Expressing Default Logic Variants in Default Logic

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Abstract

Reiter's default logic is one of the best known and most studied of the approaches to nonmonotonic reasoning. Several variants of default logic have subsequently been proposed to give systems with properties differing from the original. In this paper, we examine the relationship between default logic and its major variants. We accomplish this by translating a default theory under a variant interpretation into a second default theory, under the original Reiter semantics, wherein the variant interpretation is respected. That is, in each case we show that, given an extension of a translated theory, one may extract an extension of the original variant default logic theory. We show how constrained, rational, justified, and cumulative default logic can be expressed in Reiter's default logic. As well, we show how Reiter's default logic can be expressed in rational default logic. From this, we suggest that any such variant can be similarly treated. Consequently, we provide a unification of default logics, showing how the original formulation of default logic may express its variants. Moreover, the translations clearly express the relationships between alternative approaches to default logic. The translations themselves are shown to generally have good properties. Thus, in at least a theoretical sense, we show that these variants are in a sense superfluous, in that for any of these variants of default logic, we can exactly mimic the behaviour of a variant in standard default logic. As well, the translations lend insight into means of classifying the expressive power of default logic variants; specifically we suggest that the property of semi-monotonicity represents a division with respect to expressibility, whereas regularity and cumulativity do not.

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1 Introduction

Default logic [Reiter, 1980] is one of the best known approaches to nonmonotonic reasoning. In default logic, classical logic is augmented by *default rules* of the form $\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma}$. Such a rule is informally interpreted as "if α is true, and $\beta_1, \ldots, \beta_n$ are consistent with what is known, then conclude γ by default". An example of a default, representing the assertion "birds fly", is $\frac{Bird(x) : Fly(x)}{Fly(x)}$. Thus: "if something can be inferred to be a bird, and if that thing can be consistently assumed to fly, then infer that it does fly". The meaning of a rule then rests on notions of provability and consistency with respect to a given set of beliefs. A set of beliefs sanctioned by a set of default rules, with respect to an initial set of facts, is called an *extension* of this set of facts.

The formal definition of an extension is quite subtle (see Section 2). However, this definition has proven to be remarkably general and enduring. Consequently, much of subsequent work has concentrated on applying the formalism (see [Perrault, 1987; Baader and Hollunder, 1992; Cadoli *et al.*, 1994] for representative examples) rather than further developing it. For an exception see [Etherington, 1987b], which gives a model-theoretic characterization of extensions. The generality of the approach has also led to its being used as a means of formalising other approaches, such as inheritance networks [Etherington and Reiter, 1983] and diagnosis [Reiter, 1987]. In [Delgrande and Schaub, 2000] we suggested that default logic is an appropriate elaboration of classical logic for modelling a wide range of "commonsense" representation and reasoning problems. Full-scale implementations of default logic [Cholewiński *et al.*, 1996] have had to contend with the high complexity of reasoning in the system. However, more recently a restriction of default logic, *extended logic programs*[1] [Gelfond and Lifschitz, 1990], has received a great deal of attention, due to the availability of efficient implementations [Niemelä and Simons, 1997; Eiter *et al.*, 1997]. Finally, default logic remains a "base" general formalism in which other formalisms (such as [Poole, 1988; Giunchiglia *et al.*, 2002]) have been expressed and consequently can be compared.

The very generality of default logic means that it lacks several important properties, including *existence of extensions* [Reiter, 1980] and *cumulativity* [Makinson, 1989]. In addition, differing intuitions concerning the role of default rules have led to differing opinions concerning other properties, including *semi-monotonicity* [Reiter, 1980] and *commitment to assumptions* [Poole, 1989]. As a result, a number of modifications to the definition of a default extension have been proposed, resulting in a number of variants of default logic. Most notably these variants include *constrained default logic* [Schaub, 1992; Delgrande *et al.*, 1995], *cumulative default logic* [Brewka, 1991], *justified default logic* [Łukaszewicz, 1988], and *rational default logic* [Mikitiuk and Truszczyński, 1995].[2] In each of these variants, the definition of an extension is modified, and a system with properties differing from the original is obtained.

In this paper we examine the relationships between default logic and its variants. To accom-

[1]Extended logic programs essentially correspond to default theories in which formulas are restricted to conjunctions of literals.

[2]To be sure, there are other variants of default logic, as we later discuss. The variants covered here are arguably the best-known and studied [Antoniou, 1999]. As well, we suggest that the techniques developed here extend straightforwardly to other variants.

plish this, we make use of translations mapping a default theory under a "'variant" interpretation onto a second theory under the interpretation of the original (Reiter) approach, such that the respectively resulting extensions are in a one-to-one correspondence. We show how constrained, rational, justified, and cumulative default logic can be expressed in Reiter's default logic. In the case of the first three variant default logics, which use the language of classical logic, we add labelled formulas to the language. In the case of an *assertional default logic*, such as cumulative default logic, the situation is more complex since cumulative default logic makes use of "assertions," which extend the language of classical logic. Here we *reify* formulas; this allows us to encode the properties of assertions in classical logic. In each case we discuss properties of the underlying translation.

There has been much previous work relating default logic to other approaches to nonmonotonic reasoning, for example [Etherington, 1987a; Imielinski, 1987; Konolige, 1988; Gottlob, 1995; Janhunen, 1998; Denecker *et al.*, 2003]. Approaches such as default logic, circumscription, and autoepistemic logic were founded on varying intuitions; the aforecited references show that despite these apparently disparate intuitions, there are deep links between the approaches. The present paper does the same within the family of default logics: variant default logics are founded on divergent intuitions from the original; here we show that these seemingly divergent formalisms are nonetheless expressible by the original.

Hence we provide a unification of default logics, in that we show that the original formulation of default logic is expressive enough to subsume its variants. Thus we show that these variants are in a sense superfluous, in at least a theoretical sense, since we can exactly mimic the behaviour of any of these variants in standard default logic. Thus, for example, once one has an implementation of default logic (e.g. [Cholewiński *et al.*, 1996]), it is straightforward to obtain an implementation of a variant by implementing the translation. The reverse relation does not hold for constrained, justified, or cumulative default logic, in that one cannot express default logic in terms of these variants. However, rational default logic can be embedded in Reiter default logic, and vice versa. The translations that we provide show, in a precise sense, how each variant relates to standard default logic.

As well, our approach lends some insight into characteristics of standard default theories. For example, our translations implicitly provide specific characterisations of default theories that are guaranteed to have extensions or are guaranteed to be semi-monotonic. That is, since we map variant default logics into default logic, the theories in the image of the mapping are guaranteed to retain properties of the original variant. Further, it has been previously suggested that properties such as semi-monotonicity, regularity, and cumulativity may be used to classify default logics with respect to their expressiveness. Our results indicate however that only semi-monotonicity provides a true indication of a logic's overall expressiveness.

In the next section we introduce default logic and its variants. Since our aim is to show correspondence results, we do not discuss the properties of these default logics nor do we motivate their formulations; rather, the interested reader is referred to the cited literature. Section 3 discusses desirable properties of translations. In Section 4, we show in detail how constrained, rational, and justified default logic can be expressed in Reiter's default logic, and in Section 5 we show how cumulative default logic may be so expressed. Sections 6 and 7 provide a discussion and conclusion, respectively. Proofs of theorems are contained in an appendix.

2 Default Logic and its Variants

2.1 Default Logic

Default logic [Reiter, 1980] augments classical logic by *default rules* of the form $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$, where the constituent elements are formulas of classical propositional or first-order logic. Defaults with unbound variables are taken to stand for all corresponding instances. For simplicity, we deal just with *singular* defaults for which $n = 1$.[3] A singular rule is *normal* if β is equivalent to γ; it is *semi-normal* if β implies γ. As regards standard default logic, [Janhunen, 1999] shows that any default rule can be transformed into a set of semi-normal defaults. Moreover the great majority of applications use only semi-normal defaults, so the above assumption is a reasonable restriction. We denote the *prerequisite* α of a default $\delta = \frac{\alpha:\beta}{\gamma}$ by *Prereq*(δ), its *justification* β by *Justif*(δ) and its *consequent* γ by *Conseq*(δ). Conversely, to ease notation, in Section 4 we rely on a function δ to obtain the default rule in which a given prerequisite, justification, or consequent occurs, respectively. That is, for instance, $\delta(\mathit{Prereq}(\delta)) = \delta$. Moreover, for simplifying the technical results, we presuppose without loss of generality that default rules have unique components. We use the (unqualified) term *default logic* to refer to Reiter's original formulation; sometimes for emphasis we will redundantly refer to *standard*, or *Reiter* default logic. Variants will be referred to as *constrained* (*cumulative*, *justified*, etc.) *default logic*. Similar considerations apply to the notions of *default extension*.

As regards classical logic, the derivability operator, $\vdash$, is defined in the usual way. Accordingly, the *deductive closure* of a set S of formulas is given by $\mathit{Th}(S) = \{\phi \mid S \vdash \phi\}$.

A set of default rules D and a set of formulas W form a *default theory* (D, W) that may induce zero, one, or multiple *extensions* in the following way.

Definition 2.1 ([Reiter, 1980]) *Let (D, W) be a default theory. For any set S of formulas, let $\Gamma(S)$ be the smallest set of formulas such that*

1. $W \subseteq \Gamma(S)$,
2. $\Gamma(S) = \mathit{Th}(\Gamma(S))$,
3. *for any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in \Gamma(S)$ *and* $S \cup \{\beta\} \not\vdash \bot$ *then* $\gamma \in \Gamma(S)$.

A set of formulas E is an extension of (D, W) iff $\Gamma(E) = E$.

That is, viewing Γ as an operator, E is a fixed point of Γ. Any such extension represents a possible set of beliefs about the world at hand. For illustration, consider the default theories

$$(D_1, W_1) = \left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right) ; \quad (1)$$
$$(D_2, W_2) = \left(\left\{\frac{:B}{C}, \frac{:\neg C}{D}\right\}, \emptyset\right) . \quad (2)$$

In the literature (D_1, W_1) is often used to illustrate what is sometimes referred to as *commitment to assumption* [Poole, 1989] or *regularity* [Froidevaux and Mengin, 1994]. A default logic is

[3]Note that, with the exception of [Łukaszewicz, 1988], the variants that we deal with also employ only singular defaults.

weakly regular if each justification of an applied rule must be individually consistent with an extension; it is *strongly regular* if the justifications of applied rules must be jointly consistent with an extension. In (Reiter) default logic, (D_1, W_1) admits one extension, $Th(\{C, D\})$. Roughly speaking, B is consistent with this (purported) extension, and so the rule $\frac{:B}{C}$ is applicable, yielding C. Similarly, $\neg B$ is also consistent with this (purported) extension, and so the rule $\frac{:\neg B}{D}$ is applicable, yielding D. The application of these two rules yields the extension $Th(\{C, D\})$.

The default theory (D_2, W_2) is used to illustrate *semi-monotonicity* [Reiter, 1980]. A default logic, or class of default theories, is *semi-monotonic* just if the addition of default rules never eliminates, but rather extends or adds, new extensions. Consider first the default theory $(D_2', W_2) = \left(\left\{\frac{:\neg C}{D}\right\}, \emptyset\right)$. This theory has one extension $E_2' = Th(D)$. However, the only extension of (D_2, W_2) is $E_2 = Th(\{C\})$. E_2' fails to be an extension of (D_2, W_2) since B is consistent with E_2'; hence $\frac{:B}{C}$ is applicable, eliminating E_2' as a possible extension. Since we have $D_2' \subseteq D_2$ but $E_2' \not\subseteq E_2$, default logic fails to be semi-monotonic. Thus default logic is weakly regular and is not semi-monotonic; normal default theories on the other hand are semi-monotonic.

In the rest of this section we introduce variants of default logic, some of which will be strongly regular and some of which will be semi-monotonic.

2.2 Constrained Default Logic

In [Delgrande *et al.*, 1995] *constrained default logic* is defined. The central idea is that the justifications and consequents of a default rule jointly provide a context or assumption set for default rule application. A primary motivation for constrained default logic was to obtain a default logic that committed to its assumptions [Poole, 1989]. The definition of a *constrained extension* is as follows.

Definition 2.2 ([Delgrande *et al.*, 1995]) *Let (D, W) be a default theory. For any set of formulas T, let $\Gamma(T)$ be the pair of smallest sets of formulas (S', T') such that*

1. $W \subseteq S' \subseteq T'$,
2. $S' = Th(S')$ *and* $T' = Th(T')$,
3. *for any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta \wedge \gamma \in T'$.

A pair of sets of formulas (E, C) is a constrained extension of (D, W) iff $\Gamma(C) = (E, C)$.

The formulas in C provide a global "context", comprising a deductively closed superset of the actual extension. Defaults must be consistent with this global context in order to be applied. In our example, (D_1, W_1) has two constrained extensions, one containing C and another including D, namely, $(Th(\{C\}), Th(\{B, C\}))$ and $(Th(\{D\}), Th(\{\neg B, D\}))$. Roughly speaking, in constructing an extension, one could consider the first default, $\frac{:B}{C}$. On the assumption that this default is applicable, for any other default to be applicable, this default must have its justification not only consistent with the consequent C, but also with the justification of the first default B. Intuitively, B can be regarded as an "assumption" that must remain consistent with respect to other applicable defaults. If we consider the other possibly-applicable default, $\frac{:\neg B}{D}$, we see that this default is in fact not applicable, given the presence of $\neg B$ in the justification. Hence we obtain the extension $(\mathit{Th}(\{C\}), \mathit{Th}(\{B, C\}))$. Similar reasoning beginning with the second default yields the second extension. Accordingly, theory (D_2, W_2) has two constrained extensions, $(\mathit{Th}(\{C\}), \mathit{Th}(\{B, C\}))$ and $(\mathit{Th}(\{D\}), \mathit{Th}(\{\neg C, D\}))$.

In constrained default logic, for a default to be applicable, it must be consistent with the justifications of applied default *taken together*. If instead, for a default to be applicable, it must be consistent with the justifications of applied default *taken individually*, one obtains Łukaszewicz's approach, discussed below.

2.3 Rational Default Logic

The definition of rational default is quite close to that of constrained default logic. The following is an alternative characterisation of *rational extensions*, originally proposed in [Mikitiuk and Truszczyński, 1993], given in [Linke and Schaub, 1997]:

Definition 2.3 ([Mikitiuk and Truszczyński, 1993]) *Let (D, W) be a default theory. For any set of formulas T let $\Gamma(T)$ be the pair of smallest sets of formulas (S', T') such that*

1. $W \subseteq S' \subseteq T'$,
2. $S' = \mathit{Th}(S')$ *and* $T' = \mathit{Th}(T')$,
3. *for any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $T \cup \{\beta\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta \wedge \gamma \in T'$.

A pair of sets of formulas (E, C) is a rational extension of (D, W) iff $\Gamma(C) = (E, C)$.

This definition is the same as that of constrained default logic, except for the consistency check. As with constrained default logic, (D_1, W_1) has two rational extensions, one containing C and one including D, namely, $(\mathit{Th}(\{C\}), \mathit{Th}(\{B, C\}))$ and $(\mathit{Th}(\{D\}), \mathit{Th}(\{\neg B, D\}))$. However, theory (D_2, W_2) has only one rational extension $(\mathit{Th}(\{C\}), \mathit{Th}(\{B, C\}))$.

2.4 Justified Default Logic

Historically, justified default logic was the earliest of the variants of default logic to be proposed. A central motivation behind justified default logic was to obtain a default logic that is semi-monotonic and thus guarantees the existence of extensions. Łukaszewicz [Łukaszewicz, 1988] modifies default logic by attaching constraints to extensions in order to strengthen the applicability condition of default rules. A *justified extension* (called a *modified extension* in [Łukaszewicz, 1988]) is defined as follows.

Definition 2.4 ([Łukaszewicz, 1988]) *Let (D, W) be a default theory. For any pair of sets of formulas (S, T) let $\Gamma(S, T)$ be the pair of smallest sets of formulas S', T' such that*

1. $W \subseteq S'$,
2. $\mathit{Th}(S') = S'$,

3. *for any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $S \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ *for every* $\eta \in T \cup \{\beta\}$ *then* $\gamma \in S'$ *and* $\beta \in T'$.

A set of formulas E *is a justified extension of* (D, W) *for a set of formulas* J *iff* $\Gamma(E, J) = (E, J)$.

So a default rule $\frac{\alpha:\beta}{\gamma}$ applies if all justifications of other applying default rules are consistent with the considered extension E and the consequent γ, and if additionally γ and β are consistent with E. Unlike the contextual information in constrained default logic and rational default logic, the set of justifications J need not be deductively closed nor consistent.

In our examples, (D_1, W_1) has one justified extension, $(Th(\{C, D\}), \{B, \neg B\})$. However, theory (D_2, W_2) has two justified extensions, one with C and one containing D, or more precisely, $(Th(\{C\}), \{B\})$ and $(Th(\{D\}), \{\neg C\})$.

We summarise our running examples in Table 1. For simplicity, we describe each extension by the consequents of its generating default rules.

default logic	(D_1, W_1)		(D_2, W_2)	
standard	C, D		C	
constrained	C	D	C	D
rational	C	D	C	
justified	C, D		C	D

Table 1: Summary of results obtained from default theories (D_1, W_1) and (D_2, W_2).

2.5 Cumulative Default Logic

Brewka [Brewka, 1991] describes a variant of default logic where the applicability condition for default rules is strengthened, and the justification for adopting a default conclusion is made explicit. The intent behind cumulative default logic was to obtain a default logic that satisfied the principle of cumulativity and strong regularity, where cumulativity is the property wherein the addition of a derived conclusion to a set of facts does not change the set of conclusions. In order to keep track of implicit assumptions, Brewka introduces *assertions*, or formulas labeled with the set of justifications and consequents of the default rules which were used for deriving them. Intuitively, assertions represent formulas along with the reasons for believing them.

Definition 2.5 ([Brewka, 1991]) *Let* $\alpha, \gamma_1, \ldots, \gamma_m$ *be formulas. An assertion* ξ *is any expression of the form* $\langle \alpha, \{\gamma_1, \ldots, \gamma_m\} \rangle$, *where* $\alpha = Form(\xi)$ *is called the asserted formula and the set* $\{\gamma_1, \ldots, \gamma_m\} = Supp(\xi)$ *is called the support of* α.[4]

We let $\mathcal{A}$ denote the set of assertions over a given language.

To correctly propagate the supports, the classical inference relation is extended as follows.

[4]The two projections extend to sets of assertions in the obvious way. We sometimes misuse *Supp* for denoting the support of an asserted formula, e.g. $\langle \alpha, Supp(\alpha) \rangle$.

Definition 2.6 ([Brewka, 1991]) *Let $\mathcal{S}$ be a set of assertions. Then $\widehat{Th}(\mathcal{S})$, the set of assertional consequences of $\mathcal{S}$, is the smallest set of assertions such that*

1. $\mathcal{S} \subseteq \widehat{Th}(\mathcal{S})$,
2. *if* $\xi_1, \ldots, \xi_n \in \widehat{Th}(\mathcal{S})$ *and* $Form(\xi_1), \ldots, Form(\xi_n) \vdash \gamma$ *then* $\langle \gamma, Supp(\xi_1) \cup \cdots \cup Supp(\xi_n) \rangle \in \widehat{Th}(\mathcal{S})$.

An *assertional default theory* is a pair $(D, \mathcal{W})$, where D is a set of default rules and $\mathcal{W}$ is a set of assertions. An *assertional extension* is defined as follows.

Definition 2.7 ([Brewka, 1991]) *Let $(D, \mathcal{W})$ be an assertional default theory. For any set of assertions $\mathcal{S}$ let $\Gamma(\mathcal{S})$ be the smallest set of assertions $\mathcal{S}'$ such that*

1. $\mathcal{W} \subseteq \mathcal{S}'$,
2. $\widehat{Th}(\mathcal{S}') = \mathcal{S}'$,
3. *for any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\langle \alpha, Supp(\alpha) \rangle \in \mathcal{S}'$ *and* $Form(\mathcal{S}) \cup Supp(\mathcal{S}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \in \mathcal{S}'$.

A set of assertions $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ iff $\Gamma(\mathcal{E}) = \mathcal{E}$.

For illustration, consider the assertional default theory (often used for illustrating the failure of *cumulativity* [Makinson, 1989] in default logic)

$$(D_3, W_3) \quad = \quad \left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A} \right\}, \emptyset \right). \tag{3}$$

This theory has one assertional extension, including $\langle A, \{A\} \rangle$ as well as $\langle A \vee B, \{A\} \rangle$. Adding the latter assertion to the set of assertional facts yields the assertional default theory

$$(D_4, W_4) \quad = \quad \left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A} \right\}, \{ \langle A \vee B, \{A\} \rangle \} \right) \tag{4}$$

which has the same assertional extension. Note that without the support $\{A\}$ for $A \vee B$, one obtains a second assertional extension with $\langle \neg A, \{\neg A\} \rangle$. This is what happens in the previously-described default logics.

It is well-known that cumulative and constrained extensions are equivalent (with respect to asserted consequences of default rules), whenever the underlying facts contain no support. Similar relationships are given among original and Q-default logic [Giordano and Martinelli, 1994], justified and affirmative [Linke and Schaub, 1997], rational and CA-default logic [Giordano and Martinelli, 1994], respectively (cf. [Linke and Schaub, 1997]).

3 A Note on Translations between Default Theories

Translation functions provide a means of comparing formalisms. Previously such functions have been used to compare the expressive power of different approaches, in that it may be possible to show that a translation involving one approach suitably captures a second. Here we translate a default theory under one interpretation into a second default theory under another interpretation. Since many variant approaches to default logic share the same syntax as regular default logic, when we refer to a default theory below, we will implicitly mean a (syntactic) default theory along with an understood semantics. Thus stating that (D, W) is a default theory will come with the understanding that (D, W) is a default theory under a specific interpretation, whether constrained, justified, or whatever.

The success of our endeavour will be measured in part by properties of our translation functions. To this end, there are various desiderata that can be specified for a translation function. In this paper, we adopt (with modifications) three criteria proposed by Tomi Janhunen [Janhunen, 1999], who has investigated translations among specific subclasses of Reiter's default logic; as well we use a version of monotonicity that is adapted for default theories. These desiderata are given as follows. We understand $(D_1, W_1) \subseteq (D_2, W_2)$ to mean $D_1 \subseteq D_2$ and $W_1 \subseteq W_2$.

Definition 3.1 *Let (D, W) be a default theory where W is a set of formulas (or, in Section 5, assertions) in some language $\mathcal{L}$, and D is a set of default rules. A function $\mathcal{T} : DL_1 \rightarrow DL_2$, where DL_1 and DL_2 are classes of default theories, is:*

1. faithful *iff for all $(D, W) \in DL_1$, the consistent extensions of (D, W) and $\mathcal{T}((D, W))$ are in a one-to-one correspondence and coincide up to the propositional language of W;*
2. polynomial *iff for $(D, W) \in DL_1$ the time required to compute $\mathcal{T}((D, W))$ is polynomial in the size of D and W;*
3. modular *iff for all $(D, W) \in DL_1$, we have $\mathcal{T}((D, W)) = (D', W' \cup \mathcal{T}(\emptyset, W))$ where $\mathcal{T}((D, \emptyset)) = (D', W')$;*
4. monotonic *iff: if $D_1 \subseteq D_2$ and $W_1 \subseteq W_2$ then $\mathcal{T}((D_1, W_1)) \subseteq \mathcal{T}((D_2, W_2))$.*

In a faithful translation, (D, W) is a theory under one particular interpretation and $\mathcal{T}((D, W))$ is a theory under another; faithfulness specifies that there is a one-to-one correspondence between extensions of these theories, each under its own interpretation. This criterion extends the notion of faithfulness in [Janhunen, 1999] to different systems of default logic. Polynomiality reflects a certain, coarse notion of efficiency in the translation; it is the same as in [Janhunen, 1999]. The intent of modularity is to specify that the rules in D can be translated independently of W; thus the translation of D does not need to be recomputed when W is modified. We draw the reader's attention to the fact that we generalise the notion of modularity in [Janhunen, 1999], which would require that $\mathcal{T}((\emptyset, W)) = W$. If a translation is monotonic, then a default theory can be translated incrementally.

Of these criteria, faithfulness is essential (otherwise we have not captured one default logic in another), while polynomiality (and low-order polynomiality at that) would be required for any practical implementation. Modularity and monotonicity of the translation would be similarly desirable in a practical application; as well they reflect a certain "tightness" in the relationship between two systems.

Other translation schemes can be found in [Marek and Truszczyński, 1993], where among others the notion of semi-representability is introduced. This concept deals with the representation of default theories within restricted subclasses of default theories over an extended language. Although semi-representability adheres to a fixed interpretation of default logic, one can view our results as semi-representation results among different interpretations of default theories.

4 Correspondence with Constrained, Rational, and Justified Default Logic

This section presents encodings for representing major variant default logics in Reiter's default logic. For a default theory Δ, we produce a translated theory $\mathcal{T}_x\Delta$, such that there is a one-to-one correspondence between the extensions of Δ in x-default logic and (standard) extensions of $\mathcal{T}_x\Delta$. We begin with constrained and rational default logic, whose encoding is less involved, then consider that of justified default logic. Section 5 addresses cumulative default logic, which requires a translation of a differing form.

4.1 Correspondence with Constrained Default Logic

For a language $\mathcal{L}$ over alphabet $\mathcal{P}$, let $\mathcal{L}'$ be the language over $\mathcal{P}' = \{p' \mid p \in \mathcal{P}\}$ (so implicitly there is an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$). For a formula α, let α' be the formula obtained by replacing any symbol $p \in \mathcal{P}$ by p'; in addition define for a set W of formulas, $W' = \{\alpha' \mid \alpha \in W\}$.

Definition 4.1 *For default theory (D, W), define $\mathcal{T}_c((D, W)) = (D_c, W_c)$ where*

$$W_c = W \cup W' \qquad \textit{and} \qquad D_c = \left\{ \tfrac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D \right\} .$$

Informally, we retain the justification of an applied default rule in an extension, but as a primed formula; this set of primed formulas then corresponds to the set C in Definition 2.2. Thus we essentially encode Definition 2.2 in a standard default theory. Other variants of default logic are similarly encoded, although sometimes in a somewhat more complex formulation.

For our examples in (1) and (2), we obtain:

$$\begin{aligned} \mathcal{T}_c((D_1, W_1)) &= \left(\left\{ \tfrac{: B' \wedge C'}{C \wedge B' \wedge C'}, \tfrac{: \neg B' \wedge D'}{D \wedge \neg B' \wedge D'} \right\}, \emptyset\right) \\ \mathcal{T}_c((D_2, W_2)) &= \left(\left\{ \tfrac{: B' \wedge C'}{C \wedge B' \wedge C'}, \tfrac{: \neg C' \wedge D'}{D \wedge \neg C' \wedge D'} \right\}, \emptyset\right) . \end{aligned}$$

$\mathcal{T}_c((D_1, W_1))$ yields two extensions in standard default logic, $Th(\{C \wedge B' \wedge C'\})$ as well as $Th(\{D \wedge \neg B' \wedge D'\})$. Analogously, we obtain two extensions from $\mathcal{T}_c((D_2, W_2))$, viz. $Th(\{C \wedge B' \wedge C'\})$ and $Th(\{D \wedge \neg C' \wedge D'\})$.

We have the following results relating constrained entensions and the translation $\mathcal{T}_c$.

Theorem 4.1 *For a default theory (D, W), we have that*

1. *if (E, C) is a constrained extension of (D, W) then $Th(E \cup C')$ is an extension of $\mathcal{T}_c((D, W))$;*
2. *if F is an extension of $\mathcal{T}_c((D, W))$ then $(F \cap \mathcal{L}, \{\varphi \mid \varphi' \in F \cap \mathcal{L}'\})$ is a constrained extension of (D, W).*

Theorem 4.2 *The constrained extensions of a default theory (D, W) and the extensions of the translation $\mathcal{T}_c((D, W))$ are in a one-to-one correspondence.*

The theorem asserts that the translation $\mathcal{T}_c$ is faithful. As well, it can be observed that $\mathcal{T}_c$ is polynomial (in fact linear), modular, and monotonic.

4.2 Correspondence with Rational Default Logic

As expected, the mapping of rational default logic into standard default logic is close to that of constrained default logic:

Definition 4.2 *For default theory (D, W), define $\mathcal{T}_r((D, W)) = (D_r, W_r)$ where*

$$W_r = W \cup W' \qquad \textit{and} \qquad D_r = \left\{ \frac{\alpha : \beta'}{\gamma \wedge (\beta' \wedge \gamma')} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

As before, the consequent of rules in D_r encodes the formulas in a rational extension (Definition 2.3). For our examples in (1) and (2), we obtain:

$$\begin{aligned} \mathcal{T}_r((D_1, W_1)) &= \left(\left\{ \frac{: B'}{C \wedge B' \wedge C'}, \frac{: \neg B'}{D \wedge \neg B' \wedge D'} \right\}, \emptyset \right) \\ \mathcal{T}_r((D_2, W_2)) &= \left(\left\{ \frac{: B'}{C \wedge B' \wedge C'}, \frac{: \neg C'}{D \wedge \neg C' \wedge D'} \right\}, \emptyset \right) . \end{aligned}$$

As with $\mathcal{T}_c((D_1, W_1))$, theory $\mathcal{T}_r((D_1, W_1))$ yields two extensions, one containing $C \wedge B' \wedge C'$ and the other containing $D \wedge \neg B' \wedge D'$. In contrast to $\mathcal{T}_c((D_2, W_2))$, however, we obtain a single extension from $\mathcal{T}_r((D_2, W_2))$, containing $C \wedge B' \wedge C'$.

We have the following result.

Theorem 4.3 *For a default theory (D, W), we have that*

1. *if (E, C) is a rational extension of (D, W) then $Th(E \cup C')$ is an extension of $\mathcal{T}_r((D, W))$;*
2. *if F is an extension of $\mathcal{T}_r((D, W))$ then $(F \cap \mathcal{L}, \{\varphi \mid \varphi' \in F \cap \mathcal{L}'\})$ is a rational extension of (D, W).*

As with Theorem 4.2, one can show that the extensions of a default theory (D, W) and the translation $\mathcal{T}_r((D, W))$ are in a one-to-one correspondence. Similar to $\mathcal{T}_c$, we have that $\mathcal{T}_r$ is faithful, polynomial, modular, and monotonic.

4.3 Correspondence with Justified Default Logic

Define for a language $\mathcal{L}$ over alphabet $\mathcal{P}$ and some set S, the family $(\mathcal{L}^s)_{s \in S}$ of languages over $\mathcal{P}^s = \{p^s \mid p \in \mathcal{P}\}$ for $s \in S$. For $\alpha \in \mathcal{L}$ and $s \in S$, let α^s be the formula obtained by replacing every symbol $p \in \mathcal{P}$ in α by p^s; in addition define for a set W of formulas, $W^s = \{\alpha^s \mid \alpha \in W\}$.

In what follows, we let the set of default rules D induce copies of the original language.

Definition 4.3 *For default theory (D, W), define $\mathcal{T}_j((D, W)) = (D_j, W_j)$ where*

$$W_j = W \cup \bigcup_{\zeta \in D} W^\zeta \text{ and } D_j = \left\{ \frac{\alpha : (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)}{\gamma \wedge (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)} \;\middle|\; \delta = \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

For simplicity, we write $\beta = \mathit{Justif}^\circ(\delta)$ whenever $\mathit{Justif}(\delta) = (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)$.

Abbreviating the two default rules in both examples, (1) and (2), by $\delta 1, \delta 2$ and $\delta 1, \delta 4$, respectively, we get (after removing duplicates):

$$\begin{aligned} \mathcal{T}_j((D_1, W_1)) &= \left(\left\{ \frac{: B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}, \frac{: \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}}{D \wedge \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}} \right\}, \emptyset \right) \\ \mathcal{T}_j((D_2, W_2)) &= \left(\left\{ \frac{: B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}, \frac{: \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}}{D \wedge \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}} \right\}, \emptyset \right) \end{aligned}$$

In standard default logic, $\mathcal{T}_j((D_1, W_1))$ results in one extension containing C, D, $B^{\delta 1}, C^{\delta 1}, D^{\delta 1}$, along with $\neg B^{\delta 2}, C^{\delta 2}, D^{\delta 2}$. Unlike this, $\mathcal{T}_j((D_2, W_2))$ gives two extensions, one with C, $B^{\delta 1}$, $C^{\delta 1}$, $C^{\delta 4}$ and another including D, $\neg C^{\delta 4}$, $D^{\delta 4}$, $D^{\delta 1}$.

We have the following general result.

Theorem 4.4 *For a default theory (D, W), we have that*

1. *if (E, J) is a justified extension of (D, W) then*

$$F = \mathit{Th}\left(E \;\cup\; \bigcup_{\zeta \in D} E^\zeta \;\cup\; \bigcup_{\beta \in J} \{\beta^{\delta(\beta)}\} \right)$$

is an extension of $\mathcal{T}_j((D, W))$;

2. *if F is an extension of $\mathcal{T}_j((D, W))$ then $(F \cap \mathcal{L}, J)$ is a justified extension of (D, W), where $J = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta) \text{ and } \delta \in \mathit{GD}(\mathcal{T}_j((D, W)), F)\}$.*

$\mathit{GD}(\mathcal{T}_j((D, W)), F)$ gives the set of default rules generating F; see Definition A.1 for a formal definition.

In analogy to Theorem 4.2, one can show that the extensions of a default theory (D, W) and the translation $\mathcal{T}_j((D, W))$ are in a one-to-one correspondence. The translation $\mathcal{T}_j$ is faithful, polynomial, and modular. However, we note that while polynomial, the translation results in a quadratic increase in the size of a theory; this would add a not insignificant overhead in the computation of a translated (standard) theory as compared to the original (justified default logic) theory. As well the translation is not monotonic; specifically, in general we obtain that $\mathcal{T}_j((D, W)) \not\subseteq \mathcal{T}_j((D \cup D', W))$.

4.4 Correspondence with (Standard) Default Logic

We can show that there is a self-embedding for standard default logic to standard default logic, using the encoding of the previous subsection:

Definition 4.4 *For default theory* (D, W)*, define* $\mathcal{T}_d((D, W)) = (D_d, W_d)$ *where*

$$W_d = W \cup \bigcup_{\zeta \in D} W^\zeta \text{ and } D_d = \left\{ \frac{\alpha : \beta^\delta}{\gamma \wedge (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)} \;\middle|\; \delta = \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

One can show that this mapping results in extensions that are in a one-to-one correspondence to those of the original theory. That is, one obtains a result similar to that in Theorem 4.4. The translation $\mathcal{T}_d$ then is faithful and well as being polynomial and modular. However it is not monotonic, since elements of W_d depend in part on D.

Contrasting this embedding with the one in Definition 4.3 also illustrates in a different fashion how default logic and justified default logic relate. As well, this translation allows for embedding standard default logic into rational default logic, as made precise next.

Theorem 4.5 *For a default theory* (D, W)*, we have that*

1. *if* E *is an extension of* (D, W) *then* (F, F) *is a rational extension of* $\mathcal{T}_d((D, W))$*, where* $F = Th\Big(E \cup \bigcup_{\zeta \in D} E^\zeta \cup \bigcup_{\delta \in GD((D,W),E)} \{Justif(\delta)^\delta\}\Big)$*;*
2. *if* (F, F) *is a rational extension of* $\mathcal{T}_d((D, W))$ *then* $F \cap \mathcal{L}$ *is an extension of* (D, W)*.*

As before, one can show that the extensions of a default theory (D, W) and the translation $\mathcal{T}_d((D, W))$ are in a one-to-one correspondence.

For our examples in (1) and (2), we get:

$$\begin{aligned} \mathcal{T}_d((D_1, W_1)) &= \left(\left\{ \frac{: B^{\delta 1}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}, \frac{: \neg B^{\delta 2}}{D \wedge \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}} \right\}, \emptyset\right) \\ \mathcal{T}_d((D_2, W_2)) &= \left(\left\{ \frac{: B^{\delta 1}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}, \frac{: \neg C^{\delta 4}}{D \wedge \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}} \right\}, \emptyset\right) . \end{aligned}$$

In contrast to the two rational extensions obtained from (D_1, W_1), theory $\mathcal{T}_d((D_1, W_1))$ results in one rational extension containing C, D, $B^{\delta 1}, C^{\delta 1}, D^{\delta 1}$, and $\neg B^{\delta 2}, C^{\delta 2}, D^{\delta 2}$. As well, $\mathcal{T}_d((D_2, W_2))$ gives one rational extension containing C, $B^{\delta 1}$, $C^{\delta 1}$, $C^{\delta 4}$.

Note that a corresponding mapping into justified or constrained default logic is impossible; this is not a matter of the specific translation but rather a principal impossibility.

Theorem 4.6 *There is no mapping* $\mathcal{T}$ *such that for any default theory* (D, W)*, we have that the extensions of* (D, W) *are in a one-to-one correspondence with the constrained/justified extensions of* $\mathcal{T}((D, W))$*.*

To see this, consider theory $\left(\left\{\frac{: B}{\neg B}\right\}, \emptyset\right)$, having *no* extension. On the other hand, it is well known that every default theory has at least *one* justified and constrained extension [Łukaszewicz, 1988; Delgrande *et al.*, 1995].

5 Correspondence with Cumulative Default Logic

This section presents encodings for representing cumulative default logic and cumulative extensions in default logic. The approach here is significantly different from that of the previous section, in large part because cumulative default logic deals with *assertions*, which encode those formulas that an asserted consequent depends upon. We first provide a translation that directly encodes assertions and assertional default theories in standard default logic, using reified formulas. Second we provide another translation that makes use of known correspondences between constrained default logic and cumulative default logic.

In order to be able to talk about an assertion $\langle \alpha, \{\beta_1, \ldots, \beta_n\}\rangle \in \mathcal{A}$ within a (classical, logical) theory, an assertion is ***reified***[5] [McCarthy, 1979] as an atomic formula $\langle \cdot, \cdot \rangle^{re}$, where each argument is a reified formula that does not contain an instance of $\langle \cdot, \cdot \rangle^{re}$. Thus the assertion $\langle \alpha, \{\beta_1, \ldots, \beta_n\}\rangle$ is represented in the object language as the reified formula $\langle \alpha, \beta_1 \wedge \cdots \wedge \beta_n \rangle^{re}$.[6] Let $\mathcal{L}^{re}$ be the set of reified assertions. So that translated assertions have appropriate properties, we employ a set of formulas Ax_{re} axiomatising the reified formulas:

Definition 5.1 *Ax_{re} is the least set containing instances of the following schemata:*

1. *If* $\vdash \alpha$ *then* $\langle \alpha, \emptyset \rangle^{re} \in Ax_{re}$.
2. $(\beta_1 \equiv \beta_2) \supset (\langle \alpha, \beta_1 \rangle^{re} \equiv \langle \alpha, \beta_2 \rangle^{re})$.
3. $\langle \alpha, \gamma \rangle^{re} \wedge \langle \alpha \supset \beta, \psi \rangle^{re} \supset \langle \beta, \psi \wedge \gamma \rangle^{re}$.

We have the following analogue of Definition 2.6:

Theorem 5.1 *If* $\langle \alpha_1, \beta_1 \rangle^{re}, \langle \alpha_2, \beta_2 \rangle^{re} \in R$ *and* $\{\alpha_1, \alpha_2\} \vdash \gamma$ *then* $R \cup Ax_{re} \vdash \langle \gamma, \beta_1 \wedge \beta_2 \rangle^{re}$.

From this we establish a correspondence between extensions of cumulative default logic and default logic. We first define correspondences between assertions and formulas of classical logic.

Definition 5.2
For $\mathcal{R} \subseteq \mathcal{A}$, define

$$\begin{aligned} Re\,(\mathcal{R}) &= \{\langle \alpha, \beta \rangle^{re} \mid \langle \alpha, \beta \rangle \in \mathcal{R}\}. \\ Re^{+}(\mathcal{R}) &= Re\,(\mathcal{R}) \cup Form(\mathcal{R}) \cup Supp(\mathcal{R}) \cup Ax_{re}. \end{aligned}$$

Definition 5.3
For R a set of formulas, define

$$Re^{-1}(R) = \{\langle \alpha, \beta \rangle \mid \langle \alpha, \beta \rangle^{re} \in R\}.$$

[5] In Artificial Intelligence, a common use of reification is to assert that a particular fact or formula α is true at some state s, given perhaps by $Holds(\alpha, s)$. α is then a term in the (classical, first-order) theory, maybe best thought of as a string denoting the underlying formula. Thus for a formula $Holds(p \wedge q, s)$, $\wedge$ here would be an infix function. Appropriate "behaviour" of this function then needs to be given as an axiom for the theory; for example $Holds(p \wedge q, s) \supset Holds(p, s)$. Consequently we require Definition 5.1 so that reified assertions have the right properties.

[6] We understand empty components, such as the support in $\langle \alpha, \emptyset \rangle^{re}$, to be interpreted as $\top$.

Definition 5.4 *For assertional default theory $(D, \mathcal{W})$, define $\mathcal{T}_a((D, \mathcal{W})) = (D_a, W_a)$ where*

$$W_a = Re^+(\mathcal{W}) \qquad \textit{and} \qquad D_a = \left\{ \frac{\langle \alpha, \psi \rangle^{re} : \beta \wedge \gamma}{\langle \gamma, \psi \wedge \beta \wedge \gamma \rangle^{re} \wedge \beta \wedge \gamma} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D,\ \psi \in \mathcal{L} \right\}.$$

In an assertional default theory, the set of defaults is syntactically no different than defaults in a Reiter default theory; however, the world knowledge $\mathcal{W}$ and resulting extensions are composed of sets of assertions. In a translated theory, reified assertions appear as components of (translated) defaults, in the prerequisites and consequents. Note though that the consistency check, in the justification, remains unaffected. In fact, the treatment of $\beta \wedge \gamma$ in Definition 5.4 is identical to that of $\beta' \wedge \gamma'$ in Definition 4.1. This translation then nicely shows that only the support of (reified) assertions is needed for keeping track of underlying assumptions when applying a default rule.

Consider our examples in (3) and (4):

$$\begin{aligned}
\mathcal{T}_a((D_3, W_3)) &= \left(\left\{ \frac{\langle \top, \psi \rangle^{re} : A}{\langle A, \psi \wedge A \rangle^{re} \wedge A}, \frac{\langle A \vee B, \psi \rangle^{re} : \neg A}{\langle \neg A, \psi \wedge \neg A \rangle^{re} \wedge \neg A} \;\middle|\; \psi \in \mathcal{L} \right\}, \emptyset \right) \\
\mathcal{T}_a((D_4, W_4)) &= \left(\left\{ \frac{\langle \top, \psi \rangle^{re} : A}{\langle A, \psi \wedge A \rangle^{re} \wedge A}, \frac{\langle A \vee B, \psi \rangle^{re} : \neg A}{\langle \neg A, \psi \wedge \neg A \rangle^{re} \wedge \neg A} \;\middle|\; \psi \in \mathcal{L} \right\}, \right. \\
&\qquad\qquad \left. \{\langle A \vee B, \{A\} \rangle^{re}\} \cup \{A \vee B\} \cup \{A\} \right)
\end{aligned}$$

Both theories $\mathcal{T}_a((D_3, W_3))$ and $\mathcal{T}_a((D_4, W_4))$ yield one extension in standard default logic, containing $\langle A, \{A\} \rangle^{re}$.

We have the following general result.

Theorem 5.2 *For an assertional default theory $(D, \mathcal{W})$, we have that*

1. *if $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$, then $Th\big(Re^+(\mathcal{E})\big)$ is an extension of $\mathcal{T}_a((D, \mathcal{W}))$;*
2. *if E is an extension of $\mathcal{T}_a((D, \mathcal{W}))$, then $Re^{-1}(E)$ is an assertional extension of $(D, \mathcal{W})$.*

Similar to the previous results, we also have a one-to-one correspondence between the extensions of a default theory and the extensions of the translation. Strictly speaking the translation is not faithful, since the original theory is expressed in terms of assertions, whereas the image under the translation is expressed in terms of reified formulas. However this technical difficulty is easily skirted if we agree that assertions in cumulative default logic are in fact represented as reified formulas, in which case an extension of the translated theory can be projected onto the language of the original theory.

However the translation $\mathcal{T}_a$ is clearly not polynomial. As given, Definition 5.4 yields an infinite number of defaults (due to the presence of ψ in the formula schemata). We can nonetheless work with a finite theory in the propositional case, by the expedient of noting that over the language of a (finite) assertional default theory there will be a finite alphabet of mentioned symbols, and a finite set of sets of formulas that are equivalent. (That is, the set of formulas on a finite alphabet can be partitioned into sets of equivalent formulas, and there will be a finite number of these sets.) We then replace a formula in a set of logically equivalent formulas by some canonical representative. Consequently the translated theory will be exponentially larger in size than the original.

The translation *is* modular and monotonic, desirable properties that nonetheless are overshadowed by the non-polynomiality of the translation. As well, it is not at all clear how a direct translation from cumulative default logic to default logic can avoid this exponential blowup. However, there are known correspondences between constrained default logic and cumulative default logic, and so we describe next a second translation that makes use of this correspondence and avoids the exponential blowup in the translation.

In [Delgrande *et al.*, 1995] it was shown that there is a one-to-one correspondence between extensions of a constrained default theory (D, W) and the cumulative default theory $(D, \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\})$. [Schaub, 1993] extends this to a one-to-one correspondence between *preconstrained* default theories[7] and arbitrary assertional default theories; as well it is shown that preconstrained theories can be expressed by standard constrained theories. Based on these results we define the following.

Definition 5.5 *Let $(D, \mathcal{W})$ be an assertional default theory. Define $\mathcal{T}_{cc}((D, \mathcal{W})) = (D_{cc}, W_{cc})$ where n is a new propositional symbol[8] not occurring in D, $\mathcal{W}$, and*

$$W_{cc} = \mathit{Form}(\mathcal{W}) \cup \{n \equiv (\wedge \mathit{Supp}(\mathcal{W}))\} \qquad \textit{and} \qquad D_{cc} = \left\{ \tfrac{\alpha : \beta \wedge n}{\gamma} \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D \right\} \cup \left\{ \tfrac{: n}{\top} \right\}.$$

The following is a corollary to Theorems 2.1 and 3.2 of [Schaub, 1993].

Theorem 5.3 *Let $(D, \mathcal{W})$ be an assertional default theory and $(D_{cc}, W_{cc}) = \mathcal{T}_{cc}((D, \mathcal{W}))$.*

1. *If (E, C) is a constrained extension of (D_{cc}, W_{cc}), then there is an assertional extension $\mathcal{E}$ of $(D, \mathcal{W})$ such that $E = \mathit{Form}(\mathcal{E})$ and $C = \mathit{Th}(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E}))$.*

2. *If $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ then*

$$(\mathit{Th}(\mathit{Form}(\mathcal{E}) \cup \{n \equiv (\wedge \mathit{Supp}(\mathcal{W}))\}), \mathit{Th}(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E}) \cup \{n\}))$$

   is a constrained extension of (D, W).

We thus get a one-to-one correspondence between assertional extensions and constrained extensions (modulo the introduced propositional symbol n) for corresponding theories. The composed translation $\mathcal{T}_{ac} = \mathcal{T}_{cc} \circ \mathcal{T}_c$ then gives us a second translation from cumulative default logic into default logic, mediated by a translation to constrained default logic. We observe that $\mathcal{T}_{cc}$ is not faithful, since we lose the association of supports of a formula in an extension under the translation; consequently neither is $\mathcal{T}_{ac}$ faithful. However, the asserted formulas (i.e. disregarding supports) are the same in the corresponding extensions of $(D, \mathcal{W})$ and $\mathcal{T}_{cc}((D, \mathcal{W}))$. Hence we obtain a limited faithfulness result here, with respect to the asserted formulas. Both $\mathcal{T}_{cc}$ and $\mathcal{T}_c$ are polynomial (again, linear), modular, and monotonic; hence $\mathcal{T}_{ac}$ is also linear, modular, and monotonic.

[7] A preconstrained default theory is a constrained default theory, but where a set of *constraints* is given in the specification of a theory. A preconstrained theory is of the form (D, W, C_P) where D, W are as before, and Definition 2.2 is modified so that C contains C_P.

[8] The use of n is to simply restrict the increase in theory size to a constant factor.

6 Discussion

We have obtained, for the most part, satisfactory translations of variants of default logic into default logic, as well as a translation of default logic into rational default logic. Results concerning properties of our translations are summarised in Table 2.

Embedding		Translation	Property			
From	To		Faithful	Polynomial	Modular	Monotonic
constrained	standard	$\mathcal{T}_c$	$\surd$	linear	$\surd$	$\surd$
rational	standard	$\mathcal{T}_r$	$\surd$	linear	$\surd$	$\surd$
justified	standard	$\mathcal{T}_j$	$\surd$	quadratic	$\surd$	
standard	rational	$\mathcal{T}_d$	$\surd$	quadratic	$\surd$	
cumulative	standard	$\mathcal{T}_a$	$\surd$	exponential	$\surd$	$\surd$
cumulative	standard	$\mathcal{T}_{ac}$	$\surd$ (wrt formulas)	linear	$\surd$	$\surd$

Table 2: Summary of translations.

Translating cumulative default logic into standard default logic is clearly the most problematic of the translations we consider. This is primarily due to the use of *assertions*, which record the support of an asserted formula. Hence a direct encoding (implemented by our $\mathcal{T}_a$) appears to require an exponential increase in size of a translated theory, to allow for all possible supports. We also obtain an indirect translation $\mathcal{T}_{ac}$, making use of a known correspondence with constrained default logic to obtain a translation with good properties, except that faithfulness with respect to the supports of a formula is lost. Thus each translation has its pros and cons. Of the other non-linear translations, it may be possible to improve on the provided quadratic bound, but it is not clear to us how such an improvement could be obtained.

It should be noted that the various translations are not arbitrary, but rather deal with two issues. The first concerns how consistency is handled in a default logic, while the second deals with the nature of what is asserted (whether a formula or an assertion). The general form of the translation schemes dealing with these aspects can be illustrated as follows:

1. $$\frac{\alpha\ :\ \beta}{\gamma} \quad \mapsto \quad \frac{\alpha\ :\ c(\beta)}{\gamma \wedge c(\beta)}$$

2. $$\frac{\alpha\ :\ \beta}{\gamma} \quad \mapsto \quad \frac{a(\alpha)\ :\ \beta}{a(\gamma)}$$

The first form, which encodes an alternative consistency condition, underlies all the translations in Section 4. The translation $\mathcal{T}_a$ in Section 5 uses the second general translation in addition to the first in order to manage assertions.

The mapping $\mathcal{T}_a$ of Section 5 extends straightforwardly to the variants given in [Giordano and Martinelli, 1994; Linke and Schaub, 1997], which present cumulative variants of standard, rational, and justified default logic. Thus the mappings of Section 4 can be combined with that of Section 5 in order to obtain cumulative counterparts of the variants given in Section 2. In all, this allows us to map a whole spectrum of these variants of default logic onto the original approach. In view of the results of Section 4.2, we also obtain analogous results for mapping all variants into rational default logic (including self-mappings). In this way, both Reiter's original default logic as well as rational default logic may serve as a general host system, or target system, for mappings. Note however, that based on our experience, the translations into default logic (as opposed to rational default logic) would be more straightforward. As well, while we can simulate rational default logic in default logic via a linear translation, we have been unable to do better than a quadratic translation for simulating default logic in rational default logic.

The general approach of mapping one default logic into another also raises the question of how default logics should be classified. To date, this has mainly been done by appeal to formal properties, primarily:

- semi-monotonicity
- regularity and
- cumulativity.

Recall that a default logic, or class of default theories, is *semi-monotonic* just if the addition of default rules never eliminates, but rather extends or adds, new extensions. In particular, semi-monotonicity guarantees the existence of extensions. As well, a semi-monotonic logic has computational advantages over a non-semi-monotonic logic, in that semi-monotonicity allows for the incremental construction of an extension. Regularity, or commitment to assumptions, is concerned with how the consistency of justifications is determined with respect to an extension. A default logic is *weakly regular* if each justification of an applied rule must be individually consistent with an extension; it is *strongly regular* if the justifications must be jointly consistent with an extension. *Cumulativity* is the property wherein the addition of a derived conclusion to a set of facts does not change the set of conclusions.

Intuitively, it would seem that each of these properties might be used to classify default logics with respect to their expressiveness. However our results indicate that only the first property, semi-monotonicity, provides a truly distinguishing feature marking a borderline of expressiveness. Recall [Delgrande *et al.*, 1995; Mikitiuk and Truszczyński, 1995] that justified, constrained, and cumulative default logic enjoy semi-monotonicity, whereas Reiter default logic and rational default logic do not. In parallel, our results show that the former logics can be translated into (or: *simulated by*) the latter two logics, but the converse is not possible.[9] On the other hand, justified and Reiter's default logic enjoy weak regularity, while constrained, cumulative, and rational default logic are strongly regular (which is to say, commit to assumptions). Our results show that one can mutually simulate the formation of extensions in weakly and strongly regular default logics. Nonetheless, we have seen that our encodings of weakly regular default logics are quadratic, while strongly regular systems can be encoded linearly. As a matter of fact, this is due to the multiplicity of "consistency contexts" underlying extensions in weakly regular default logics. For mimicing this in a strongly regular logic, our translations provide as many language

[9] At least, it is impossible as regards a bijection among the sets of extensions.

copies as there are possible "consistency contexts", spanned by mutually inconsistent justifications. Hence, although regularity represents no real demarcation with respect to expressiveness, it nonetheless indicates a possible representational advantage. Also, this can be seen as a representational advantage of Reiter's default logic over rational default logic. The same applies analogously for cumulative default logics. Cumulative default logic is cumulative, while the other considered default logics are not. Thus only semi-monotonicity provides a clear division between more and less expressive variants of default logics.

7 Concluding remarks

We have shown how variants of default logic can be expressed in Reiter's original approach. Similarly, we have shown that rational default logic and default logic may be encoded, one into the other. However the encoding from rational default logic to default logic seems more straightforward than vice versa, since the latter translation is neither linear nor monotonic. For the most part the provided transformations have good properties, being (with exceptions discussed in the previous section) faithful, polynomial, modular, and monotonic. This work then complements previous work in nonmonotonic reasoning which has shown links between (seeming) disparate approaches. Here we show links between (seemingly) disparate variants of default logic. As well, the translations clearly illustrate the relationships between alternative approaches to default logic.

As argued in Section 6, there is a division between default logic and rational default logic on the one hand, and the remaining variants on the other, manifesting itself through the property of semi-monotonicity. Although it has often been informally argued that the computational advantages of semi-monotonicity are offset by a loss of representational power, this claim has up to now not been formally sustained. The results reported in [Janhunen, 1999] provide another indication of the relation between semi-monotonicity and expressiveness: normal default logic is a semi-monotonic fragment of Reiter's default logic and is strictly less expressive than default logic.

Our approach can also be seen as a refinement of the investigations of complexity and/or expressiveness conducted in [Gottlob, 1992; Stillman, 1991; Marek and Truszczyński, 1993; Gottlob and Mingyi, 1994; Gogic *et al.*, 1995; Janhunen, 1999]. From the perspective of complexity, there were of course hints that mappings such as ours are possible. First, it is well-known that the reasoning problems of all considered variants are at the second level of the polynomial hierarchy [Gottlob, 1992; Stillman, 1991].[10] The same level of complexity applies to the "existence of extensions" problem in default logic and rational default logic, while it is trivial in justified and constrained default logic (and analogously for the respective assertional counterparts). In view of the same complexity of reasoning tasks, observe that our impossibility claim expressed in Theorem 4.6 is about the non-existence of corresponding sets of extensions. This does not exclude the possibility of an encoding of incoherent Reiter or rational default theories in a semi-monotonic variant that, for instance, indicates incoherence through a special-purpose symbol.

[10] To be more precise, the problem of deciding whether a propositional formula is in some or all extensions, respectively, is Σ_p^2– and Π_p^2–complete.

However, there would be no one-to-one mapping here, since for any justified or constrained extension containing this special-purpose symbol, there would be no corresponding standard or rational extension.

The most closely related work to our own is that of Tomi Janhunen [Janhunen, 1999], who has investigated translations among specific subclasses of Reiter's default logic. For instance, he gives a translation mapping arbitrary default theories into semi-normal theories, showing that semi-normal default theories are as expressive as general ones. Other translation schemes can be found in [Marek and Truszczyński, 1993], where among others the notion of semi-representability is introduced. This concept deals with the representation of default theories within restricted subclasses of default theories over an extended language. Although semi-representability adheres to a fixed interpretation of default logic, one can view our results as semi-representation results among different interpretations of default theories. As regards future research, it would be interesting to see whether the results presented here lead to new relationships in the hierarchy of non-monotonic logics established in [Janhunen, 1999]. Also, a more detailed analysis of time and space complexity is an issue for future research.

The present work may also lend insight into computational characteristics of default logic. For example, our mappings provide specific syntactic characterisations of default theories that are guaranteed to have extensions. That is, for example, constrained default theories are guaranteed to have extensions; hence default theories appearing in the image of our mapping (Definition 4.1) are guaranteed to have extensions.

Apart from the theoretical insights, an advantage of mappings such as we have given, is that it suffices to have one general implementation of default logic for capturing a whole variety of different approaches. In this respect, our results allow us to handle all sorts of default logics by standard default logic implementations, such as DeReS [Cholewiński *et al.*, 1996].

A Auxiliary definitions and results

First, we define the set of generating default rules:

Definition A.1 *Let (D, W) be a default theory and let E be a set of formulas. Define*

$$GD((D,W),E) = \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \;\middle|\; \alpha \in E, \neg\beta \notin E \right\} .$$

For the proofs, we need the following ("pseudo-iterative") alternative characterisation for an extension. Alternative characterisations of extensions for the various default logic variants are found preceding the respective proofs.

Theorem A.1 *Let (D, W) be a default theory and let E be a set of formulas.*
Define $E_0 = Th(W)$ and for $i \geq 0$

$$\begin{aligned} GD_i &= \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \;\middle|\; \alpha \in E_i, \neg\beta \notin E \right\} \\ E_{i+1} &= Th(E_i \cup \{\mathit{Conseq}(\delta) \mid \delta \in GD_i\}) \end{aligned}$$

Then E is an extension for (D, W) iff $E = \bigcup_{i=0}^{\infty} E_i$.

This characterisation is easily derived from the one in given by Reiter [Reiter, 1980]:

Theorem A.2 ([Reiter, 1980]) *Let (D, W) be a default theory and let E be a set of formulas.*

Define $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = \mathit{Th}(E_i) \cup \mathit{Conseq}(GD_i)$$

Then E is an extension for (D, W) iff $E = \bigcup_{i=0}^{\infty} E_i$.

B Proofs

B.1 Correspondence with Constrained Default Logic

We have the following alternative characterisation of a constrained default logic extension.

Theorem B.1 ([Delgrande *et al.*, 1995]) *Let (D, W) be a default theory and let E, C be sets of formulas.*

Define $E_0 = C_0 = \mathit{Th}(W)$ and for $i \geq 0$

$$\begin{aligned} GD_i^c &= \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \,\middle|\, \alpha \in E_i, \neg(\beta \wedge \gamma) \notin C \right\} \\ E_{i+1} &= \mathit{Th}(E_i \cup \{\mathit{Conseq}(\delta) \mid \delta \in GD_i^c\}) \\ C_{i+1} &= \mathit{Th}(C_i \cup \{\mathit{Conseq}(\delta) \wedge \mathit{Justif}(\delta) \mid \delta \in GD_i^c\}) \end{aligned}$$

Then (E, C) is a constrained extension of (D, W) iff $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Theorem B.2 *Let (D, W) be a default theory over $\mathcal{L}$.*

Let E and C be (deductively closed) sets of formulas over $\mathcal{L}$ and let F be the set of formulas over $\mathcal{L} \cup \mathcal{L}'$ such that $F = \mathit{Th}(E \cup C')$.

For $i \geq 0$, define E_i and C_i as in Theorem B.1 relative to (D, W), E, and C.

For $i \geq 0$, define F_i as E_i in Theorem A.1 relative to $\mathcal{T}_c((D, W))$ and F.

Then, we have for $i \geq 0$ that $E_i = F_i \cap \mathcal{L}$ and $C_i' = F_i \cap \mathcal{L}'$ and $F_i = \mathit{Th}(E_i \cup C_i')$.

Proof B.2 Observe that $E = F \cap \mathcal{L}$ and $C' = F \cap \mathcal{L}'$. We prove our claim by induction.

Base. We have $E_0 = \mathit{Th}(W) = \mathit{Th}(W \cup W') \cap \mathcal{L} = F_0 \cap \mathcal{L}$.

Analogously, we get $C_0' = \mathit{Th}(W') = \mathit{Th}(W \cup W') \cap \mathcal{L}' = F_0 \cap \mathcal{L}'$.

Lastly, $F_0 = \mathit{Th}(W \cup W') = \mathit{Th}(\mathit{Th}(W) \cup \mathit{Th}(W')) = \mathit{Th}(E_0 \cup C_0')$.

Step. Suppose we have that $E_i = F_i \cap \mathcal{L}$, $C'_i = F_i \cap \mathcal{L}'$ along with $F_i = \mathit{Th}(E_i \cup C'_i)$.

We interpolate the following lemma.

Lemma B.3 *Given the induction hypothesis, we have*

$$\begin{array}{rcl} & \frac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \in & \left\{ \frac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \,\middle|\, \frac{\alpha : \beta}{\gamma} \in D, \alpha \in F_i, \neg(\beta' \wedge \gamma') \notin F \right\} \\ \textit{iff} & \frac{\alpha : \beta}{\gamma} \in & \left\{ \frac{\alpha : \beta}{\gamma} \in D \,\middle|\, \alpha \in E_i, \neg(\beta \wedge \gamma) \notin C \right\} \end{array}$$

Proof B.3 Given that $\alpha \in \mathcal{L}$, we have $\alpha \in F_i$ iff $\alpha \in E_i$ because $E_i = F_i \cap \mathcal{L}$.

It remains to be shown that $\neg(\beta' \wedge \gamma') \notin F$ iff $\neg(\beta \wedge \gamma) \notin C$ is true. To see this, we proceed as follows. We have $\neg(\beta \wedge \gamma) \notin C$ iff $\neg(\beta' \wedge \gamma') \notin C'$ iff $\neg(\beta' \wedge \gamma') \notin F \cap \mathcal{L}'$ iff $\neg(\beta' \wedge \gamma') \notin F$. ■

Lemma B.3 implies that $\frac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \in \mathit{GD}_i$ (as in Theorem A.1) iff $\frac{\alpha : \beta}{\gamma} \in \mathit{GD}^c_i$ (as in Theorem B.1). Hence, $\gamma \wedge (\beta' \wedge \gamma') \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}_i\}$ iff $\gamma \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}^c_i\}$ and $\beta \wedge \gamma \in \{\mathit{Conseq}(\delta) \wedge \mathit{Justif}(\delta) \mid \delta \in \mathit{GD}^c_i\}$.

Given the induction hypothesis, this implies that $E_{i+1} = F_{i+1} \cap \mathcal{L}$ and $C'_{i+1} = F_{i+1} \cap \mathcal{L}'$ along with $F_{i+1} = \mathit{Th}\big(E_{i+1} \cup C'_{i+1}\big)$. ■

Proof 4.1

1. Let (E, C) be a constrained extension of (D, W). According to Theorem B.1, we then have that $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$, where E_i and C_i are defined as in Theorem B.1. Define $F = \mathit{Th}(E \cup C')$ and F_i as E_i in Theorem A.1 but relative to $\mathcal{T}_c((D, W))$ and F.

$$\begin{array}{rcll} F &=& \mathit{Th}(E \cup C') & \\ &=& \mathit{Th}(\bigcup_{i=0}^{\infty} E_i \cup \bigcup_{i=0}^{\infty} C'_i) & \\ &=& \mathit{Th}(\bigcup_{i=0}^{\infty} (E_i \cup C'_i)) & \\ &=& \mathit{Th}(\bigcup_{i=0}^{\infty} \mathit{Th}(E_i \cup C'_i)) & \\ &=& \mathit{Th}(\bigcup_{i=0}^{\infty} F_i) & \text{(according to Theorem } B.2) \\ &=& \bigcup_{i=0}^{\infty} F_i & \text{(since } F_k \subseteq F_{k+1} \text{ and } F_k = \mathit{Th}(F_k) \text{ for } k \geq 0) \end{array}$$

   Hence F is an extension of $\mathcal{T}_c((D, W))$.

2. Let F be an extension of $\mathcal{T}_c((D, W))$. According to Theorem A.1, we then have that $F = \bigcup_{i=0}^{\infty} F_i$, where F_i is defined as E_i in Theorem A.1 but relative to $\mathcal{T}_c((D, W))$ and F. Define $E = F \cap \mathcal{L}$ and $C = \{\varphi \mid \varphi' \in F \cap \mathcal{L}'\}$ and E_i and C_i as in Theorem B.1.

$$\begin{array}{rcl} E &=& F \cap \mathcal{L} \\ &=& (\bigcup_{i=0}^{\infty} F_i) \cap \mathcal{L} \\ &=& \bigcup_{i=0}^{\infty} (F_i \cap \mathcal{L}) \\ &=& \bigcup_{i=0}^{\infty} E_i \end{array} \qquad \begin{array}{rcll} C' &=& F \cap \mathcal{L}' & \\ &=& (\bigcup_{i=0}^{\infty} F_i) \cap \mathcal{L}' & \\ &=& \bigcup_{i=0}^{\infty} (F_i \cap \mathcal{L}') & \\ &=& \bigcup_{i=0}^{\infty} C'_i & \text{(according to Theorem } B.2) \end{array}$$

   Hence $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$, that is, (E, C) is a constrained extension of (D, W).

■

Proof 4.2 To see that we have a one-to-one correspondence, consider the two cases of Theorem 4.1:

1. If $(E_1, C_1) \neq (E_2, C_2)$ then clearly $Th(E_1 \cup C_1') \neq Th(E_2 \cup C_2')$.

2. Conversely, if $F_1 \neq F_2$, then there is some $\delta \in D_c$ such that $\delta \in GD(\mathcal{T}_c((D, W)), F_1) \setminus GD(\mathcal{T}_c((D, W)), F_2)$. Suppose that $(F_1 \cap \mathcal{L}, \{\varphi \mid \varphi' \in F_1 \cap \mathcal{L}'\}) = (F_2 \cap \mathcal{L}, \{\varphi \mid \varphi' \in F_2 \cap \mathcal{L}'\})$. This implies *Prereq*$(\delta) \in F_2 \cap \mathcal{L}$ and *Justif*$(\delta) \in \{\varphi \mid \varphi' \in F_2 \cap \mathcal{L}'\}$. Consequently, $\delta \in GD(\mathcal{T}_c((D, W)), F_1)$, a contradiction.

■

B.2 Correspondence with Rational Default Logic

We have the following alternative characterisation of a rational default logic extension.

Theorem B.4 ([Mikitiuk and Truszczyński, 1993]) *Let (D, W) be a default theory and let E, C be sets of formulas.*

Define $E_0 = C_0 = Th(W)$ and for $i \geq 0$

$$
\begin{aligned}
GD_i^r &= \left\{ \tfrac{\alpha : \beta}{\gamma} \in D \,\middle|\, \alpha \in E_i, \neg\beta \notin C \right\} \\
E_{i+1} &= Th(E_i \cup \{\mathit{Conseq}(\delta) \mid \delta \in GD_i^r\}) \\
C_{i+1} &= Th(C_i \cup \{\mathit{Conseq}(\delta) \wedge \mathit{Justif}(\delta) \mid \delta \in GD_i^r\})
\end{aligned}
$$

Then (E, C) is a rational extension of (D, W) iff $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Given the proximity of Definition 2.3 to Definition 2.2, the proof of Theorem 4.3 is basically the same as that given in Section B.1. We thus concentrate below on the part specific to rational default logic, playing the role of Theorem B.2:

Theorem B.5 *Let (D, W) be a default theory over $\mathcal{L}$.*

Let E and C be (deductively closed) sets of formulas over $\mathcal{L}$ and let F be a set of formulas over $\mathcal{L} \cup \mathcal{L}'$ such that $F = Th(E \cup C')$.

For $i \geq 0$, define E_i and C_i as in Theorem B.4 relative to (D, W), E, and C.

For $i \geq 0$, define F_i as E_i in Theorem A.1 relative to $\mathcal{T}_r((D, W))$ and F.

Then, we have for $i \geq 0$ that $E_i = F_i \cap \mathcal{L}$ and $C_i' = F_i \cap \mathcal{L}'$ and $F_i = Th(E_i \cup C_i')$.

Proof B.5 Observe that since $F = Th(E \cup C')$, we have $E = F \cap \mathcal{L}$ and $C' = F \cap \mathcal{L}'$.

We prove our claim by induction.

Base. We have $E_0 = Th(W) = Th(W \cup W') \cap \mathcal{L} = F_0 \cap \mathcal{L}$, $C_0' = Th(W') = Th(W \cup W') \cap \mathcal{L}' = F_0 \cap \mathcal{L}'$, and $F_0 = Th(W \cup W') = Th(Th(W) \cup Th(W')) = Th(E_0 \cup C_0')$.

Step. Suppose $F_i = \mathit{Th}(E_i \cup C'_i)$ and so $E_i = F_i \cap \mathcal{L}$ and $C'_i = F_i \cap \mathcal{L}'$.

First of all, this implies that $\mathit{Th}(E_i) = \mathit{Th}(F_i) \cap \mathcal{L}$ and $\mathit{Th}(C'_i) = \mathit{Th}(F_i) \cap \mathcal{L}'$.

Next, we have the following lemma.

Lemma B.6 *Given the induction hypothesis, we have*

$$\begin{array}{rcl} \frac{\alpha : \beta'}{\gamma \wedge (\beta' \wedge \gamma')} & \in & \left\{ \frac{\alpha : \beta'}{\gamma \wedge (\beta' \wedge \gamma')} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D, \alpha \in F_i, \neg\beta' \not\in F \right\} \\ \textit{iff} \qquad \frac{\alpha : \beta}{\gamma} & \in & \left\{ \frac{\alpha : \beta}{\gamma} \in D \;\middle|\; \alpha \in E_i, \neg\beta \not\in C \right\} \end{array}$$

Proof B.6 Our claim holds if: $\neg\beta' \not\in F$ iff $\neg\beta \not\in C$ is true. To see this, we proceed as follows. We have $\neg\beta \not\in C$ iff $\neg\beta' \not\in C'$ iff $\neg\beta' \not\in F \cap \mathcal{L}'$ iff $\neg\beta' \not\in F$. ■

Continuing the proof of the theorem, Lemma B.6 implies that $\frac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \in \mathit{GD}_i$ (as in Theorem A.1) iff $\frac{\alpha : \beta}{\gamma} \in \mathit{GD}^r_i$ (as in Theorem B.4). Hence, $\gamma \wedge (\beta' \wedge \gamma') \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}_i\}$ iff $\gamma \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}^r_i\}$ and $\beta \wedge \gamma \in \{\mathit{Conseq}(\delta) \wedge \mathit{Justif}(\delta) \mid \delta \in \mathit{GD}^r_i\}$.

In all, this implies that $E_{i+1} = F_{i+1} \cap \mathcal{L}$, $C'_{i+1} = F_{i+1} \cap \mathcal{L}'$, and $F_{i+1} = \mathit{Th}\big(E_{i+1} \cup C'_{i+1}\big)$. ■

B.3 Correspondence with Justified Default Logic

We have the following alternative characterisation of a justified default logic extension.

Theorem B.7 ([Łukaszewicz, 1988]) *Let (D, W) be a default theory and let E, J be sets of formulas.*

Define $E_0 = \mathit{Th}(W)$, $J_0 = \emptyset$ and for $i \geq 0$

$$\begin{array}{rcl} \mathit{GD}^j_i & = & \left\{ \frac{\alpha : \beta}{\gamma} \in D \;\middle|\; \alpha \in E_i, \forall \eta \in J \cup \{\beta\}. \neg(\eta \wedge \gamma) \not\in E \right\} \\ E_{i+1} & = & \mathit{Th}\big(E_i \cup \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}^j_i\}\big) \\ J_{i+1} & = & J_i \cup \{\mathit{Justif}(\delta) \mid \delta \in \mathit{GD}^j_i\} \end{array}$$

Then (E, J) is a justified extension of (D, W) iff $(E, J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$.

Recall from Section 4 that without loss of generality we deal with default rules have unique components. This greatly facilitates this proof since a justification uniquely determines the default rule in which it occurs. Thus, we have for istance $\delta(\mathit{Justif}(\delta)) = \delta$ for every $\delta \in D$.

Theorem B.8 *Let (D, W) be a default theory over $\mathcal{L}$ and let $J \subseteq \mathit{Justif}(D)$.*

Let E be a deductively closed set of formulas over $\mathcal{L}$ and let F be a set of formulas over $\mathcal{L} \cup \bigcup_{\zeta \in D} \mathcal{L}^\zeta$ such that $F = \mathit{Th}\Big(E \cup \bigcup_{\zeta \in D} E^\zeta \cup \bigcup_{\beta \in J} \{\beta^{\delta(\beta)}\}\Big)$ and $E = F \cap \mathcal{L}$ and $J = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta)$ and $\delta \in \mathit{GD}(\mathcal{T}_j((D, W)), F)\}$.

For $i \geq 0$, define E_i and J_i as in Theorem B.7 relative to (D, W), E, and J.

For $i \geq 0$, define F_i as E_i in Theorem A.1 relative to $\mathcal{T}_j((D, W))$ and F.

Then, we have for $i \geq 0$ that $F_i = \mathit{Th}\Big(E_i \cup \bigcup_{\zeta \in D} E_i^\zeta \cup \bigcup_{\beta \in J_i} \{\beta^{\delta(\beta)}\}\Big)$ and $E_i = F_i \cap \mathcal{L}$ and $J_i = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta)$ and $\delta \in \mathit{GD}_{i-1}\}$.

Proof B.8 We prove our claim by induction.

Base. We have $E_0 = Th(W) = Th\Big(W \cup \bigcup_{\zeta\in D} W^\zeta\Big) \cap \mathcal{L} = F_0 \cap \mathcal{L}$ and

$$\begin{aligned} F_0 &= Th\Big(W \cup \bigcup_{\zeta\in D} W^\zeta\Big) \\ &= Th\Big(Th(W) \cup \bigcup_{\zeta\in D} Th\big(W^\zeta\big)\Big) \\ &= Th\Big(E_0 \cup \bigcup_{\zeta\in D} E_0^\zeta\Big) \end{aligned}$$

By definition, we have $J_0 = \emptyset = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta) \text{ and } \delta \in \emptyset\}$.

Step. Suppose we have that $F_i = Th\Big(E_i \cup \bigcup_{\zeta\in D} E_i^\zeta \cup \bigcup_{\eta\in J_i}\{\eta^{\delta(\eta)}\}\Big)$ and $E_i = F_i \cap \mathcal{L}$ and $J_i = \{\eta \mid \eta = \mathit{Justif}^\circ(\delta) \text{ and } \delta \in GD_{i-1}\}$.

We introduce the following lemma in order to complete the proof.

Lemma B.9 *Given the induction hypothesis, we have for* $\delta = \frac{\alpha:\beta}{\gamma}$ *that*

$$\begin{array}{ccc} \frac{\alpha:(\beta^\delta\wedge\gamma^\delta)\wedge(\bigwedge_{\zeta\in D}\gamma^\zeta)}{\gamma\wedge(\beta^\delta\wedge\gamma^\delta)\wedge(\bigwedge_{\zeta\in D}\gamma^\zeta)} & \in & \left\{ \frac{\alpha:(\beta^\lambda\wedge\gamma^\lambda)\wedge(\bigwedge_{\zeta\in D}\gamma^\zeta)}{\gamma\wedge(\beta^\lambda\wedge\gamma^\lambda)\wedge(\bigwedge_{\zeta\in D}\gamma^\zeta)} \,\middle|\, \lambda = \frac{\alpha:\beta}{\gamma} \in D, \alpha \in F_i, \neg(\beta^\lambda \wedge \gamma^\lambda \wedge \bigwedge_{\zeta\in D}\gamma^\zeta) \notin F \right\} \\ \textit{iff} \qquad\qquad \delta & \in & \left\{ \frac{\alpha:\beta}{\gamma} \in D \,\middle|\, \alpha \in E_i, \forall\eta \in J \cup \{\beta\}.\neg(\eta\wedge\gamma) \notin E \right\} \end{array}$$

Proof B.9 Given that $\alpha \in \mathcal{L}$, we have $\alpha \in F_i$ iff $\alpha \in E_i$ because $E_i = F_i \cap \mathcal{L}$.

It remains to be shown that $\neg(\beta^\delta \wedge \gamma^\delta \wedge \bigwedge_{\zeta\in D}\gamma^\zeta) \notin F$ iff $\forall\eta \in J\cup\{\beta\}.\neg(\eta\wedge\gamma) \notin E$ is true. To see this, we proceed as follows. We have $\forall\eta \in J \cup \{\beta\}.\neg(\eta \wedge \gamma) \notin E$ iff $\neg(\beta \wedge \gamma) \notin E$ and $\neg(\eta \wedge \gamma) \notin E$ for every $\eta \in J$. Since E is deductively closed $\neg(\beta \wedge \gamma) \notin E$ is equivalent to the (redundant) condition $\neg(\beta\wedge\gamma) \notin E$ and $\neg\gamma \notin E$. We thus have that $\forall\eta \in J\cup\{\beta\}.\neg(\eta\wedge\gamma) \notin E$ holds iff

1. $\neg(\beta \wedge \gamma) \notin E$,
2. $\neg\gamma \notin E$, and
3. $\neg(\eta \wedge \gamma) \notin E$ for every $\eta \in J$.

Due to the isomorphism between $\mathcal{L}$ and $\mathcal{L}^\zeta$ for every $\zeta \in D$, this is equivalent to

1. $\neg(\beta^\delta \wedge \gamma^\delta) \notin E^\delta$,
2. $\neg\gamma^{\delta(\eta)} \notin E^{\delta(\eta)}$ for every $\eta \in \mathit{Justif}(D) \setminus J$, and
3. $\neg(\eta^{\delta(\eta)} \wedge \gamma^{\delta(\eta)}) \notin E^{\delta(\eta)}$ for every $\eta \in J$.

We now proceed itemwise:

1. $\neg(\beta^\delta \wedge \gamma^\delta) \notin E^\delta$
   (a) Suppose $\beta \notin J$.
   Then, by definition of F, $\neg(\beta^\delta \wedge \gamma^\delta) \notin E^\delta$ is equivalent to $\neg(\beta^\delta \wedge \gamma^\delta) \notin F \cap \mathcal{L}^\delta$.

(b) Suppose $\beta \in J$.

Given that E^δ is deductively closed (by virtue of E being deductively closed), $\neg(\beta^\delta \wedge \gamma^\delta) \notin E^\delta$ is equivalent to $E^\delta \not\models \neg\beta^\delta \vee \neg\gamma^\delta$, which is equivalent to $E^\delta \cup \{\beta^\delta\} \not\models \neg\beta^\delta \vee \neg\gamma^\delta$. That is, $\neg(\beta^\delta \wedge \gamma^\delta) \notin \mathit{Th}\big(E^\delta \cup \{\beta^\delta\}\big)$. By definition of F, this is equivalent to $\neg(\beta^\delta \wedge \gamma^\delta) \notin F \cap \mathcal{L}^\delta$.

In both cases, we obtain that $\neg(\beta^\delta \wedge \gamma^\delta) \notin E^\delta$ is equivalent to $\neg(\beta^\delta \wedge \gamma^\delta) \notin F \cap \mathcal{L}^\delta$.

2. $\neg\gamma^{\delta(\eta)} \notin E^{\delta(\eta)}$ for every $\eta \in \mathit{Justif}(D) \setminus J$.

   By definition of F, this is equivalent to $\neg\gamma^{\delta(\eta)} \notin F \cap \mathcal{L}^{\delta(\eta)}$.

3. $\neg(\eta^{\delta(\eta)} \wedge \gamma^{\delta(\eta)}) \notin E^{\delta(\eta)}$ for every $\eta \in J$. Consider $\eta \in J$.

   Given that $E^{\delta(\eta)}$ is deductively closed (by virtue of E being deductively closed), $\neg(\eta^{\delta(\eta)} \wedge \gamma^{\delta(\eta)}) \notin E^{\delta(\eta)}$ is equivalent to $\neg\gamma^{\delta(\eta)} \notin \mathit{Th}\big(E^{\delta(\eta)} \cup \{\eta^{\delta(\eta)}\}\big)$. By definition of F, this is equivalent to $\neg\gamma^{\delta(\eta)} \notin F \cap \mathcal{L}^{\delta(\eta)}$.

This case analysis shows that $\forall \eta \in J \cup \{\beta\}.\neg(\eta \wedge \gamma) \notin E$ holds iff $\neg(\beta^\delta \wedge \gamma^\delta) \notin F \cap \mathcal{L}^\delta$ and $\neg\gamma^{\delta(\eta)} \notin F \cap \mathcal{L}^{\delta(\eta)}$ is true for every $\eta \in \mathit{Justif}(D)$ (joining the result of 2. and 3.). By definition of F, the latter is furthermore equivalent to $\neg(\beta^\delta \wedge \gamma^\delta) \notin F$ and $\neg\gamma^{\delta(\eta)} \notin F$ for every $\eta \in \mathit{Justif}(D)$. Given the strict separation of F via the sublanguages and the fact that F is deductively closed the latter is equivalent to $\neg(\beta^\delta \wedge \gamma^\delta \wedge \bigwedge_{\zeta \in D} \gamma^\zeta) \notin F$. ■

Lemma B.9 implies for $\delta = \frac{\alpha : \beta}{\gamma}$ that $\frac{\alpha : (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)}{\gamma \wedge (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)} \in \mathit{GD}_i$ (as in Theorem A.1) iff $\delta \in \mathit{GD}_i^j$ (as in Theorem B.7). Hence, $\gamma \wedge (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta) \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}_i\}$ iff $\gamma \in \{\mathit{Conseq}(\delta) \mid \delta \in \mathit{GD}_i^j\}$ and $\beta \in \{\mathit{Justif}(\delta) \mid \delta \in \mathit{GD}_i^j\}$.

Given the induction hypothesis, this implies that $E_{i+1} = F_{i+1} \cap \mathcal{L}$ and $J_{i+1} = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta) \text{ and } \delta \in \mathit{GD}_i\}$ and $F_{i+1} = \mathit{Th}\Big(E_{i+1} \cup \bigcup_{\zeta \in D} E_{i+1}^\zeta \cup \bigcup_{\eta \in J_{i+1}} \{\eta^{\delta(\eta)}\}\Big)$. ■

Proof 4.4

1. Let (E, J) be a justified extension of (D, W). According to Theorem B.7, we then have that $(E, J) = (\bigcup_{i=0}^\infty E_i, \bigcup_{i=0}^\infty J_i)$, where E_i and J_i are defined as in Theorem B.7. Define $F = \mathit{Th}\Big(E \cup \bigcup_{\zeta \in D} E^\zeta \cup \bigcup_{\beta \in J} \{\beta^{\delta(\beta)}\}\Big)$ and F_i as E_i in Theorem A.1 but relative to $\mathcal{T}_j((D, W))$ and F.

$$
\begin{aligned}
F &= \mathit{Th}\Big(E \cup \textstyle\bigcup_{\zeta \in D} E^\zeta \cup \bigcup_{\beta \in J} \{\beta^{\delta(\beta)}\}\Big) \\
&= \mathit{Th}\Big(\textstyle\bigcup_{i=0}^\infty E_i \cup \bigcup_{\zeta \in D} \bigcup_{i=0}^\infty E_i^\zeta \cup \bigcup_{\beta \in \bigcup_{i=0}^\infty J_i} \{\beta^{\delta(\beta)}\}\Big) \\
&= \mathit{Th}\Big(\textstyle\bigcup_{i=0}^\infty E_i \cup \bigcup_{i=0}^\infty \bigcup_{\zeta \in D} E_i^\zeta \cup \bigcup_{i=0}^\infty \bigcup_{\beta \in J_i} \{\beta^{\delta(\beta)}\}\Big) \\
&= \mathit{Th}\Big(\textstyle\bigcup_{i=0}^\infty (E_i \cup \bigcup_{\zeta \in D} E_i^\zeta \cup \bigcup_{\beta \in J_i} \{\beta^{\delta(\beta)}\})\Big) \\
&= \mathit{Th}(\textstyle\bigcup_{i=0}^\infty F_i) && \text{(according to Theorem } B.8) \\
&= \textstyle\bigcup_{i=0}^\infty F_i && \text{(since } F_k \subseteq F_{k+1} \text{ and } F_k = \mathit{Th}(F_k) \text{ for } k \geq 0)
\end{aligned}
$$

   Hence F is an extension of $\mathcal{T}_j((D, W))$.

2. Let F be an extension of $\mathcal{T}_j((D,W))$. According to Theorem A.1, we then have that $F = \bigcup_{i=0}^{\infty} F_i$, where F_i is defined as E_i in Theorem A.1 but relative to $\mathcal{T}_j((D,W))$ and F. Define $E = F \cap \mathcal{L}$ and $J = \{\beta \mid \beta = \mathit{Justif}^{\circ}(\delta) \text{ and } \delta \in \mathit{GD}(\mathcal{T}_j((D,W)), F)\}$ and E_i and J_i as in Theorem B.7.

$$\begin{array}{rclrcll}
E & = & F \cap \mathcal{L} & J & = & \{\beta \mid \beta = \mathit{Justif}^{\circ}(\delta), \delta \in \mathit{GD}(\mathcal{T}_j((D,W)), F)\} & \\
 & = & (\bigcup_{i=0}^{\infty} F_i) \cap \mathcal{L} & & = & \{\beta \mid \beta = \mathit{Justif}^{\circ}(\delta), \delta \in \bigcup_{i=0}^{\infty} \mathit{GD}_i\} & \\
 & = & \bigcup_{i=0}^{\infty}(F_i \cap \mathcal{L}) & & = & \bigcup_{i=0}^{\infty}\{\beta \mid \beta = \mathit{Justif}^{\circ}(\delta), \delta \in \mathit{GD}_i\} & \\
 & = & \bigcup_{i=0}^{\infty} E_i & & = & \bigcup_{i=1}^{\infty} J_i & \text{(according to Theorem } B.8) \\
 & & & & = & \bigcup_{i=0}^{\infty} J_i &
\end{array}$$

Hence $(E,J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$, that is, (E,J) is a justified extension of (D,W).

■

B.4 Correspondence with (Standard) Default Logic

Given the proximity of Definition 4.4 to Definition 4.3, the proof of Theorem 4.5 is basically the same as that given in Section B.3. We thus concentrate below on the part specific to the encoding of Reiter's and rational default logic:

First of all, observe that for default theories of the form $\mathcal{T}_d((D,W))$, we have $E_i = C_i$ in Theorem B.4.

Theorem B.10 *Let (D,W) be a default theory over $\mathcal{L}$.*

Let E be a deductively closed set of formulas over $\mathcal{L}$ and let F be a set of formulas over $\mathcal{L} \cup \bigcup_{\zeta \in D} \mathcal{L}^{\zeta}$ such that $F = \mathit{Th}\left(E \cup \bigcup_{\zeta \in D} E^{\zeta} \cup \bigcup_{\delta \in \mathit{GD}((D,W),E)} \{\mathit{Justif}(\delta)^{\delta}\}\right)$ and $E = F \cap \mathcal{L}$.

For $i \geq 0$, define E_i as in Theorem A.1 relative to (D,W) and E.

For $i \geq 0$, define F_i as $E_i (= C_i)$ in Theorem B.4 relative to $\mathcal{T}_d((D,W))$ and F.

Then, we have for $i \geq 0$ that $F_i = \mathit{Th}\left(E_i \cup \bigcup_{\zeta \in D} E_i^{\zeta} \cup \bigcup_{\delta \in \mathit{GD}_{i-1}} \{\mathit{Justif}(\delta)^{\delta}\}\right)$ and $E_i = F_i \cap \mathcal{L}$, where GD_i is defined as in Theorem A.1.

Proof B.10 We prove our claim by induction.

Base. Identical to the Base step in Proof B.8.

Step. The induction step is analogous to the one in Proof B.8, except that it relies on the following following lemma.

Lemma B.11 *Given the induction hypothesis, we have for $\delta = \frac{\alpha : \beta}{\gamma}$ that*

$$\begin{array}{ccc}
\frac{\alpha : \beta^{\delta}}{\gamma \wedge (\beta^{\delta} \wedge \gamma^{\delta}) \wedge (\bigwedge_{\zeta \in D} \gamma^{\zeta})} & \in & \left\{ \frac{\alpha : \beta^{\lambda}}{\gamma \wedge (\beta^{\lambda} \wedge \gamma^{\lambda}) \wedge (\bigwedge_{\zeta \in D} \gamma^{\zeta})} \,\middle|\, \lambda = \frac{\alpha : \beta}{\gamma} \in D, \alpha \in F_i, \neg\beta^{\lambda} \notin F \right\} \\
\mathit{iff} \qquad \delta & \in & \left\{ \frac{\alpha : \beta}{\gamma} \in D \,\middle|\, \alpha \in E_i, \neg\beta \notin E \right\}
\end{array}$$

Proof B.11 Given that $\alpha \in \mathcal{L}$, we have $\alpha \in F_i$ iff $\alpha \in E_i$ because $E_i = F_i \cap \mathcal{L}$.

It remains to be shown that $\neg\beta^\delta \notin F$ iff $\neg\beta \notin E$. The latter is equivalent to $\neg\beta^\delta \notin E^\delta$. We distinguish the following two cases.

1. $\delta \notin GD((D,W),E)$. Then, by definition of F, $\neg\beta^\delta \notin E^\delta$ is equivalent to $\neg\beta^\delta \notin F \cap \mathcal{L}^\delta$.

2. $\delta \in GD((D,W),E)$.

   Given that E^δ is deductively closed (by virtue of E being deductively closed), $\neg\beta^\delta \notin E^\delta$ is equivalent to $\neg\beta^\delta \notin Th\big(E^\delta \cup \{\beta^\delta\}\big)$. By definition of F, this is equivalent to $\neg\beta^\delta \notin F \cap \mathcal{L}^\delta$.

This case analysis shows that $\neg\beta \notin E$ holds iff $\neg\beta^\delta \notin F \cap \mathcal{L}^\delta$. By definition of F, this is equivalent to $\neg\beta^\delta \notin F$. ■

■

B.5 Correspondence with Cumulative Default Logic

We have the following alternative characterisation of a cumulative default logic extension.

Theorem B.12 ([Brewka, 1991]) *Let $(D,\mathcal{W})$ be an assertional default theory and let $\mathcal{E}$ be a set of assertions.*

Define $\mathcal{E}_0 = \mathcal{W}$ and for $i \geq 0$

$$\begin{aligned} GD_i^a &= \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \mid \langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i, Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ \mathcal{E}_{i+1} &= \widehat{Th}(\mathcal{E}_i) \ \cup \ \{ \ \textit{CumConseq}(\delta) \mid \delta \in GD_i^a \} \\ &= \widehat{Th}(\mathcal{E}_i) \ \cup \ \{ \ \langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \mid \delta = \tfrac{\alpha:\beta}{\gamma} \in D, \ \delta \in GD_i^a \}. \end{aligned}$$

Then $\mathcal{E}$ is an assertional extension of $(D,\mathcal{W})$ iff $\mathcal{E} = \bigcup_{i=0}^{\infty} \mathcal{E}_i$.

We define the closure operator restricted to reified assertions as follows:

Definition B.1 *Let R be a set of reified assertions.*

Define $Th^{re}(R) = \{\beta \mid R \cup Ax_{re} \vdash \beta$ and $\beta \in \mathcal{L}^{re}.\}$

Proof 5.1 $\{\alpha_1, \alpha_2\} \vdash \gamma$ by assumption, so $\vdash \alpha_1 \supset (\alpha_2 \supset \gamma)$, and thus $\langle \alpha_1 \supset (\alpha_2 \supset \gamma), \emptyset\rangle^{re} \in Ax_{re}$ by Definition 5.1.1.

As well, since $\langle \alpha_1, \beta_1\rangle^{re} \in R$ we have $\langle \alpha_1, \beta_1\rangle^{re} \wedge \langle \alpha_1 \supset (\alpha_2 \supset \gamma), \emptyset\rangle^{re} \supset \langle \alpha_2 \supset \gamma, \beta_1\rangle^{re} \in Ax_{re}$ by Definition 5.1.3. Thus $R \cup Ax_{re} \vdash \langle \alpha_2 \supset \gamma, \beta_1\rangle^{re}$ by modus ponens.

Since $\langle \alpha_2, \beta_2\rangle^{re} \in R$ by assumption, and $\langle \alpha_2, \beta_2\rangle^{re} \wedge \langle \alpha_2 \supset \gamma, \beta_1\rangle^{re} \supset \langle \gamma, \beta_1 \wedge \beta_2\rangle^{re} \in Ax_{re}$, we obtain $R \cup Ax_{re} \vdash \langle \gamma, \beta_1 \wedge \beta_2\rangle^{re}$ by modus ponens. ■

Lemma B.13 *Let $\mathcal{R}$ be a set of assertions. Then*

1. $Re\left(\widehat{Th}(\mathcal{R})\right) = Th^{re}(Re\,(\mathcal{R}))$

2. $Re^{+}\left(\widehat{Th}(\mathcal{R})\right) = Th\big(Re^{+}(\mathcal{R})\big)$

3. $\mathcal{R} = Re^{-1}(Re\,(\mathcal{R}))$ *.*

Proof B.13 Immediate from Definition 2.6 and Theorem 5.1. ■

Lemma B.14 *Let R be a set of reified assertions.*
Then $Re^{-1}(Th^{re}(R)) = \widehat{Th}\big(Re^{-1}(R)\big)$.

Proof B.14 Immediate from Definition 2.6 and Theorem 5.1. ■

Lemma B.15 *Let E be an extension of $\mathcal{T}_a((D, \mathcal{W}))$. Then*

$$E \vdash \alpha \quad \textit{iff} \quad Supp\big(Re^{-1}(E)\big) \cup Form\big(Re^{-1}(E)\big) \vdash \alpha$$

where α mentions no reified formula.

Proof B.15

1. Assume that $E \vdash \alpha$.

   Then from the compactness of classical logic, there are ϕ_i, $1 \le i \le n$ for some n, such that $\{\phi_1, \dots, \phi_n\} \subseteq E$ and $\{\phi_1, \dots, \phi_n\} \vdash \alpha$ As well, every ϕ_i mentions no reified formula. Moreover, without loss of generality, we can assume that every such ϕ_i is either a member of W_a or is the consequent of a generating default from D_a.

   But by the specification of W_a and D_a, we have that for every such ϕ_i there is a reified formula $\langle \psi_1, \psi_2 \rangle^{re}$ such that $\psi_1 \vdash \phi_i$ or $\psi_2 \vdash \phi_i$.

   Thus by classical monotonicity we obtain that $Supp\big(Re^{-1}(E)\big) \cup Form\big(Re^{-1}(E)\big) \vdash \alpha$.

2. Conversely, assume that $Supp\big(Re^{-1}(E)\big) \cup Form\big(Re^{-1}(E)\big) \vdash \alpha$

   We show that

   $$Supp\big(Re^{-1}(E)\big) \cup Form\big(Re^{-1}(E)\big) \subseteq E, \tag{5}$$

   from which our result follows from the monotonicity of classical logic.

   Equation (5) follows if we can show that, if $E \vdash \langle \phi_1, \phi_2 \rangle^{re}$ then $E \vdash \phi_1$ and $E \vdash \phi_2$.

   So assume that $E \vdash \langle \phi_1, \phi_2 \rangle^{re}$; then there is a minimum i, according to Theorem A.2, such that $\langle \phi_1, \phi_2 \rangle^{re} \in E_i$.

   Base. If $i = 0$ then $\langle \phi_1, \phi_2 \rangle^{re} \in W_a$.

   This implies that $\langle \phi_1, \phi_2 \rangle \in \mathcal{W}$. Hence $\phi_1, \phi_2 \in Form(\mathcal{W}) \cup Supp(\mathcal{W})$, and so $Form(\mathcal{W}) \cup Supp(\mathcal{W}) \vdash \phi_1 \wedge \phi_2$, from which we obtain $W_a \vdash \phi_1 \wedge \phi_2$, and so $W_a \vdash \phi_1$ and $W_a \vdash \phi_2$.

Step. For the induction hypothesis, assume that the result holds for $i = k$.

For $i = k + 1$ we have by assumption that $\langle \phi_1, \phi_2 \rangle^{re} \in E_{k+1}$.

Using Theorem A.2, there are two cases to consider.

(a) $\langle \phi_1, \phi_2 \rangle^{re} \in Th(E_k)$.
Thus $E_k \vdash \langle \phi_1, \phi_2 \rangle^{re}$, and by the induction hypothesis we have that $E_k \vdash \phi_1$ and $E_k \vdash \phi_2$.

(b) In accordance with Definition 5.4, there is a default rule δ with consequent $\langle \gamma, \psi \wedge \beta \wedge \gamma \rangle^{re} \wedge \beta \wedge \gamma$ applied at step $k + 1$, and where $\langle \phi_1, \phi_2 \rangle^{re} = \langle \gamma, \psi \wedge \beta \wedge \gamma \rangle^{re}$.
Trivially, since $\phi_1 = \gamma$ we have $\phi_1 \in E_{k+1}$. As well, we have $\beta \in E_{k+1}$.
Last, from the applicability conditions for δ, we obtain that $E_k \vdash \langle \alpha, \psi \rangle^{re}$. By the induction hypothesis we get that $E_k \vdash \psi$, from which, together with the preceding we obtain that $E_{k+1} \vdash \psi \wedge \beta \wedge \gamma$, that is $E_{k+1} \vdash \phi_2$. This completes the induction and the proof of the lemma.

■

Proof 5.2

1. Let $\mathcal{E}$ be an assertional extension of default theory $(D, \mathcal{W})$. If $\mathcal{E}$ is inconsistent then $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$. Hence, from [Brewka, 1991, Lemma 2.7], we have $Form(\mathcal{W}) \cup Supp(\mathcal{W}) \vdash \bot$. Since $Form(\mathcal{W}) \cup Supp(\mathcal{W}) \subseteq W_a$, we have $W_a \vdash \bot$, from which we obtain that (D_a, W_a) has a single (inconsistent) extension.

   So let $\mathcal{E}$ be a consistent assertional extension of default theory $(D, \mathcal{W})$. We show that $Re^+(\mathcal{E})$ and $Re^+(\mathcal{E}_i)$, $i \geq 0$, are equivalent to conditions satisfying an extension of (D_a, W_a) as given in Theorem A.2.

   We use induction for the sets $\mathcal{E}_i$, $i \geq 0$.

   Base:

$$\begin{aligned} Re^+(\mathcal{E}_0) &= Re\,(\mathcal{E}_0) \cup Form(\mathcal{E}_0) \cup Supp(\mathcal{E}_0) \\ &= Re\,(\mathcal{W}) \cup Form(\mathcal{W}) \cup Supp(\mathcal{W}) \\ &= W_a. \end{aligned}$$

Step:

$$\begin{aligned}
Re^{+}(\mathcal{E}_{i+1}) &= Re\,(\mathcal{E}_{i+1}) \cup Form(\mathcal{E}_{i+1}) \cup Supp(\mathcal{E}_{i+1}) \qquad \text{by defn of } Re^{+}(\cdot) \\
&= Re\left(\widehat{Th}(\mathcal{E}_i) \cup \{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}\right) \cup \\
&\qquad Form\left(\widehat{Th}(\mathcal{E}_i) \cup \{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}\right) \cup \\
&\qquad\quad Supp\left(\widehat{Th}(\mathcal{E}_i) \cup \{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}\right) \quad \text{by Theorem B.12} \\
&= \left[Re\left(\widehat{Th}(\mathcal{E}_i)\right) \cup Form\left(\widehat{Th}(\mathcal{E}_i)\right) \cup Supp\left(\widehat{Th}(\mathcal{E}_i)\right)\right] \quad \cup \\
&\qquad [Re\,(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}) \cup \\
&\qquad\quad Form(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}) \cup \\
&\qquad\qquad Supp(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\})] \\
&\qquad\qquad\qquad \text{rearranging terms} \\
&= Re^{+}\left(\widehat{Th}(\mathcal{E}_i)\right) \cup Re^{+}(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}) \\
&= Th^{re}\left(Re^{+}(\mathcal{E}_i)\right) \cup Re^{+}(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}) \quad \text{by Lemma B.13}
\end{aligned}$$

Expanding the rightmost term above we get:

$$\begin{aligned}
& Re^{+}(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\}) \\
&= Re^{+}(\{\ \langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \mid \tfrac{\alpha:\beta}{\gamma} \in D, \\
&\qquad\qquad \langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i, Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}). \\
&= (\{\langle \gamma, Supp(\alpha) \wedge \beta \wedge \gamma\rangle^{re},\ \gamma,\ Supp(\alpha) \wedge \beta \wedge \gamma \ \mid \\
&\qquad \tfrac{\alpha:\beta}{\gamma} \in D, \langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i, Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\})
\end{aligned}$$

We thus have:

$$\{\langle \gamma, Supp(\alpha) \wedge \beta \wedge \gamma\rangle^{re},\ \gamma,\ Supp(\alpha) \wedge \beta \wedge \gamma\} \subseteq Re^{+}(\{\ \textit{\textbf{CumConseq}}(\delta) \mid \delta \in GD_i^a\})$$

iff

(a) There exists $\frac{\alpha:\beta}{\gamma} \in D$ where

(b) $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i$, and

(c) $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.

Proceeding itemwise we have:

(a) $\frac{\alpha:\beta}{\gamma} \in D$ iff $\frac{\langle \alpha, Supp(\alpha)\rangle^{re} : \beta \wedge \gamma}{\langle \gamma, Supp(\alpha) \wedge \beta \wedge \gamma\rangle^{re} \wedge \beta \wedge \gamma} \in D_a$ from Definition 5.4.

(b) $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i$ iff $\langle \alpha, Supp(\alpha)\rangle^{re} \in Re^{+}(\mathcal{E}_i)$ by the induction hypothesis.

(c) We obtain that

$$Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \quad \text{iff} \quad Re^{+}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$$

as follows:

$\mathcal{E} \vdash \bot$ iff $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$ from [Brewka, 1991].
$\mathcal{E} \vdash \bot$ iff $Re\,(\mathcal{E}) \vdash \bot$ from Lemma B.13.
Thus, $Re\,(\mathcal{E}) \vdash \bot$ iff $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$.
Clearly $Re\,(\mathcal{E}) \cup Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$ iff $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$.
Thus $Re^{+}(\mathcal{E}) \vdash \bot$ iff $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \vdash \bot$,
and so $Re^{+}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \vdash \bot \quad$ iff $\quad Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \vdash \bot$
which was to be shown.

Substituting this in, and continuing with the proof of the inductive step, we have:

$$\begin{aligned}
Re^{+}(\mathcal{E}_{i+1}) &= Th^{re}\big(Re^{+}(\mathcal{E}_i)\big) \bigcup \{\langle\gamma, Supp(\alpha) \wedge \beta \wedge \gamma\rangle^{re},\ \gamma,\ Supp(\alpha) \wedge \beta \wedge \gamma \mid \\
&\qquad \tfrac{\langle\alpha, Supp(\alpha)\rangle^{re} : \beta\wedge\gamma}{\langle\gamma, Supp(\alpha)\wedge\beta\wedge\gamma\rangle^{re}\wedge\beta\wedge\gamma} \in D_a, \\
&\qquad\qquad \langle\alpha, Supp(\alpha)\rangle^{re} \in Re^{+}(\mathcal{E}_i)\,,\, Re^{+}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\} \\
&= Th^{re}\big(Re^{+}(\mathcal{E}_i)\big) \bigcup \{\langle\gamma, Supp(\alpha) \wedge \beta \wedge \gamma\rangle^{re},\ \beta \wedge \gamma \mid \\
&\qquad \tfrac{\langle\alpha, Supp(\alpha)\rangle^{re} : \beta\wedge\gamma}{\langle\gamma, Supp(\alpha)\wedge\beta\wedge\gamma\rangle^{re}\wedge\beta\wedge\gamma} \in D_a, \\
&\qquad\qquad \langle\alpha, Supp(\alpha)\rangle^{re} \in Re^{+}(\mathcal{E}_i)\,,\, Re^{+}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\} \\
&\qquad \text{since } Supp(\alpha) \in Th\big(Re^{+}(\mathcal{E}_i)\big) \text{ by the induction hypothesis} \\
&= Th^{re}\big(Re^{+}(\mathcal{E}_i)\big) \ \cup\ \{\ \textit{Conseq}(\delta) \mid \delta \in GD(\mathcal{T}_a((D, \mathcal{W})), Re^{+}(\mathcal{E}))\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\bigcup_{i=0}^{\infty} Re^{+}(\mathcal{E}_i) &= \bigcup_{i=0}^{\infty} (Re\,(\mathcal{E}_i) \cup Form(\mathcal{E}_i) \cup Supp(\mathcal{E}_i)) \qquad \text{by Definition 5.2} \\
&= Th^{re}\left(\bigcup_{i=0}^{\infty}(Re\,(\mathcal{E}_i) \cup Form(\mathcal{E}_i) \cup Supp(\mathcal{E}_i))\right) \\
&= Th^{re}\left(\bigcup_{i=0}^{\infty} Re\,(\mathcal{E}_i) \cup \bigcup_{i=0}^{\infty} Form(\mathcal{E}_i) \cup \bigcup_{i=0}^{\infty} Supp(\mathcal{E}_i)\right) \\
&= Th^{re}\left(Re\left(\bigcup_{i=0}^{\infty} \mathcal{E}_i\right) \cup Form\left(\bigcup_{i=0}^{\infty} \mathcal{E}_i\right) \cup Supp\left(\bigcup_{i=0}^{\infty} \mathcal{E}_i\right)\right) \\
&= Th^{re}(Re\,(\mathcal{E}) \cup Form(\mathcal{E}) \cup Supp(\mathcal{E})) \\
&= Th^{re}\big(Re^{+}(\mathcal{E})\big) \\
&\equiv Re^{+}(\mathcal{E})\,.
\end{aligned}$$

So $Re^{+}(\mathcal{E})$, $Re^{+}(\mathcal{E}_i)$, $0 \leq i$ satisfies the conditions of an extension, given in Theorem A.2.

2. Define $\mathcal{E} = Re^{-1}(E)$ and $\mathcal{E}_i = Re^{-1}(E_i)$ for every $i \geq 0$. We need to show that $\mathcal{E}$ and $\mathcal{E}_i$, $i \geq 0$, so defined satisfy the conditions for an assertional extension given in Theorem B.12.

   If $E \vdash \bot$ then $Re^{-1}(E) = \mathcal{E} = \mathcal{A}$, and $\mathcal{E}$ is the sole (inconsistent) assertional extension of $(D, \mathcal{W})$. Consequently, assume that E is consistent.

   We use induction for the sets $\mathcal{E}_i$, $i \geq 0$.

   Base:

$$\mathcal{E}_0 = Re^{-1}(E_0) = Re^{-1}(W_a) = Re^{-1}(Re\,(\mathcal{W})) \equiv \mathcal{W}.$$

   (The final step follows from Lemma B.13.)

   Step:

$$\begin{array}{rcll} \mathcal{E}_{i+1} & = & Re^{-1}(E_{i+1}) & \text{by the definition of } \mathcal{E}_{i+1} \\ & = & Re^{-1}\Big(Th(E_i) \quad \bigcup \quad \{\langle\gamma, \psi \wedge \beta \wedge \gamma\rangle^{re} \wedge \beta \wedge \gamma \mid & \\ & & \qquad \frac{\langle\alpha,\psi\rangle^{re} : \beta\wedge\gamma}{\langle\gamma,\psi\wedge\beta\wedge\gamma\rangle^{re}\wedge\beta\wedge\gamma} \in D_a, \langle\alpha, \psi\rangle^{re} \in E_i, E \cup \{\beta \wedge \gamma\} \not\vdash \bot\Big\}\Big) & \\ & & \qquad \text{by Theorem A.2.} & \end{array}$$

   Let RS be the expression following the main $\cup$ in the preceding. So:

$$\begin{array}{rcll} \mathcal{E}_{i+1} & = & Re^{-1}\Big(Th^{re}(E_i) \quad \bigcup \quad RS\Big) & \\ & = & Re^{-1}(Th^{re}(E_i)) \quad \bigcup \quad Re^{-1}(RS) & \\ & = & \widehat{Th}\big(Re^{-1}(E_i)\big) \quad \bigcup \quad Re^{-1}(RS) & \text{by Lemma B.14} \\ & = & \widehat{Th}(\mathcal{E}_i) \quad \bigcup \quad Re^{-1}(RS) & \text{by the induction hypothesis} \end{array}$$

   Further:

$$\begin{array}{rcl} Re^{-1}(RS) & = & Re^{-1}\left(\{\langle\gamma, \psi \wedge \beta \wedge \gamma\rangle^{re} \wedge \beta \wedge \gamma \mid \right. \\ & & \qquad \left. \frac{\langle\alpha,\psi\rangle^{re} : \beta\wedge\gamma}{\langle\gamma,\psi\wedge\beta\wedge\gamma\rangle^{re}\wedge\beta\wedge\gamma} \in D_a,\ \langle\alpha, \psi\rangle^{re} \in E_i,\ E \cup \{\beta \wedge \gamma\} \not\vdash \bot\Big\}\right) \\ & = & Re^{-1}\left(\{\langle\gamma, \psi \wedge \beta \wedge \gamma\rangle^{re} \wedge \beta \wedge \gamma \mid \right. \\ & & \qquad \left. \frac{\alpha : \beta}{\gamma} \in D,\ \langle\alpha, \psi\rangle^{re} \in E_i,\ E \cup \{\beta \wedge \gamma\} \not\vdash \bot\Big\}\right) \end{array}$$

   We have that $Re^{-1}(\{\langle\gamma, \psi \wedge \beta \wedge \gamma\rangle^{re} \wedge \beta \wedge \gamma\}) = \{\langle\gamma, \psi \wedge \beta \wedge \gamma\rangle\}$.

As well, by the induction hypothesis we have that $\langle \alpha, \psi \rangle^{re} \in E_i$ iff $\langle \alpha, \psi \rangle \in \mathcal{E}_i$. Substituting in the preceding, we continue:

$$\begin{aligned}
Re^{-1}(RS) &= \Big\{ \langle \gamma, \psi \wedge \beta \wedge \gamma \rangle \mid \tfrac{\alpha : \beta}{\gamma} \in D,\ \langle \alpha, \psi \rangle \in \mathcal{E}_i,\ E \cup \{\beta \wedge \gamma\} \not\vdash \bot \Big\} \\
&= \Big\{ \langle \gamma, \psi \wedge \beta \wedge \gamma \rangle \;\Big|\; \tfrac{\alpha : \beta}{\gamma} \in D,\ \langle \alpha, \psi \rangle \in \mathcal{E}_i, \\
&\qquad \mathit{Supp}\big(Re^{-1}(E)\big) \cup \mathit{Form}\big(Re^{-1}(E)\big) \cup \{\beta \wedge \gamma\} \not\vdash \bot \Big\} \quad \text{by Lemma B.15} \\
&= \Big\{ \langle \gamma, \psi \wedge \beta \wedge \gamma \rangle \mid \tfrac{\alpha : \beta}{\gamma} \in D,\ \langle \alpha, \psi \rangle \in \mathcal{E}_i,\ \mathit{Supp}(\mathcal{E}) \cup \mathit{Form}(\mathcal{E}) \cup \{\beta \wedge \gamma\} \not\vdash \bot \Big\}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{E}_{i+1} &= \widehat{Th}(\mathcal{E}_i) \quad \bigcup \\
&\qquad \Big\{ \langle \gamma, \psi \wedge \beta \wedge \gamma \rangle \mid \tfrac{\alpha : \beta}{\gamma} \in D,\ \langle \alpha, \psi \rangle \in \mathcal{E}_i,\ \mathit{Supp}(\mathcal{E}) \cup \mathit{Form}(\mathcal{E}) \cup \{\beta \wedge \gamma\} \not\vdash \bot \Big\} \\
&= \widehat{Th}(\mathcal{E}_i) \cup \Big\{ \langle \gamma, \mathit{Supp}(\alpha) \wedge \beta \wedge \gamma \rangle \;\Big|\; \tfrac{\alpha : \beta}{\gamma} \in GD_i^a \Big\}
\end{aligned}$$

This completes the induction.

Lastly,

$$\mathcal{E} = Re^{-1}(E) = Re^{-1}\left(\bigcup_{i=0}^{\infty} E_i \right) = \bigcup_{i=0}^{\infty} Re^{-1}(E_i) = \bigcup_{i=0}^{\infty} \mathcal{E}_i.$$

Thus $\mathcal{E}$ satisfies the conditions of an assertional extension as given in Theorem B.12.

■

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