CMPT 407/710 - Complexity Theory: Problem Set 1

Note: Please attempt all the problems. You can ask me for hints, but please avoid the temptation to just search the internet for solutions! Working through the problems by yourself will help you understand the material better, and help prepare you for the quizzes. (The problems marked by * are a bit more challenging; these problems are optional.)

1. In this question, you're asked to fill in the steps of the proof of the following result about the language $PAL = \{w \in \{0,1\}^* \mid \ w \text{ is a palindrome}\}$.
   Theorem *Any one-tape TM deciding the language PAL requires time $\Omega(n^2)$.*

   NOTE: To prove this lower bound, we need to argue that *every* one-tape TM (which may not even be known to us) that runs in time $o(n^2)$ will fail to correctly decide PAL. It is not enough to show that a particular one-tape TM for PAL takes $\Omega(n^2)$ time steps.

   For the proof, we need the notion of a *crossing sequence*. Given a computation of some one-tape TM M on an input w, let us number the tape cells as follows: the portion of the tape staring with the input w is numbered by $0, 1, 2, \dots$; the portion of the tape to the left of the input w is numbered by negative integers $-1, -2, -3, \dots$.For any integer i, we observe the behaviour of our TM as it crosses between the tape cells i and $i+1$, moving right or left across the cut. Let $S_w(i)$ denote the string that records the states of the TM and the directions of its tape head as the tape head moves between cells i and $i+1$. For example, the string $S_w(i)$ can look like $(q_1, R), (q_2, L), (q_1, R), (q_1, L)$; this means that the TM is in state q_1 as it crosses from cell i to $i+1$ for the first time, then the TM is in state q_2 as it crosses from $i+1$ to i next time, and so on.

   The string $S_w(i)$ is called a crossing sequence.

   (a) Show that for any two strings $x = x_1x_2$ and $y = y_1y_2$ such that $|x_1| = |y_1| = i$ and $S_x(i-1) = S_y(i-1)$, if a TM M accepts x and y, then it also accepts the strings x_1y_2 and y_1x_2.

   (b) Let M be any one-tape TM deciding PAL. Let x and y be any two distinct strings of length n each. Consider the inputs $X = x0^nx^R$ and $Y = y0^ny^R$, where 0^n means a string of n zeros, and x^R means the reverse of the string x. Show that for every $n < i < 2n$, it must be the case that $S_X(i) \neq S_Y(i)$.

   (c) For the TM M as in the previous item, show that there exists a constant $\epsilon > 0$ such that the following holds: For every $n < i < 2n$, the number of strings of the form $X = x0^nx^R$ (for x of length n) that have $|S_X(i)| < \epsilon n$ is less than $2^{n/2}$.

(d) Using what you've shown in the previous item, argue that there will always exist a string $X = x0^n x^R$, for large enough n, such that $|S_X(i)| \geq \epsilon n$ for every $n < i < 2n$. Conclude that the TM M will take time $\Omega(n^2)$ on that string X.

2. Show that the language $L = \{0\}$ is complete for P, under polytime reductions.

3. Show that if $\mathsf{P} = \mathsf{NP}$, then $\mathsf{EXP} = \mathsf{NEXP}$.

4. Define the class $\mathsf{LINSPACE} = \mathsf{Space}(n)$. Show that $\mathsf{LINSPACE} \neq \mathsf{P}$.

5. When we proved in class that $\mathsf{Time}(t) \subseteq \mathsf{Size}(t^2)$, we implicitly used the assumption that we are given a one-tape TM M running in time t. What happens if M is a k-tape TM running in time t? Can you still get a simulating circuit for M of size $O(t^2)$? Explain!

6. Call a Turing machine M *oblivious* if M's sequence of tape-head movements depends only on the input *size*. That is, M is oblivious if for every input $x \in \{0,1\}^n$ and $i \in \mathbb{N}$, the location of each of M's heads at the ith step of computation on input x is only a function of $|x|$ and i.

   (a) Show that every language $L \in \mathsf{Time}(t(n))$, for a time-constructible $t(n)$, can be decided by an oblivious TM in time $O(t^2(n))$.

   (b) * Using the construction of an efficient universal TM (given in Lecture 2), show that every language $L \in \mathsf{Time}(t(n))$, for a time-constructible $t(n)$, can be decided by an oblivious TM in time $O(t(n) \cdot \log t(n))$.

7. * Show that there is a universal *nondeterministic* TM U such that, given the description of a nondeterministic TM M and input x, if M runs on x in time at most t, then U simulates M on x in time at most $c_M \cdot t$, where c_M is a constant depending on the machine M (its number of tapes, alphabet size, and the number of states).

   (That is, in contrast to the *deterministic* case, the universal *nondeterministic* TM does not suffer the extra $\log t$ factor slowdown.)