CMPT 881: Pseudorandomness Prof. Valentine Kabanets

Lecture 6: Random Walk on Graphs and Spectral Graph Theory

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1 Random Walk on Graphs

Theorem 1 *If G is a d-regular undirected multigraph, then*

1. *All eigenvalues of A have absolute value at most 1. So, write $\lambda_1 = 1$ and $|\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ (in descending order of absolute value). Also, Define $\lambda_i(G) = |\lambda_i|$.*
2. *G is connected iff the eigenvalue 1 has multiplicity 1.*
3. *if G is connected then G is bipartite iff -1 is also eigenvalue of A.*
4. *G is connected and nonbipartite then $|\lambda_2| \leq (1 - \frac{1}{n.d.\ diameter(\ G\ )})$.*

Proof: (1) Let λ be any eigenvalue of A and v be the corresponding eigenvector, then we have $Av = \lambda v$. Let $|v_m| = \max_i\{|v_i|\}$. Then,

$$\begin{aligned} |\lambda v_m| &= \left|\sum_{i=1}^{n} a_{mi} v_i\right| \\ |\lambda|\,|v_m| &\leq \sum_{i=1}^{n} |a_{mi}|\,|v_i| \\ &\leq \left(\sum_{i=1}^{n} |a_{mi}|\right) |v_m| \\ |\lambda|\,|v_m| &\leq v_m \end{aligned}$$

Since $|v_m| \neq 0$, we get

$$|\lambda| \leq 1.$$

(2) ***if* part**

Suppose G is disconnected. Let $I \subset V$ be a connected component of G. Define v such that

$$v_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$(Av)_i = \sum_{j=1}^{n} a_{ij} v_j = \sum_{\substack{j=1 \\ j \in I}}^{n} a_{ij} = (v)_i$$

Thus, v is an eigenvector corresponding to eigenvalue 1. We have already shown that $u = (1/n, 1/n, \ldots, 1/n)$ is eigenvector for eigenvalue 1. Now, v is linearly independent of u because I is proper subgraph of G. Hence, eigenvalue 1 has multiplicity more than 1.

***Only if* part**

Suppose v is linearly independent of $(1,1,\ldots,1)$ and $Av = v$. Define $v_m = max_i\{v_i\}$. Now, consider $v_m = \sum_{i=1}^{n} a_{mi}v_i$. If $a_{mi} \neq 0$ and $v_i \neq v_m$ for $i \neq m$ then $\sum_{i=1}^{n} a_{mi}v_i < v_m$, which is impossible. Thus, if $a_{mi} \neq 0$ then $v_i = v_m$. Define $I = \{j|v_j = v_m\}$. Since v is linearly independent of $(1,1,\ldots,1)$, I is proper subset of vertices of G and I is disconnected from rest of the graph G. If not, then there exist vertex $k \in G - I$ and vertex $l \in I$ such that $(k,l) \in E$. This would mean $a_{kl} \neq 0$ and thus $v_k = v_l$ and $k \in I$. Thus, G has more than one components.

(3) ***Only if* part**

Suppose $G = (A,B,E)$ is bipartite then define $v = (v_1, v_2, \ldots, v_n)$ where

$$v_i = \begin{cases} 1 & \text{if } i \in L \\ -1 & \text{if } i \in R \end{cases}$$

Now if $v_i \in L$, then $(Av)_i = \sum_{j=1}^{n} a_{ij}v_j = \sum_{\substack{j=1 \\ j\in R}}^{n} a_{ij}v_j = -1$. Similarly, if $v_i \in R$ then $(Av)_i = 1$. Thus, $Av = -v$ and -1 is eigenvalue of A.

***if* part**

Suppose that -1 is eigenvalue of A and v be the corresponding eigenvector, then we have $Av = -v$. Define $|v_m| = max_i\{|v_i|\}$. By using argument similar to the one in proof of part(3) we can show that if $(i,m) \in E$ then of $v_i = -v_m$ and if $(i,j) \in E, (j,m) \in E$, then $v_i = -v_j = v_m$. Thus, we can divide the vertices of G into two sets $A = \{i|v_i = v_m\}$ and $B = \{i|v_i = -v_m\}$ such that $(i,j) \notin E$ when both i and j are in A or both are in B. Thus, G is bipartite.

(4) Exercise.

■

Lemma 2 $\lambda_2(G) = \max_{x\perp u} \frac{\|Ax\|}{\|x\|}$, *where $x \perp u$ denotes x is orthogonal to u.*

Proof: Let $v_1, v_2, v_3, \ldots, v_m$ are eigenvectors of A corresponding to eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$. W.l.o.g. we can assume that $\|v_i\| = 1$. Let $x \perp u$. So, $x = c_2v_2 + c_3v_3 + \ldots + c_nv_n$ ($v_2, \ldots, v_n$ are eigenvectors other than u). Then we have,

$$\begin{aligned} x &= c_2v_2 + c_3v_3 + \ldots + c_nv_n \\ Ax &= c_2Av_2 + c_3Av_3 + \ldots + c_nAv_n \\ Ax &= c_2\lambda_2v_2 + c_3\lambda_3v_3 + \ldots + c_n\lambda_nv_n \\ \|Ax\|^2 &= (c_2\lambda_2)^2 + (c_3\lambda_3)^2 + \ldots + (c_n\lambda_nx)^2 \\ &\leq (\lambda_2(G))^2(c_2^2 + c_3^2 + \ldots + c_n^2) \\ &= (\lambda_2(G))^2(\|x\|)^2 \end{aligned}$$

Thus,

$$\begin{aligned} \|Ax\| &\leq \lambda_2(G)\|x\| \\ \lambda_2(G) &\geq \frac{\|Ax\|}{\|x\|} \end{aligned}$$

Equality is achieved when $x = v_2$. ■

1.1 Analysis of Random Walk on Graph

We said that the normalized adjacency matrix A of an undirected multigraph G defines a random walk on G. Also, this Markov chain has an unique stationary distribution u. Here, we will show that starting with any initial distribution the Markov chain converges to the stationary distribution in poly(n, d) steps.

Let π be any initial probability distribution of the Markov chain. We claim that π can be written as sum of u and $u^{\perp}$, where $u^{\perp}$ is orthogonal to u. After one step of random walk the distribution will be $A\pi$.

$$A\pi = Au + Au^{\perp}$$

Since, u is the stationary distribution

$$A\pi = u + Au^{\perp}$$

By induction,

$$A^l\pi = u + A^l u^{\perp}$$

By the previous lemma,

$$\begin{aligned}
\|Au^{\perp}\| &\leq \lambda_2(G)\|u^{\perp}\| \\
\|A^l u^{\perp}\| &\leq \lambda_2(G)^l\|u^{\perp}\| \\
\text{So,}\|A^l(\pi - u)\| &\leq \lambda_2(G)^l\|(\pi - u)\| \\
\|A^l\pi - u\| &\leq \lambda_2(G)^l\|u^{\perp}\|
\end{aligned}$$

We claim that $\|u^{\perp}\| \leq 1$.(Check this!)
Thus,

$$\|A^l\pi - u\| \leq \lambda_2(G)^l$$

If G is nonbipartite, then

$$\begin{aligned}
\|A^l\pi - u\| &\leq \left(1 - \frac{1}{n.d.\ \text{diameter}(G)}\right)^l \\
&\leq e^{\frac{-l}{nd\ \text{diameter(G)}}} \\
&\leq \epsilon,\ \text{for } l = \log(1/\epsilon).n.d.diameter(G) \text{ for all } \epsilon > 0
\end{aligned}$$

diameter of a graph on n vertices is most n. Thus, $l =$ poly(n, d).

2 Expander Graph

Informally, an expander graph is sparse, yet well connected graph. Sets of relatively small size have large number of neighbours. This is called expansion property of expander graphs.

Definition 3 *$G(V, E)$ on n vertices has the vertex expansion (k, a) if $\forall S \in V, |S| \leq k$*

$$|N(S)| \stackrel{def}{=} |\{u | \exists v \in S \text{ s.t. } (u, v) \in E\}| \geq a\,|S|$$

We will discuss only d-regular multigraphs(graphs in which every vertex has degree d).

Definition 4 *An (k, a) family of expander graphs is an infinite family $\{G_i\}$ of multigraphs which has following properties:*

1. *Every graph G_i is d-regular on n_i vertices and n_i does not grow too fast.*
2. *Every graph G_i has (k, a) vertex expansion.*

Theorem 5 *$\forall d \geq 3$, a random d-regular graph is an $(\Omega(n), d - 1.01)$ expander with high probability*

We will give the proof for bipartite graph which is simpler.

Definition 6 *A bipartite graph $G = (L, R, E)$ where $|L| = |R| = n$ has an (k, a) vertex expansion if $\forall S \subset L, |S| \leq k$, we have $|N(S)| \geq a\,|S|$.*

Theorem 7 *$\forall d \geq 3$, a random left d-regular bipartite graph on $(n + n)$ vertices will be an $(\alpha n, d - 1.01)$ expander with $p \geq 1/2$.*

Proof: In our random graph model, for every vertex $v \in L$ we pick its d neighbours at random. Fix any $k = \alpha n$ (α to be decided later). Now, A fails to be a $(k, d - 1.01)$ expander graph if $\exists S \subset V$ such that $|S| \leq k$ and $N(S) < (d - 1.01)\,|S|$. Let p be the probability that there exists a set $S \subset L, |S| = k$ s.t. $|N(S)| < k(d - 1.01)$.

$$\begin{aligned}
p &\leq \tbinom{n}{k} \Pr\left[S \subset L, |S| \leq k \text{ and } |N(S)| \leq k(d - 1.01)\right] \\
&\leq \tbinom{n}{k} \Pr\left[\text{ There are more than } 1.01k \text{ repetitions in } kd \text{ selections}\right] \\
&\leq \tbinom{n}{k} \binom{kd}{1.01k} \left(\frac{kd}{n}\right)^{1.01k} \\
&\leq \left(\frac{en}{k}\right)^{k} \left(\frac{ekd}{1.01k}\right)^{1.01k} \left(\frac{kd}{n}\right)^{1.01k}
\end{aligned}$$

By choosing α small enough, we can show that $p \leq 1/2$. ■