebra of the fifteen operators \( M_{\mu} , P_\mu, \Phi = M_{65} , K_\mu = 2R(M_{6u} - M_{5u}) \) is exactly that of the conformal group in Minkowski space. Indeed, the operators \((1/2R)(\bar{m}_6 + \bar{m}_5)\) do go over to the gradients \( i\partial_x^\mu \) in the limit \( R \to \infty \).

5.2. Conformal Covariance of Massless Fields

In analogy with the representation of rotations (2.9), the variation induced on a field \( \chi(\eta) \) is

\[
\delta \chi = \chi'(\eta) - \chi(\eta) = -iE^6j_{6A} \chi.
\]

For the spinor equation (3.9), one finds that the generators (5.1) are symmetry operators on the space of solutions, provided the mass is zero and \( \ell = -3/2 \); if \( O \) denotes the matrix differential operator for the massless spinor equation, then

\[
[O, j_{6A}] = i(\eta_A/R) O + (\ell + 3/2)((\eta_A \eta_B / R^2) + (\eta_B / R)).
\]

However, one finds that if the scalar field equation is given by (3.1) with (3.2b),
then it is not conformally covariant if the parameter $m$ is simply set equal to zero. There are two possible ways to remedy this flaw: either the form of the field equation can be modified so as to ensure covariance when $m = 0$ [23, 24], or one can work with the field equation (3.1) with the Casimir operator $I_1$ taking on the discrete value $-2/R^2$ [8]. Using either interpretation, the conformally covariant scalar equation is given by

$$
(1/2R^2) m_{AB} m^{AB} \chi + (2/R^2) \chi = 0,
$$

(5.8)
so that $j_{aA}$ for scalars is a symmetry operator on the space of solutions of (5.8).

$$
[L, j_{aA}] = (2i\eta_A/R) L,
$$

(5.9)
where $L$ is the differential operator in (5.8). The net effect of this modification is the addition of an extra term $\chi^2/R^2$ to the Lagrangian density (4.20a), while in the flat-space limit one still recovers the massless scalar field. Since the conjugate momentum (4.20b) is unaffected the commutation relations remain the same, while in the conservation laws (4.22) the extra term $-\chi^2/R^2$ must be added to $\mathcal{L}_4$.

5.3. Conformal Group Conservation Laws

For the case of spinors, the general form (4.11) of the quantum field theory generator $F$ yields five additional conservation laws from the substitution of the field variation (5.6). The extra conserved quantities can be written

$$
K = \mathcal{L} = j_{aA}^A dS_{CD},
$$

(5.10a)
with the divergenceless integrand

$$
S_{aA}^{CD} = (\eta^B/R) S_{AB}^{CD} - i\pi^{CD} \eta_{aA} \chi.
$$

(5.10b)
In terms of the inner product for spinors, with $\mathcal{L} = 0$ for solutions $\chi$, the quantities (5.10) are

$$
M_{aA} = (\chi, j_{aA} \chi) = \int_{\lambda = \lambda_0} (\bar{\psi} \gamma^A \bar{m}_{aA} \psi) \lambda^{-3} d\lambda,
$$

(5.11)
where $\bar{m}_{aA}$ are the generators (5.2). In keeping with the discussion following (5.5), these can be combined with the $M_{5u}$ to form the analogs in de Sitter space of the quantum field theory generators of dilations and special conformal transformations:

$$
\Phi = M_{65}, \quad K_u = 2R(M_{5u} - M_{5a}).
$$

These have as limits for $R \to \infty$ the usual forms of the conformal generators for spinor fields in Minkowski space.

The fact that the generator $F$ given by (4.11) yields conserved quantities for spinors
is a reflection of the form invariance (4.7) of $\mathcal{L}$. Equivalently, this means that the variation of the Lagrangian density is identically a divergence, without the help of the field equations, or in other words that $\mathcal{L}$ transforms as a field (see [13]). This property of $\mathcal{L}$ turns out not to be the case for scalar fields.

Instead, for scalars, with $\mathcal{L}$ given by

$$\mathcal{L} = -(1/4R^2) m_{CDX}m^{CDX} + \chi^2/R^2,$$ (5.12)

one finds that inserting the rule (5.6) with $\ell = -1$ leads to the variation

$$\delta \mathcal{L} = -iE^A[(\eta^{\alpha}/R) m_{AB} + 4i\eta_A/R] \mathcal{L} - (1/2R^3) m_{AB}(\eta^B\chi^2)$$ (5.13)

without the help of the field equations. The first term above is a divergence, since

$$[(\eta^{\alpha}/R) m_{AB} + 4i\eta_A/R] \mathcal{L} = m_{AB}(\eta^B\mathcal{L}/R).$$

Equation (5.13) means that $\mathcal{L}$ transforms in part as a scalar field with $\ell = -4$. But because of the extra term in (5.13) the version of (5.10b) for scalars reads

$$S_{\alpha A}^{CD} = (\eta^{\beta}/R) S_{AB}^{CD} - i\pi^{CD}/\eta_{A\chi} + (1/4R^3) \delta_A^C\eta^D\chi^2,$$ (5.14)

and it is this object whose divergence vanishes by virtue of the field equation (5.8). The conserved quantities formed by integrating (5.14) are

$$\Phi = M_{\delta 5} = \int_{\lambda = \lambda_0} \left[-iD_\alpha \bar{m}_{\delta 5} \chi$$

$$- (1/2R\lambda)(R^2\lambda^2 - R^2 + \chi^2/2R^2) \lambda^{-3} d^3y,$$

$$K_j = 2R(M_{\delta j} - M_{\delta j}) = \int_{\lambda = \lambda_0} \left[-iD_\alpha \bar{m}_{\delta j} \chi$$

$$+ 2(R/\lambda)(\lambda - 1) y^j_{\delta 5} \mathcal{L}_4 - (1/R\lambda) y^j_{\delta 5} \mathcal{L}_4 \lambda^{-3} d^3y,$$ (5.15)

$$K_{A} = 2R(M_{\delta A} - M_{\delta A}) = \int_{\lambda = \lambda_0} \left[-iD_\alpha \bar{m}_{\delta A} \chi$$

$$- (1/2R^3\lambda)(R^2\lambda^2 + R^2 + \chi^2) \lambda^{-3} d^3y.$$

It is only in the last quantity above that the extra term in (5.14) gives a contribution that survives when $R \rightarrow \infty$. In this limit, the quantities (5.15) go over to the usual expressions for massless scalar particles in Minkowski space [12], including the term in $K_4$ due to the Lagrangian density in Minkowski space not transforming as a field for special conformal transformations. The new feature in curved space that comes to light in (5.15) is that now $\mathcal{L}_4$ does not transform as a field either for special conformal transformations or dilations.
The presence of extra terms due to the lack of form invariance of $\mathcal{L}$ in Minkowski space under special conformal transformations was put in a more general setting than that of just scalar fields by Callan, Coleman, and Jackiw [13]. By examining the variation of $\mathcal{L}$ for general spin and invoking Poincaré invariance, they showed that if dilation invariance also held, the construction of a divergenceless special conformal current is contingent upon the field virial $(\partial \mathcal{L} / \partial \psi) \delta \psi (\delta \psi + is \psi)$ equalling a divergence identically.

The analogous procedure here is to assume rotational invariance, which from (4.13) has the form

$$m_{AB} \mathcal{L} = (\partial \mathcal{L} / \partial \chi) j_{AB} \chi + \pi^{CD} m_{CD} (j_{AB} \chi).$$

Using this relation, the variation of $\mathcal{L}$ induced by (5.6) is, in general,

$$\delta \mathcal{L} = m_{AB} (\eta^B \mathcal{L} / R) - (4i \eta_A / R) \mathcal{L} - (i \eta_A / R) (\partial \mathcal{L} / \partial \chi)$$

$$- (i \eta_A / R) \pi^{CD} m_{CD} \chi + (2i \eta_A / R) \pi^{CD} m_{AB} \chi$$

$$+ (2i \eta_A / R) \pi^{CD} m_{AB} \chi + (2i \eta_A / R) \pi^{CD} m_{AB} \chi.$$

Now in the analogous expression in Minkowski space, all the terms except the divergence and the field virial can be set equal to zero by dilation invariance alone. However, here the analogs of dilations and special conformal transformations are unified, in that their conservation depends on the conservation of the five-vector $M_{\alpha \lambda}$. In fact, it is easy to see that what would be the field virial in this case is not a divergence for scalars, so that the situation is more complicated in de Sitter space than in Minkowski space. Nevertheless, by using the particular Lagrangian density for each field, one is still able to calculate $\delta \mathcal{L}$ explicitly in each case, and so determine the correct form of the conserved current in five-space.

References

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