CMPT 881 - Pseudorandomness: Problem Set 1
Due: October 28 (at the beginning of the class)

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Reminder: you are encouraged to work in groups of two or three; however, you must turn in your own write-up and note with whom you worked. You may consult the course notes and optional texts. Please attempt all problems.

1. **Random walks and Electrical networks** Let G be a connected, undirected graph on n vertices, and let s and t be any two distinct vertices of G. For any vertex v of G, let p_v denote the probability that a random walk on G starting at the vertex v will visit s before visiting t. (Using a more picturesque language, imagine that G represents a map of some city. A drunken tourist is trying to get to his hotel, represented by the vertex s, by performing a random walk in the city. The vertex t represents a bar. For each point in the city, we are interested in the probability that our tourist will get to his hotel before he gets to the bar (where he would stay all night)).

   (a) Let $\hat{G}$ be the electrical network corresponding to the graph G (where each edge of G becomes a unit resistance). For each vertex v of $\hat{G}$, let ϕ_v denote the voltage difference between v and t, when a source of one volt is applied to $\hat{G}$ between s and t so that $\phi_s = 1$ and $\phi_t = 0$. Prove that, for each vertex v of G,

   $$p_v = \phi_v.$$

   (You will need to prove the *uniqueness* of a solution to a certain system of linear equations.)

   (b) Let G be a graph on n vertices $1, 2, \ldots, n$ lying on a line, i.e., with edges $(i, i+1)$, for $1 \leq i \leq n-1$. Using the connection between random walks and electrical networks established in the previous question, compute p_i for each $1 \leq i \leq n$ (i.e., give a formula for p_i).

   (c) Generalize the conclusion of question (i) above to the case of *weighted* graphs, where each edge e of G has some weight w_e. A random walk on weighted graphs is defined as follows: If you are at a vertex v that is connected to d neighbours with edges $e_1, \ldots, e_d$ having weights $w_1, \ldots, w_d$, respectively, then the probability of moving to the ith neighbour is w_i/W_v, where $W_v = \sum_{j=1}^{d} w_j$ is the total weight of the edges leaving v.

2. **Existence of expanders** A d-regular bipartite graph on $n + n$ vertices is a bipartite graph on the vertex set $L \cup R$, where $|L| = |R| = n$ and $L \cap R = \emptyset$, such that each vertex in L has d neighbours in R (and no neighbours in L), and each vertex in R has d neighbours in L (and no neighbours in R). Using the probabilistic method, prove that for every $d \geq 3$, there is a

family of d-regular bipartite graphs on $n + n$ vertices that are $(\alpha n, A)$-expanders, for some constant $\alpha > 0$ and constant $A > 1$, and all sufficiently large n. Try to make A as close to the degree d of the graph as possible (say, $A = d - 1.01$), at the expense of making α a very small constant. Note that your expander graph must "expand" every small subset S of vertices by a factor A, where S may contain vertices from both L and R.

3. **Eigenvalues of special graphs** For each of the following graphs G with the normalized adjacency matrices A, prove the correctness of the formula for the eigenvalues of A.

   (a) $G = K_n$, a complete graph on n vertices. The eigenvalues are $\lambda_1 = 1, \lambda_2 = \dots \lambda_n = -\frac{1}{n-1}$.

   (b) $G = Q_n$, an n-dimensional cube, i.e., a graph on vertices $V = \{v \mid v \in \{0,1\}^n\}$, where $u, v \in V$ are connected by an edge iff u and v differ in exactly one coordinate (the Hamming distance between u and v is exactly one). The eigenvalues are $\lambda_k = 1 - \frac{2k}{n}$, for $k = 0, 1, \dots, n$, with multiplicities $\binom{n}{k}$. [Hint: Consider the set of vectors $B = \{\chi_v \mid v \in \{0,1\}^n\}$, where the uth coordinate of χ_v is $\chi_v(u) = (-1)^{(v,u)}$; here $(u, v) = \sum_{i=1}^{n} u_i v_i \mod 2$ is the inner product of u and v modulo 2. Prove that B is an orthogonal basis of the 2^n-dimensional real space $\mathbb{R}^{2^n}$. Then consider what happens when the adjacency matrix of Q_n is multiplied by a vector χ_v, for each $v \in \{0,1\}^n$.]

   (c) **[Bonus]** $G = C_n$, a cycle on n vertices (i.e., $1 \to 2 \to 3 \to \cdots \to n \to 1$). The eigenvalues are $\lambda_k = \cos \frac{2\pi k}{n}$, for $k = 0, 1, \dots, n-1$.