Meshes and Subdivision

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CMPT 464/764: Geometric Modeling in Computer Graphics

Lecture 5, given by Prof. Ali Mahdavi-Amiri

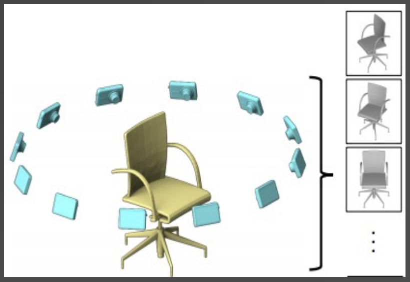
Outline on 3D representations

- Implicit reps
- Parametric reps
- Meshes (subdivision)
- Point clouds
- Volumes
- Projective reps
- Structured reps

Smooth curves and surfaces

Discrete representations

3D → 2D



Parts + relations = structures
Encompasses all low-level reps

Today

- Implicit reps
- Parametric reps

Smooth curves and surfaces

- Meshes (subdivision)
- Point clouds
- Volumes

Discrete representations

- Projective reps

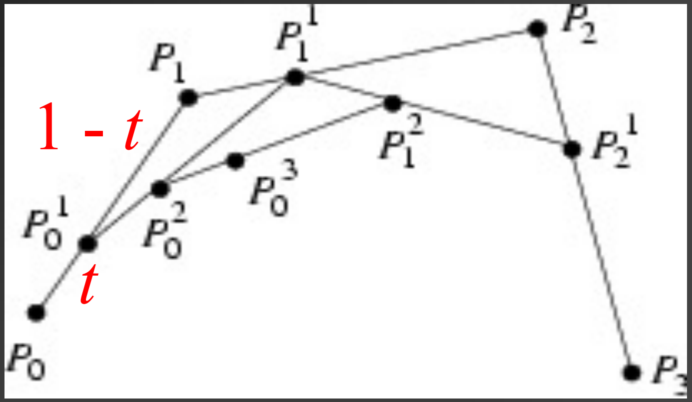
3D → 2D

- Structured reps

Parts + relations = structures

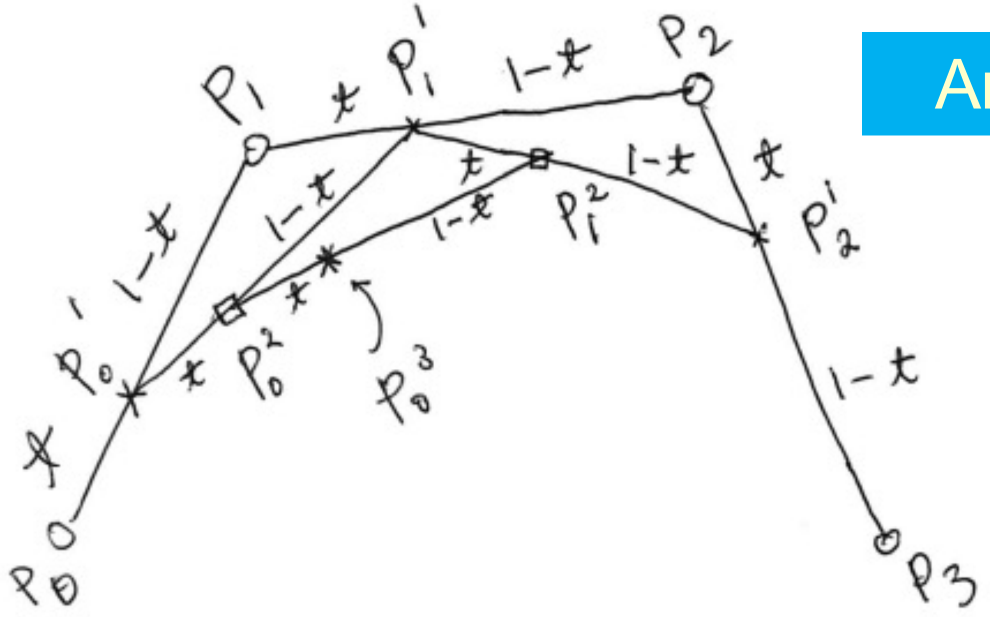
Encompasses all low-level reps

Recall exercise: identify this curve?



- $0 \leq t \leq 1$: express P_0^1, P_1^1, P_2^1 as linear combinations of P_0, …, P_3
- Then express P_0^2 and P_1^2 as linear combinations of P_0^1, P_1^1, and P_2^1
- Finally, express P_0^3 as a linear combination of P_0^2 and P_1^2
- **What is this curve?**

Answer: Cubic Bezier Curve



Bernstein Polynomials of degree 3 :

t^3

$3t^2(1-t)$

$3t(1-t)^2$

$(1-t)^3$

$$P_0^1 = tP_1 + (1-t)P_0$$

$$P_1^1 = tP_2 + (1-t)P_1$$

$$P_2^1 = tP_3 + (1-t)P_2$$

$$P_0^2 = tP_1^1 + (1-t)P_0^1 = t^2P_2 + 2t(1-t)P_1 + (1-t)^2P_0$$

$$P_1^2 = tP_2^1 + (1-t)P_1^1 = t^2P_3 + 2t(1-t)P_2 + (1-t)^2P_1$$

$$P_0^3 = tP_1^2 + (1-t)P_0^2 = \underline{t^3}P_3 + \underline{3t^2(1-t)}P_2 + \underline{3t(1-t)^2}P_1 + \underline{(1-t)^3}P_0$$

Standard derivation of Cubic Bézier

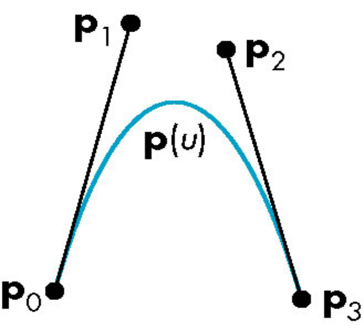
- Defined by four control points P_0, P_1, P_2, and P_3

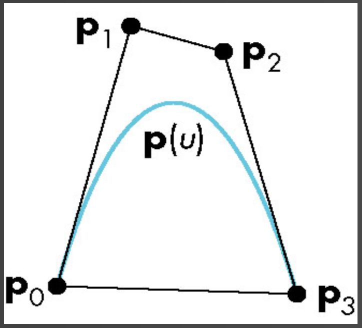
$x(0) = P_0$

$x(1) = P_3$

$x'(0) = 3(P_1 - P_0)$

$x'(1) = 3(P_3 - P_2)$





[Angel 02]

Standard derivation of Cubic Bézier

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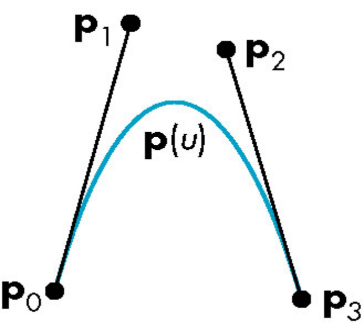
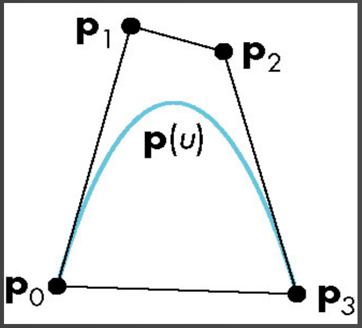




[Angel 02]

- **Convex hull property**: Bézier curve lies within the convex hull of the four control points – good control
- Convex hull of a set of points on the plane: tightest convex polygon enclosing the set – why would it be useful in graphics?

Convex hull property

- A cubic curve satisfies the convex hull property if it lies within the convex hull of its four control points

- Convex hull property is satisfied if and only if the basis polynomials $b_1(t)$, $b_2(t)$, $b_3(t)$, $b_4(t)$ satisfy:

  1. $0 \leq b_1(t), b_2(t), b_3(t), b_4(t) \leq 1$ for $t \in [0, 1]$, and

  2. $b_1(t) + b_2(t) + b_3(t) + b_4(t) = 1$

- Then each point of the curve is a **convex combination** of the control points

- The basis $b_i(t)$ form a **partition of unity**

Cubic Bezier change-of-basis matrix

Symmetric matrix!

$$M_{Bezier} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Exercise: derive the Bezier change of basis matrix, by learning from derivation for cubic Hermite from last week

Bézier bases: Berstein polynomials

$B_0(t) = (1 – t)^3$, $B_1(t) = 3t(1 – t)^2$,

$B_2(t) = 3t^2(1 – t)$, $B_3(t) = t^3$

- Well-known as the **Bernstein Polynomials** of degree 3
- Bernstein polynomials of degree *n*

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

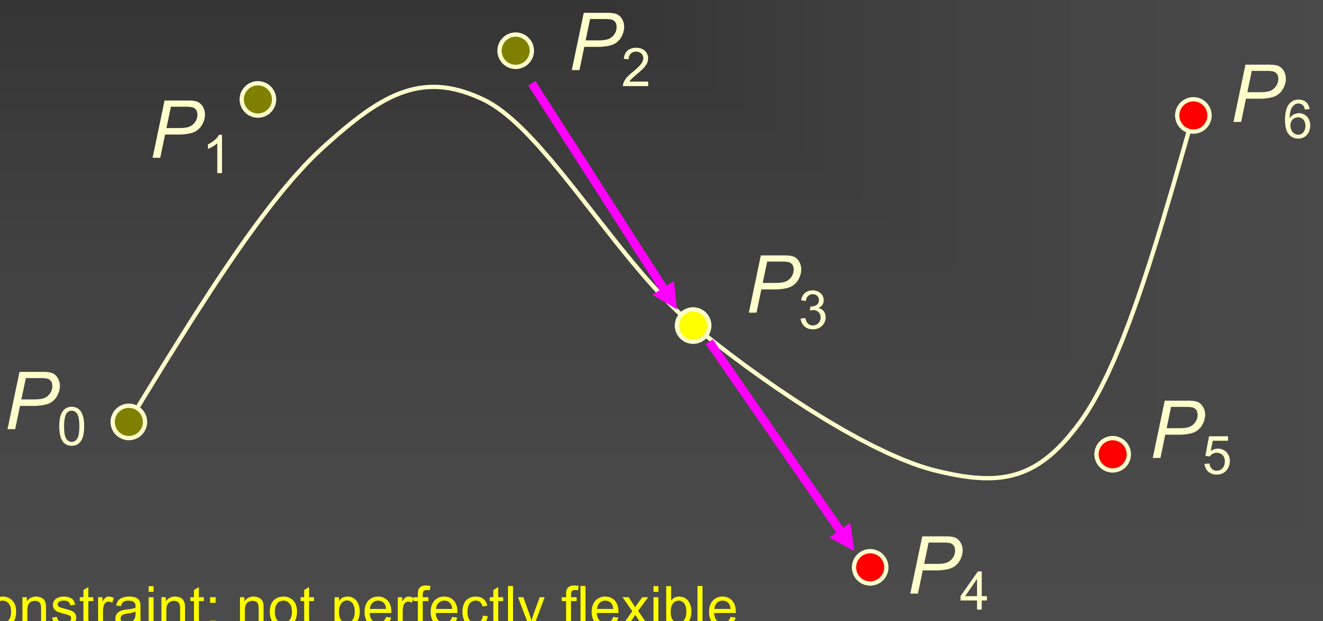
- We have (a recursion)

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

- Partition of unity easy to see: $\Sigma_i B_i(t) = [t + (1 – t)]^n$

Piecewise cubic Bézier curves

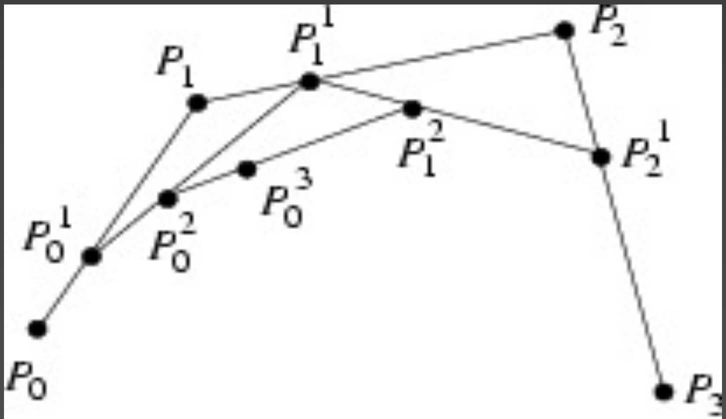
- How to ensure **C**^{1} or **G**^{1} for piecewise Bézier curves?
- Each segment is parameterized over [0, 1] as usual

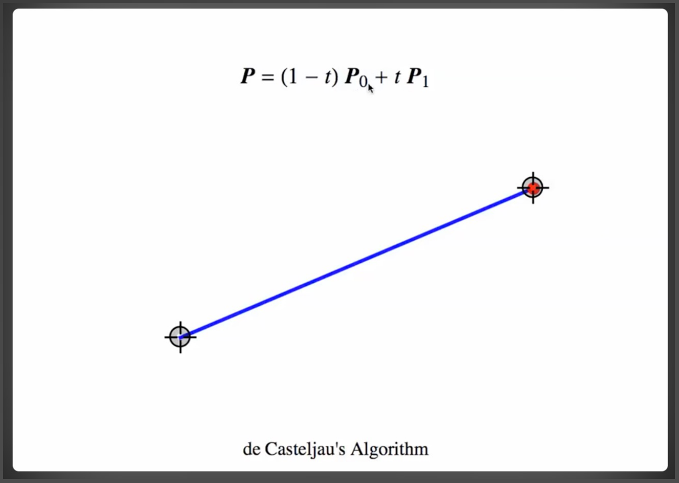


A constraint: not perfectly flexible

How would you have rendered Bezier?

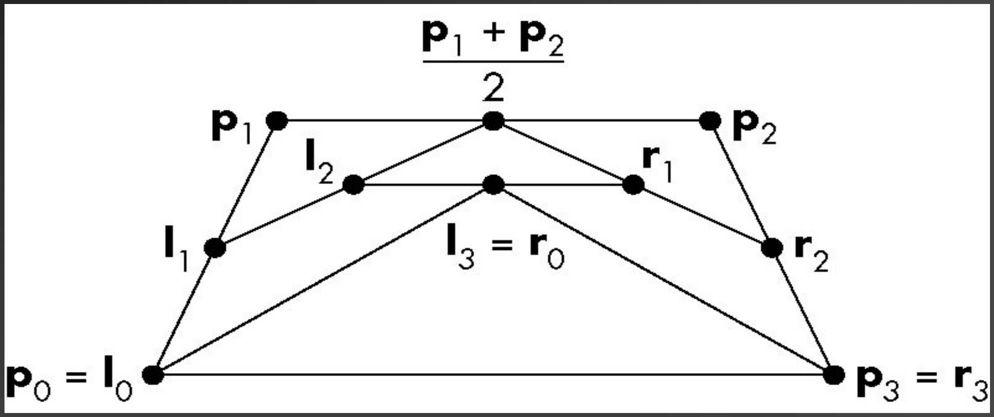
- Treat it as a generic polynomial curve and apply standard polynomial evaluation
- But Bezier curves are special and there is a nice alternative, using the de Casteljau's procedure below (also Youtube link)





https://www.youtube.com/watch?v=YATikPP2q70

Bézier curve via de Casteljau



- Original four control points P_0, P_1, P_2, P_3 become seven new control points l_0, l_1, l_2, $l_3 = r_0$, r_1, r_2, r_3
- Each set of new control points control half of the Bezier curve
- In the limit, the control points obtained form the Bézier curve determined by P_0, P_1, P_2, P_3

A proof (aside)

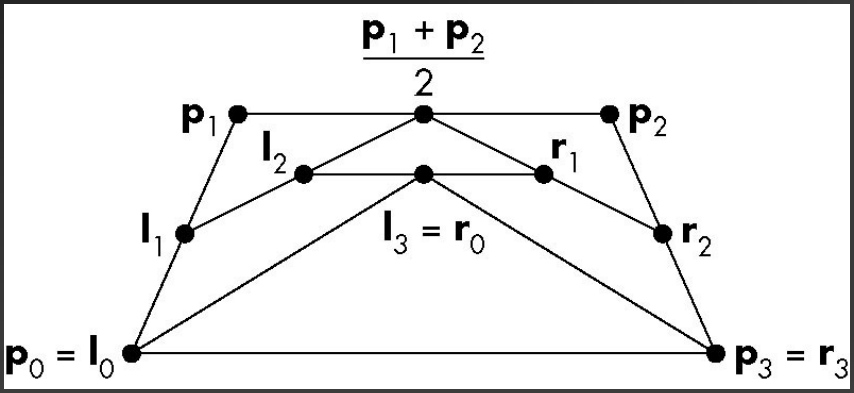
- Bézier curve **$p(t) = TM_BP$**
- $p(1/2) = P_0/8 + 3P_1/8 + 3P_2/8 + P_3/8 = l_3$
- Reparameterize first half of $p(t)$: $t\in[0,1/2]$ to $q(s)$: $s\in[0,1]$

$$q(s) = p(s/2) = [1 \quad s/2 \quad s^2/4 \quad s^3/8]M_BP =$$

$$S\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} M_BP = SM_B\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 1/8 & 3/8 & 3/8 & 1/8 \end{bmatrix} P = SM_B\begin{bmatrix} l_0 \\ l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

- Second half of $p(t)$ is similar

de Casteljau = subdivision



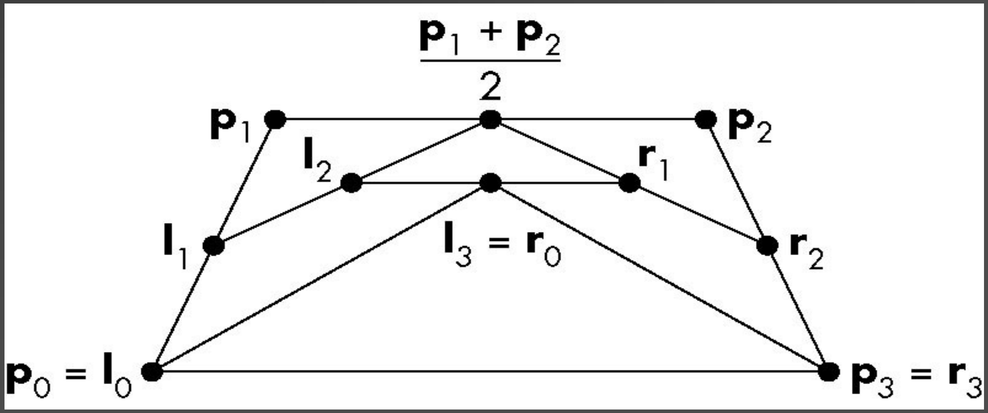
$$\begin{bmatrix} l_0 \\ l_1 \\ l_2 \\ l_3 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 1/8 & 3/8 & 3/8 & 1/8 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

In general, $\mathbf{p}^{(k+1)} = S\mathbf{p}^{(k)}$, S is a **subdivision matrix**

- This is a **subdivision** scheme:
  - Subdivide to obtain new points (**refinement** procedure)
  - New points (*l*’s and *r*’s) are **weighted averages** of the old (*P*’s)
  - Note: de Casteljau’s is not interpolatory except at the boundary

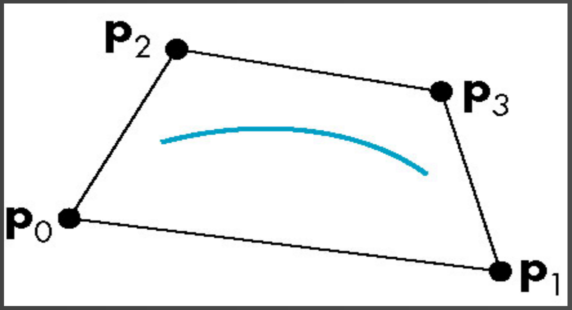
Cubic Bézier via subdivision

- Keep subdividing until sufficiently fine, then connect adjacent control points obtained to form polygonal curve
- A recursive algorithm
- Involve only additions and divisions by 2 — shifts
- **Very fast**
- **Multi-resolution!**

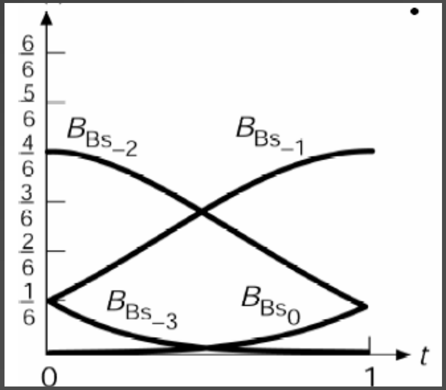


Second example: cubic B-splines

- Each cubic B-spline segment is specified by four control points
- Has the convex hull property
- No interpolation in general
- **Big advantage: C^2 continuous**
- The cubic B-spline change of basis matrix

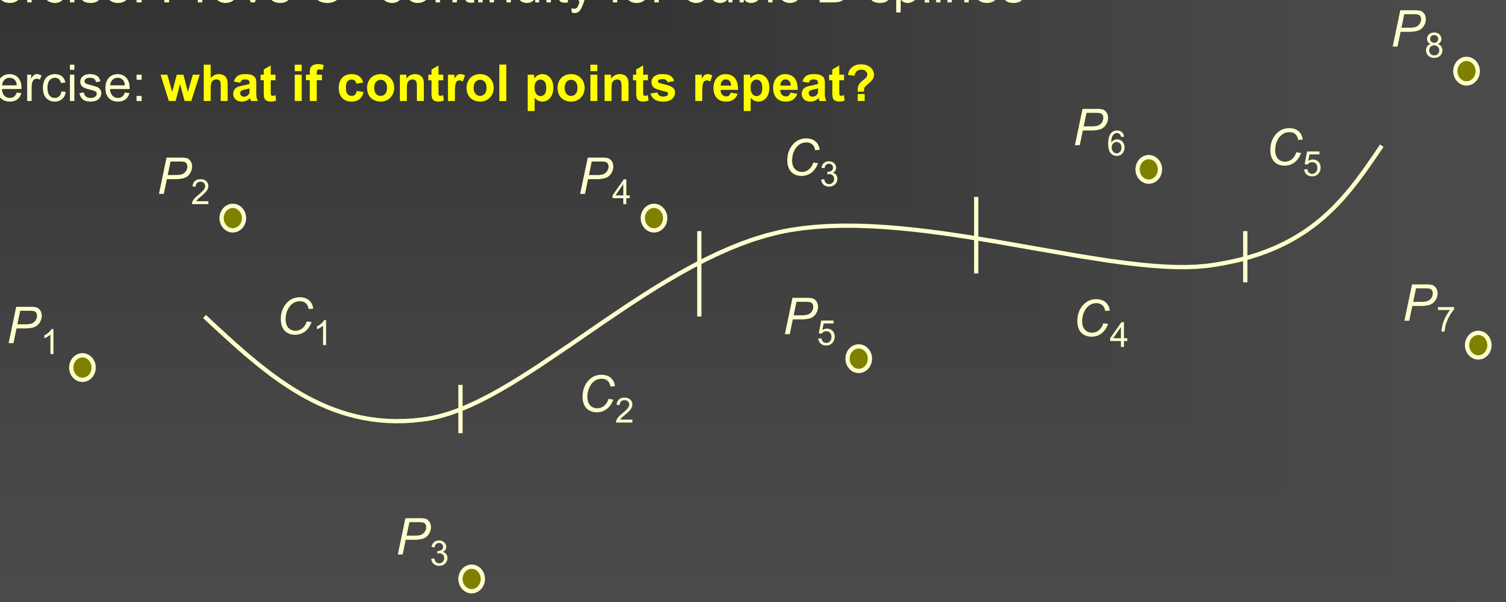


$$M_{B-spline} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$



Piece-wise cubic B-splines

- Two consecutive segments share three control points
- *m* control points → *m* – 3 segments
- Exercise: Prove **C**^{2} continuity for cubic B-splines
- Exercise: **what if control points repeat?**



B-Splines through subdivision

- B-splines can also be generated via subdivision, in the same form

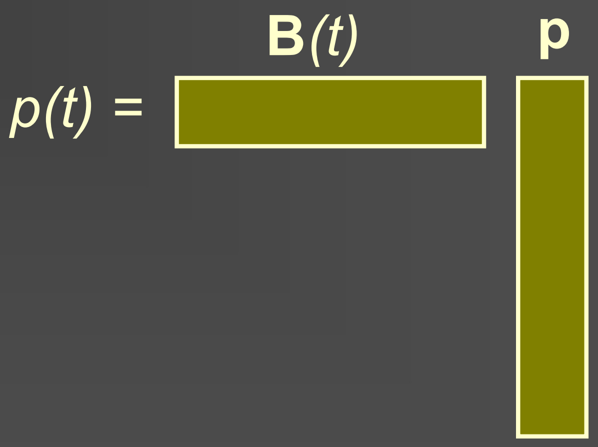
$$\mathbf{c}^{(k+1)} = S\mathbf{c}^{(k)}$$

- Consider any curve represented in l-th degree B-spline basis (the B's)

$$\boxed{p(t) = \sum_i p_i B_l^i(t)}$$

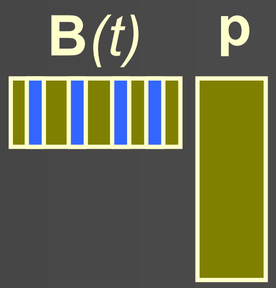
where l is the B-spline degree, i the index, and p_i's are control points.

- In matrix form, we have $p(t) = \mathbf{B}(t)\ \mathbf{p}$, where
  $\mathbf{p}$: column vector of control points
  $\mathbf{B}(t)$: row vector of B-spline bases



B-Splines via subdivision

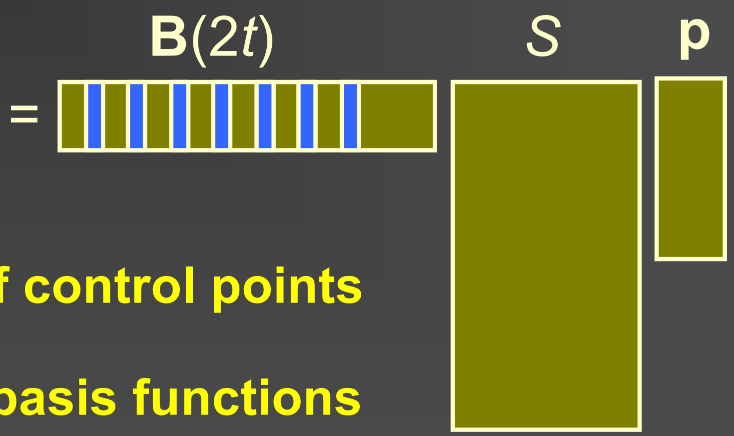
- Continue from matrix representation: $p(t) = \mathbf{B}(t)\ \mathbf{p}$ =



- Eventually, we shall rewrite

$$p(t) = \mathbf{B}(t)\ \mathbf{p} = \mathbf{B}(2t)\ S\ \mathbf{p}$$

where



  - S is the subdivision matrix
  - $\mathbf{p}' = S\ \mathbf{p}$ is the **new, refined set of control points**
  - $\mathbf{B}(2t)$ represent **refined B-spline basis functions**
- Let us focus on **uniform B-splines**

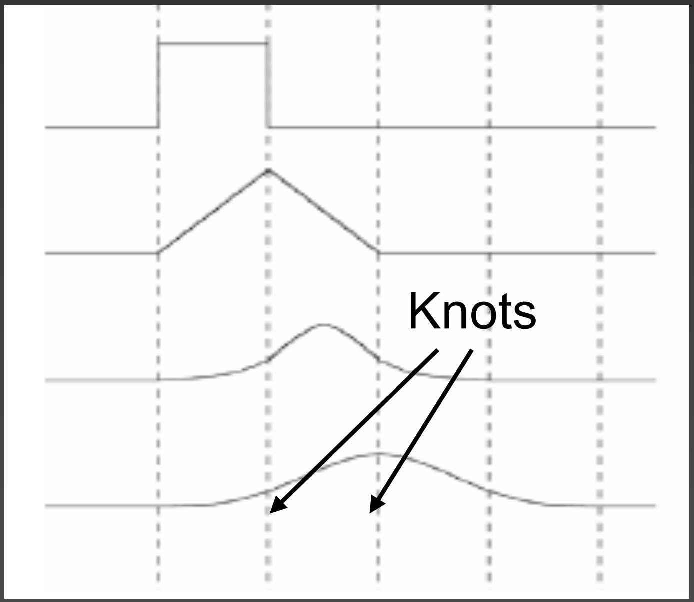
What are splines?

- An *m*-th degree spline is a **piecewise polynomial** of degree *m* that is $\mathbf{C}^{m-1}$
- A spline curve is defined by a **knot sequence;** the knots are at parametric *t* values where the **polynomial pieces join**
- Most common are **uniform** knot spacing, i.e., $t = 0, 1, 2, \ldots$ Nonuniform knot spacing or repeated knots are also possible
- A **spline basis** often serves as a blending function with **local control**
- Resulting spline curve is given by a set of control points blended by **shifted or translated** versions of the spline basis

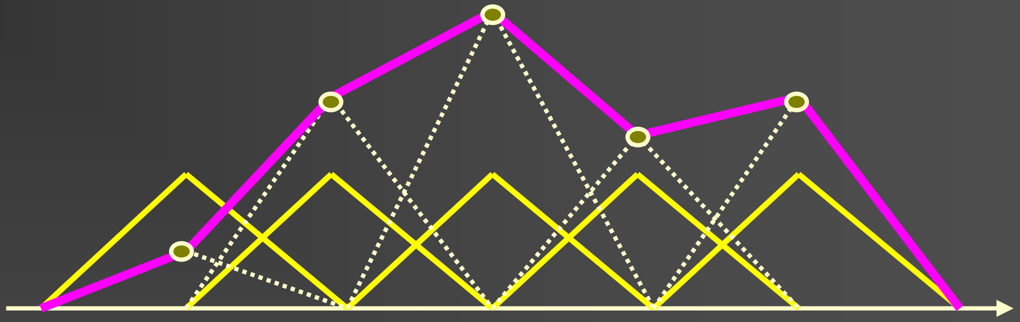
Example: uniform B-splines

- B-splines: one particular class of spline curves

Degree 0-3 uniform B-splines



Note **local control** and **increased continuity**

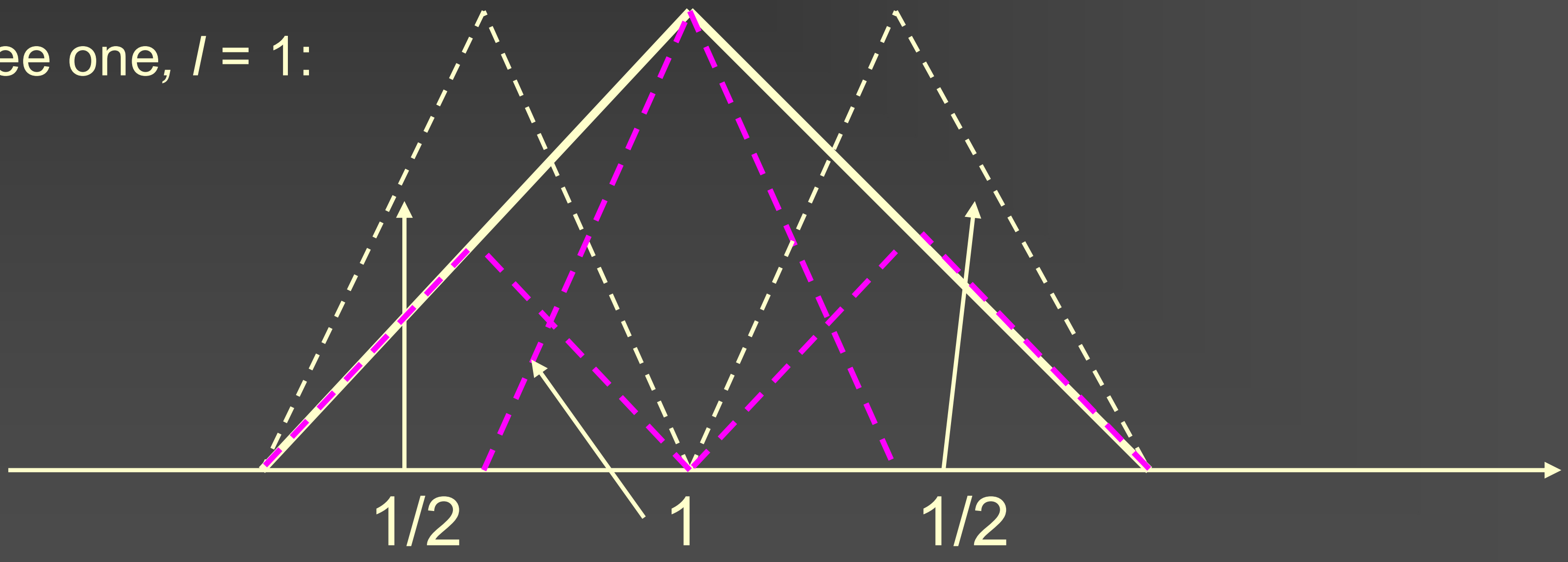


A piecewise linear curve (C^0) obtained by blending five uniform degree-1 B-splines with control points

Key property of uniform B-splines

- A uniform B-spline can be written as a **linear combination** of translated (k) and **dilated** or **compressed** ($2t$) copies of itself
- This is the key to connect B-splines to subdivision

Degree one, $l = 1$:



Technical details (aside)

- B-spline of degree *l*, $B_l(t)$, is $\mathbf{C}^{l-1}$ continuous, $l \geq 1$
- The *i*-th B-spline, B^i_l, is simply a **translate** of the B-spline $B_l(t)$ or $B_l^0(t)$: $B^i_l(\mathrm{t}) = B_l(t - i)$ — right shift of *i* units
- B-splines satisfy the **refinement equation**

$$B_l(t) = \frac{1}{2^l}\sum_{k=0}^{l+1}\binom{l+1}{k} B_l(2t-k)$$

— binomial coefficients

$$B(t) \xrightarrow{\text{compress then translate by } k/2} B(2t-k)$$

  - A uniform B-spline can be written as a linear combination of translated (*k*) and **dilated** or **compressed** (2*t*) copies of itself
  - This is the key to connect B-splines to subdivision

B-spline via subdivision

- Using the refinement equation from last slide, we have

$$\mathbf{B}(t) = \mathbf{B}(2t)\, S$$

where the entries of S are given by

$$S_{2i+k,i} = s_k = \frac{1}{2^l}\binom{l+1}{k}$$

- Thus, $p(t) = \mathbf{B}(t)\,\mathbf{p} = \mathbf{B}(2t)\, S\mathbf{p}$

We have changed B-spline bases $\mathbf{B}(t)$ to $\mathbf{B}(2t)$, where each element of $\mathbf{B}(2t)$ is half as wide as one in $\mathbf{B}(t)$ and the sequence in $\mathbf{B}(2t)$ are spaced twice as dense

Refinement of B-splines

Linear B-spline case; this extends to B-splines of any degree.

What have we done?

- **Refined the B-spline basis** functions, and at the same time,
- **Refined the set of control points p**
- Twice as many new control points $\mathbf{p}' = S\mathbf{p}$:
  - One new point (an **odd point**) is inserted between two consecutive control points in **p**
  - Each control point in **p** (an **even point**) is either retained (**interpolatory**) or moved (**approximating**) in **p**’
- *S* is the **subdivision matrix**

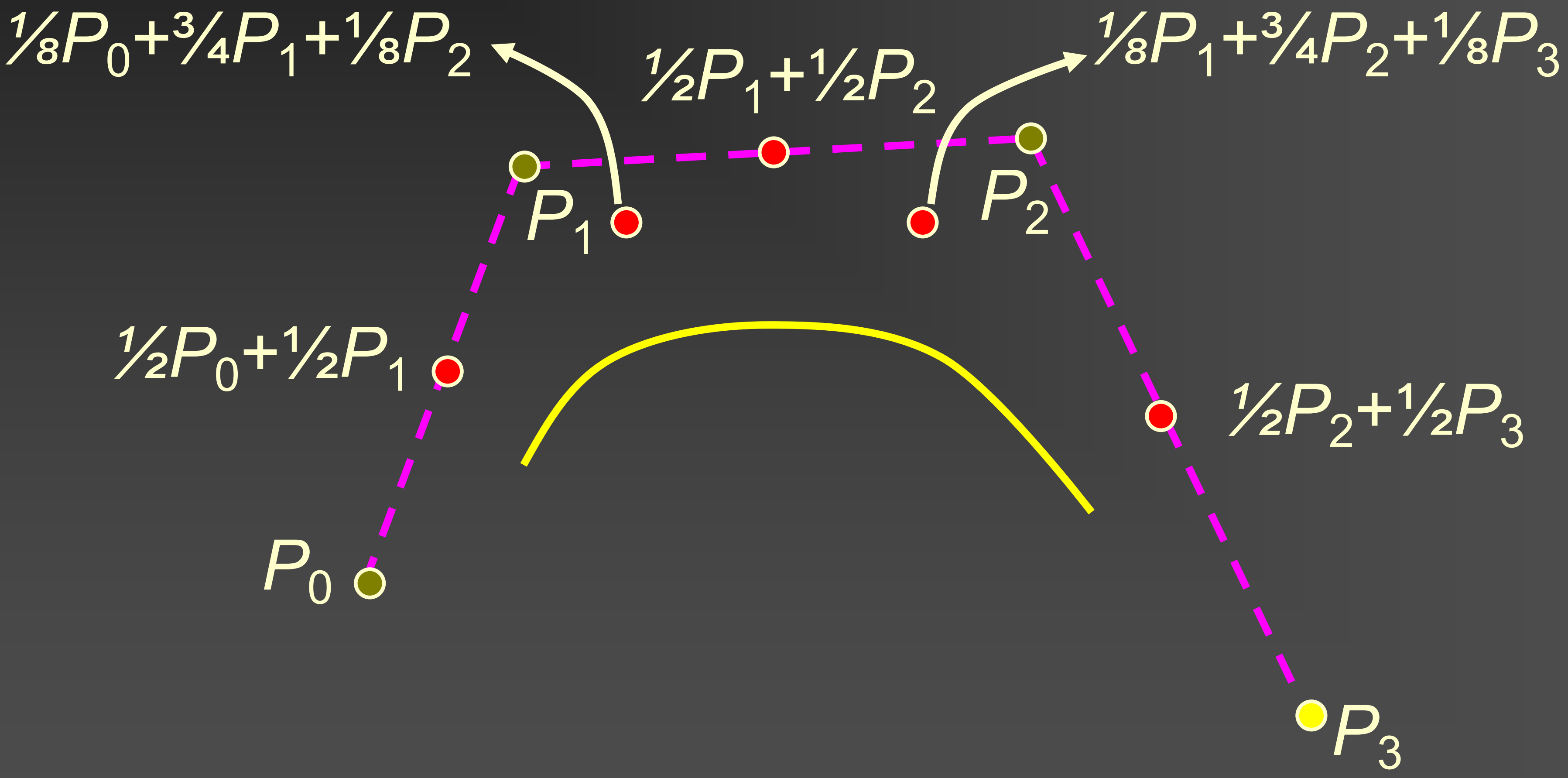
Subdivision matrix *S*

$$\begin{array}{r}
\\
\\
\\
\\
\text{even points}\\
\text{odd points}\\
\\
\\
\\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0\\
1/2 & 1/2 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 1/2 & 1/2 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 1/2 & 1/2 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1/2 & 1/2\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\qquad
\begin{bmatrix}
4/8 & 0 & 0 & 0 & 0\\
6/8 & 1/8 & 0 & 0 & 0\\
4/8 & 4/8 & 0 & 0 & 0\\
1/8 & 6/8 & 1/8 & 0 & 0\\
0 & 4/8 & 4/8 & 0 & 0\\
0 & 1/8 & 6/8 & 1/8 & 0\\
0 & 0 & 4/8 & 4/8 & 0\\
0 & 0 & 1/8 & 6/8 & 1/8\\
0 & 0 & 0 & 4/8 & 4/8\\
0 & 0 & 0 & 1/8 & 6/8\\
0 & 0 & 0 & 0 & 4/8
\end{bmatrix}$$

for linear uniform B-spline

for uniform cubic B-splines

Cubic B-splines via subdivision



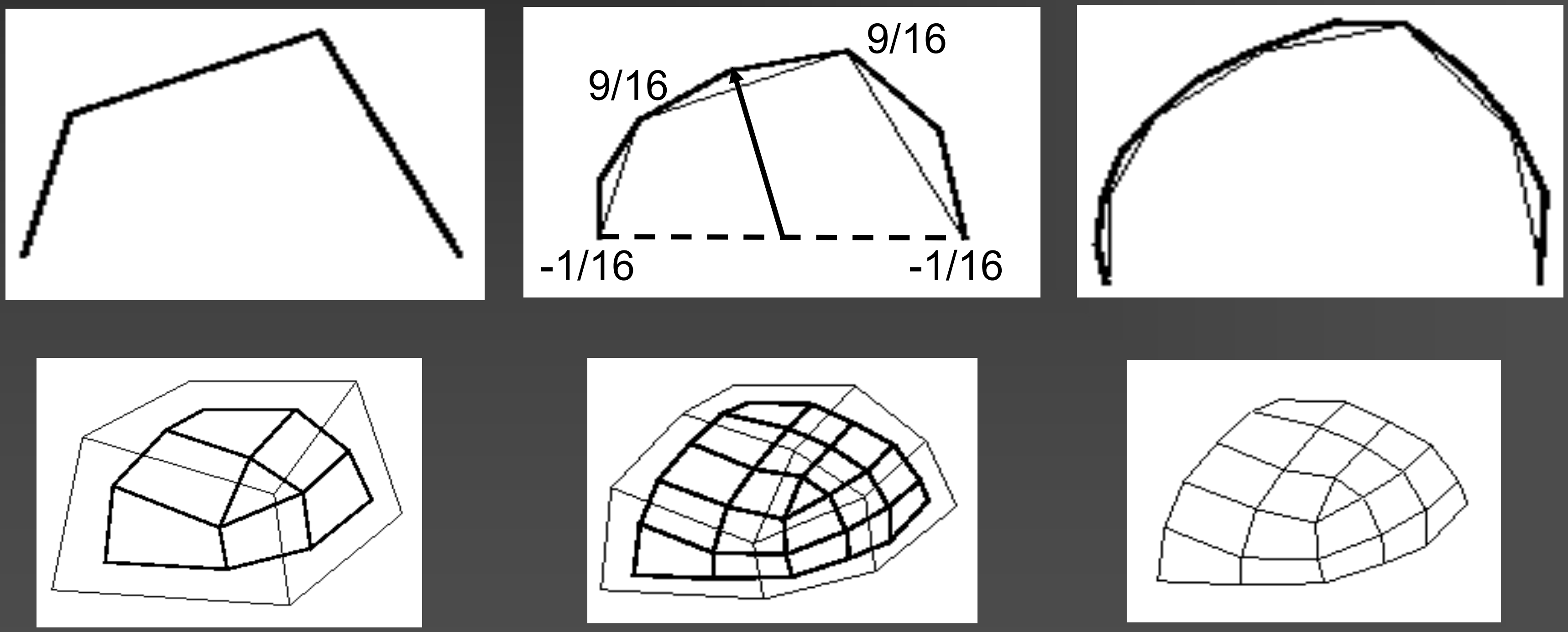
Convergence of subdivision (aside)

$$\mathbf{p}^{j} = S^{j}\,\mathbf{p}^{0}$$

- The recursively refined set of control points converge to the actual spline curve $p(t) = \sum \mathbf{p}_i \, B_i^j(t)$
- Have **geometric rate of convergence**, i.e., difference decrease by constant factor (see notes) — $||\varepsilon^{j}|| < c\gamma^{j}$
- Can thus obtain spline curves via subdivision, just like de Casteljau for Bezier curves!

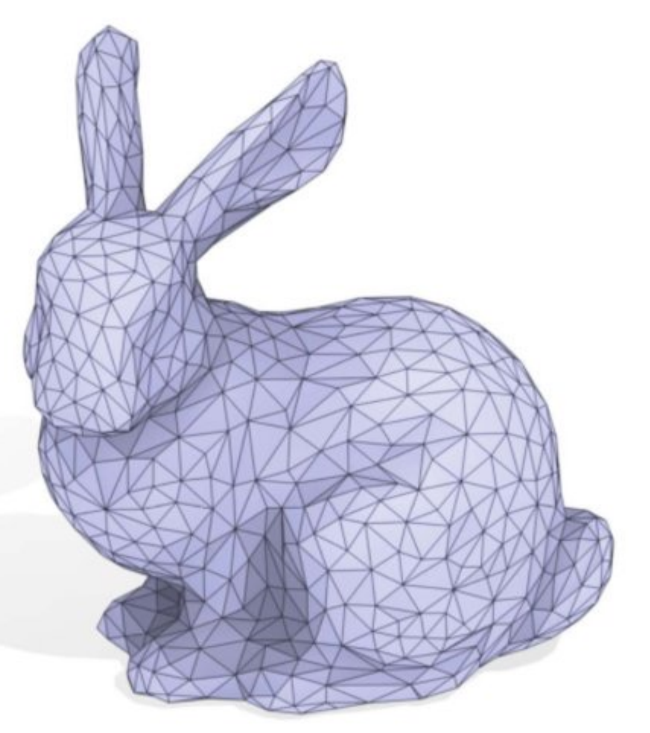
Idea of subdivision

- A subdivision curve (or surface) is the **limit** of a sequence of successively refined control polygon (or control **mesh**)



What are (polygonal) meshes?

- Polygonal mesh: composed of a set of polygons pasted along their edges – triangles most common
- Still most popular in graphics and CAD

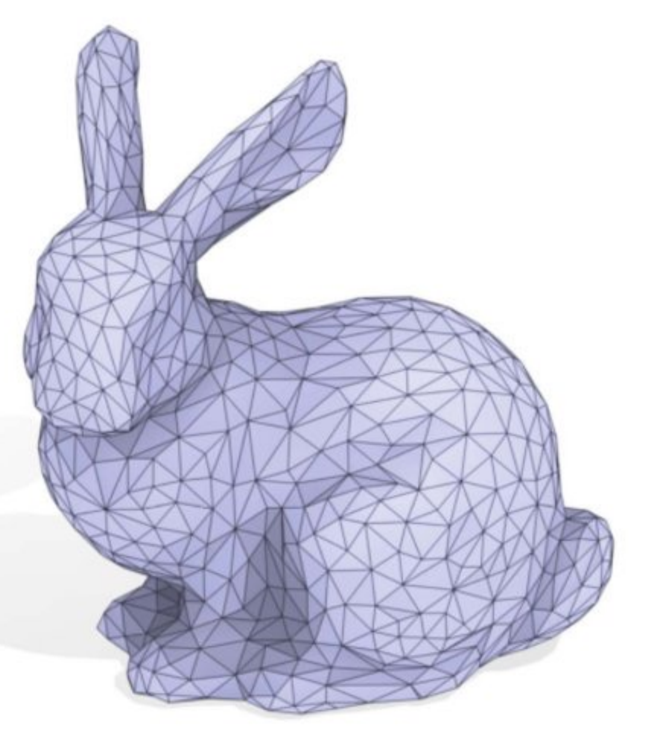


A triangle bunny mesh

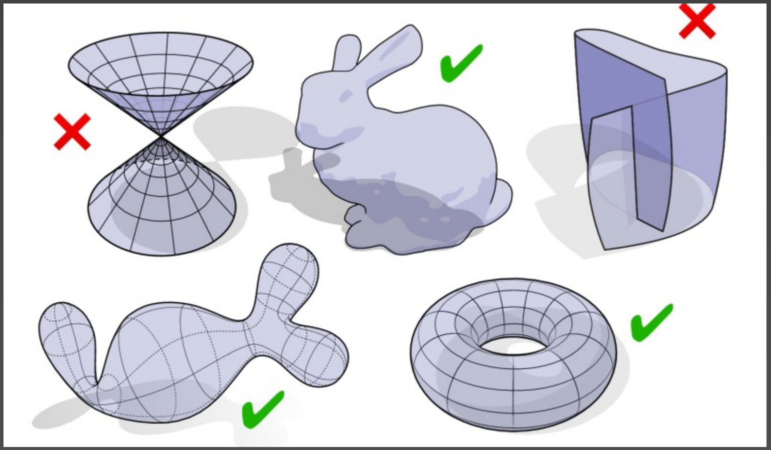
What are (polygonal) meshes?

- Polygonal mesh: composed of a set of polygons pasted along their edges – triangles most common
- Still most popular in graphics and CAD
- Basic mesh components and properties:

vertices, edges, faces, valences, normal, curvature, boundaries, **manifold or not**

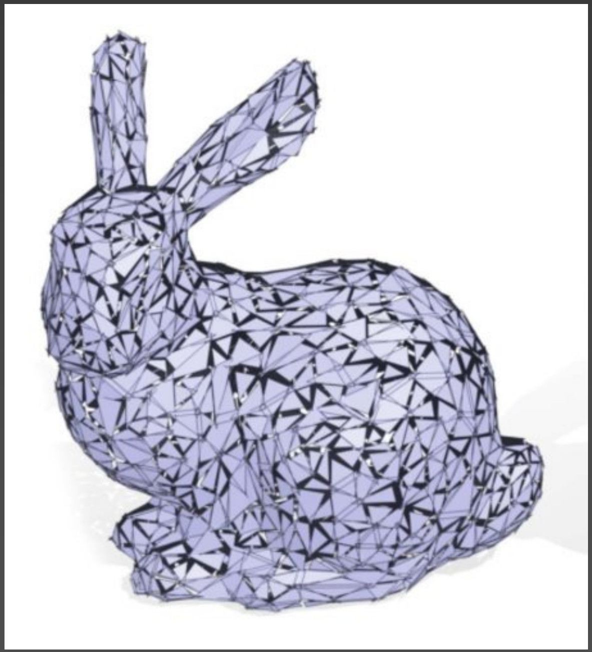


A triangle bunny mesh

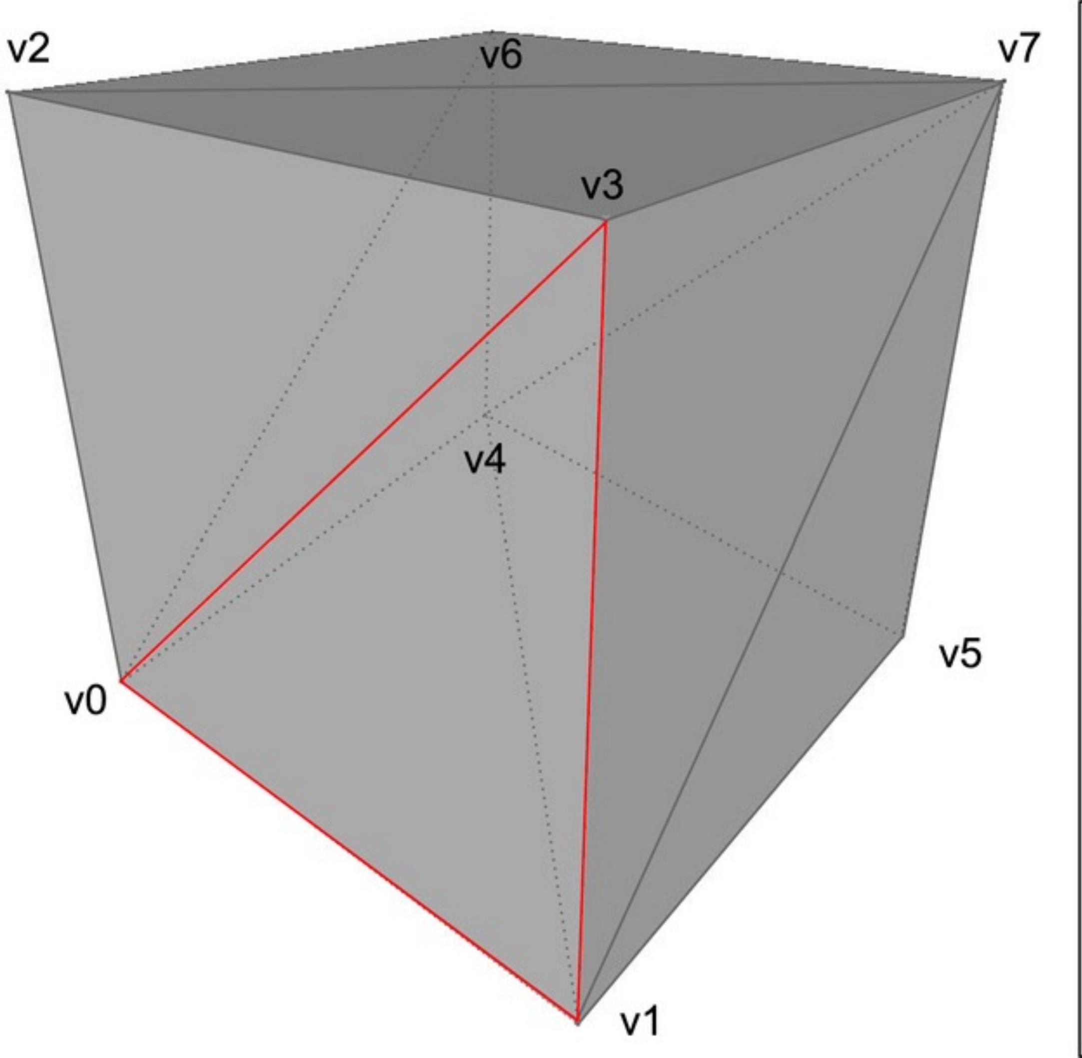


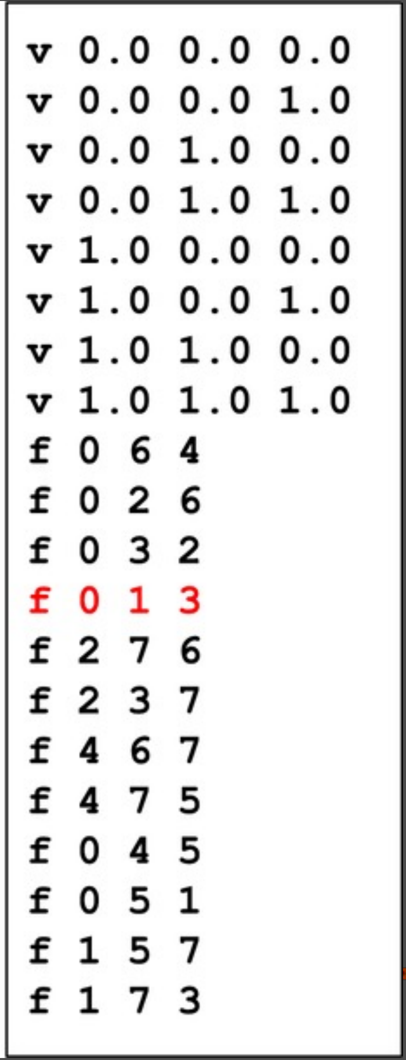
Polygon soup

- For each triangle, just store 3 coordinates, no connectivity information
- Not much different from point clouds
- MobileNeRF is a polygon soup
- 3DGS is a dense soup of Gaussians



Mesh storage format: OBJ

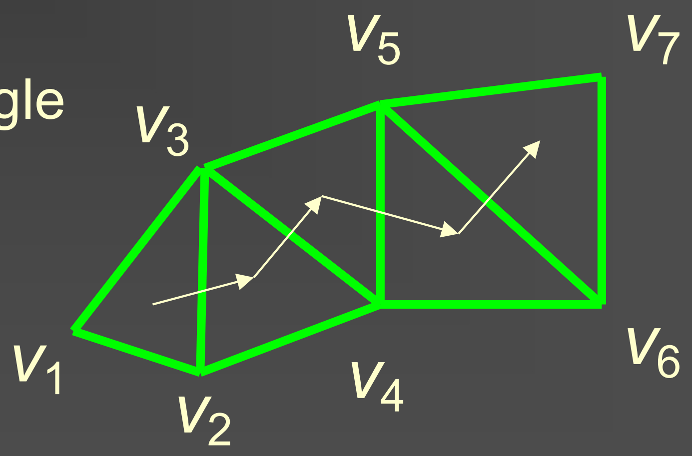




```
v 0.0 0.0 0.0
v 0.0 0.0 1.0
v 0.0 1.0 0.0
v 0.0 1.0 1.0
v 1.0 0.0 0.0
v 1.0 0.0 1.0
v 1.0 1.0 0.0
v 1.0 1.0 1.0
f 0 6 4
f 0 2 6
f 0 3 2
f 0 1 3
f 2 7 6
f 2 3 7
f 4 6 7
f 4 7 5
f 0 4 5
f 0 5 1
f 1 5 7
f 1 7 3
```

More efficient storage: triangle strips

- A triangle strip gives a compact way of representing a set of triangles
- For *n* triangles in a strip, instead of passing through and transform 3*n* vertices, only need *n*+2 vertices
- In a sequence, e.g., v_1, v_2, v_3, v_4, …, first three vertices form the first triangle; each subsequent vertex forms a new triangle with its preceding two vertices
- Many algorithms exist to "stripify" a triangle mesh into **long triangle strips**



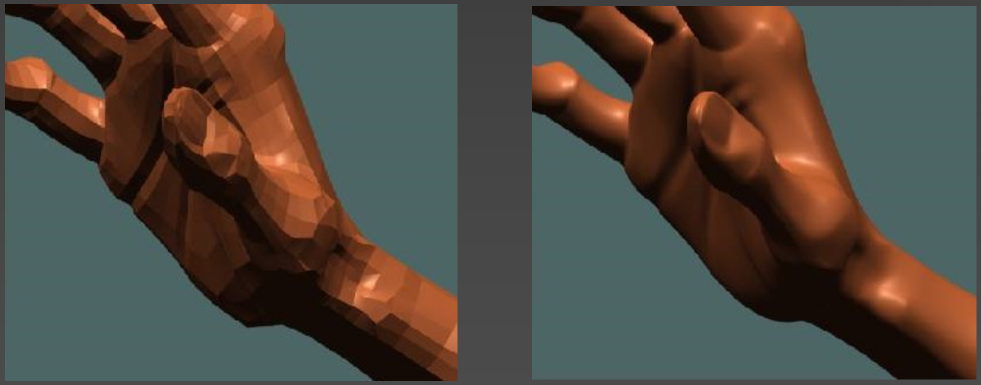
Back to subdivision

- An effective and efficient way to model and render smooth curves and surfaces, e.g., Bezier and B-splines, via **local refinement**
- Two aspects:
    - **Topological rule**: where to insert new vertices? Are old vertices kept?
    - **Geometrical rule**: spatial location of the new vertices – typically given as an average of nearby new or old vertices
- First introduced to graphics by Ed Catmull and Chaikin in the 1970's
- One of the most intensely studied subjects of geometric modeling (1990's) and ubiquitous in modeling and animation software now

Subdivision surfaces in animation

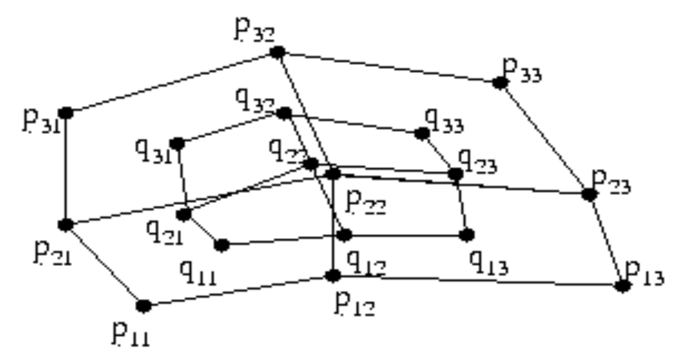
- Geri’s game: Academy award for animated short (1998)
- Subdivision surfaces in Geri’s game:

  http://mrl.nyu.edu/~dzorin/sig99/derose/sld001.htm



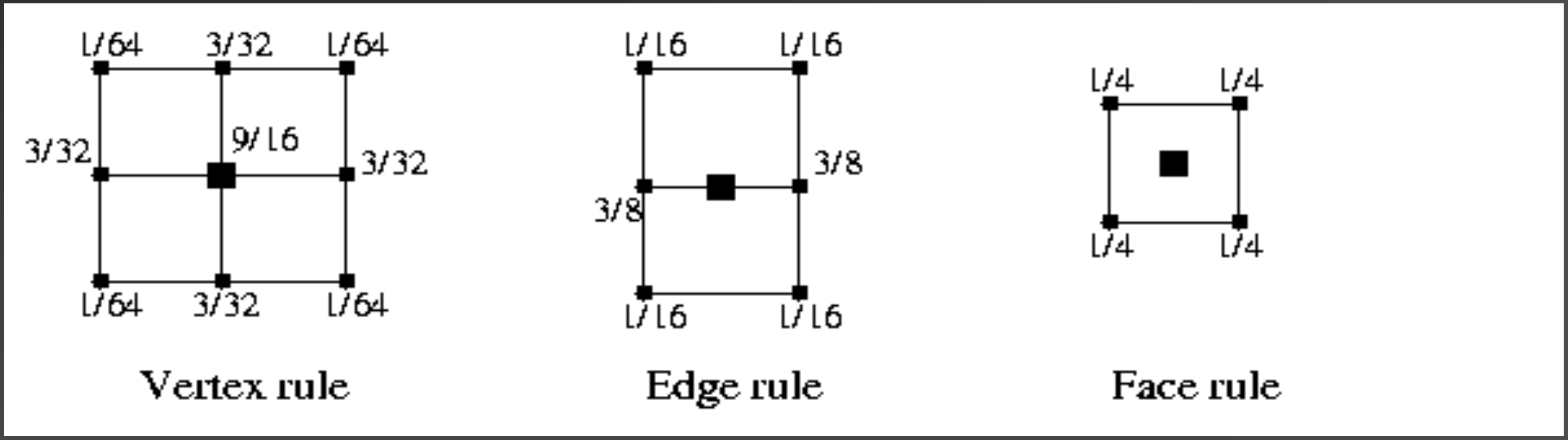
Surface example: Catmull-Clark

- Works on quadrilateral meshes
- Topological rules:
    - One new point per face and edge; retain the old vertices
    - Connect face point with all adjacent edge points
    - Connect old vertex with all adjacent edge points



Catmull-Clark subdivision

- Geometric rules (**subdivision masks** shown below)

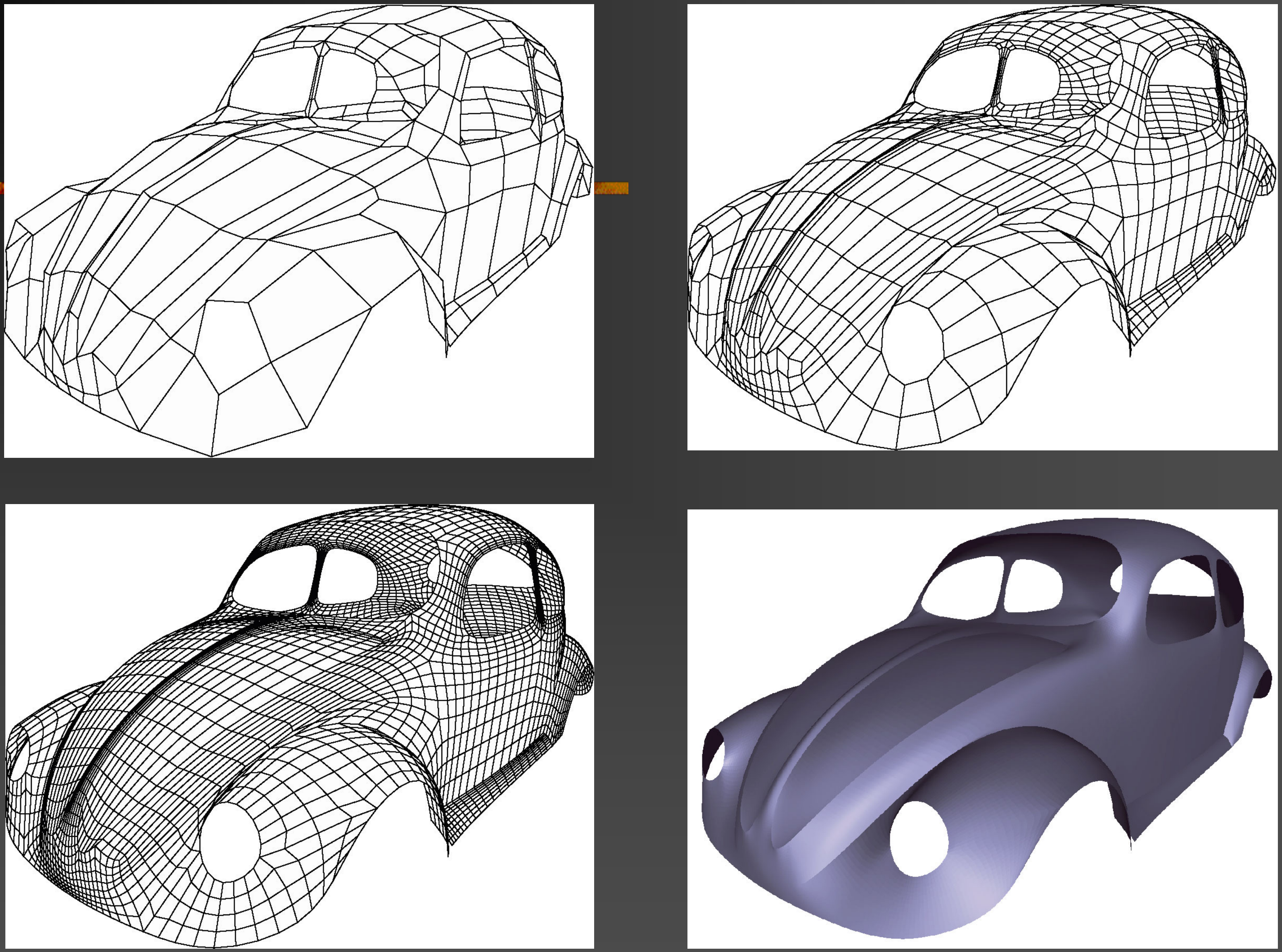


- This is all nice if the quadrilateral mesh connectivity is **regular**, i.e., a rectangular grid, but not always the case

Extraordinary vertices

- In a quadrilateral mesh, a vertex whose valence is not 4 is called an **extraordinary vertex**
- In a triangle mesh, an extraordinary vertex has valence ≠ 6
- Geometric rules for extraordinary vertices are different

Exercise: For a closed triangle mesh, can all vertices have degree 6?



Catmull-Clark and B-splines

- Even if original mesh has faces other than quadrilaterals, after one subdivision, all faces become quadrilaterals
- **Number of extraordinary vertices never increase**
- Over rectangular (regular) region, the limit is **bicubic B-spline surface**, i.e., $\mathbf{C}^2$
- Continuity at extraordinary vertices: $\mathbf{C}^1$
- There are many other types of subdivision surfaces with different schemes giving different levels of continuity

Advantages

- **Efficient to compute/render** with simple algorithms: weighted averages within a local neighborhood
- Flexible **local control** of surface features
- **Provable smoothness** if well designed
- **One-piece and seamless**; can model surfaces with arbitrary topology (same topology as control mesh) with relative ease
- **Compact** representation: base mesh + (fixed) rules
- Natural **level-of-detail** (hierarchical) representation

Subdivision surface vs. mesh

- Subdivision surfaces are **smooth limit surfaces**

- But in practice, e.g., rendering, only a few subdivisions are needed to produced a **mesh** that is dense enough

- **Polygonal meshes**: a much more general geometric representation
  - Does not have to result from subdivision – **irregular connectivity** vs. **subdivision connectivity**
  - Typically obtained from discretization of math representation or reconstruction out of a **point cloud**

Derivation of B-spline basis

- … via **convolution**
- Recall: B-spline bases defined by a **knot sequence**
- In uniform case (**uniform B-splines**), i.e., uniform spacing of the knots, B-spline basis can be defined via repeated **convolution**

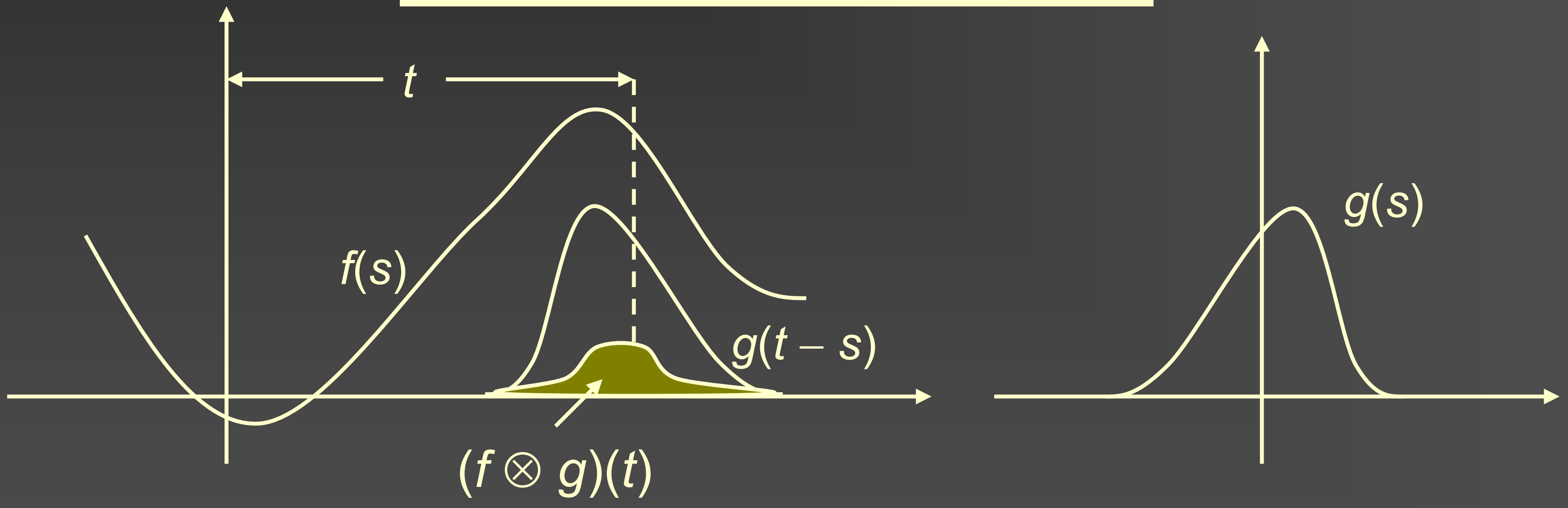
$$B_l(t) = (B_{l-1} \otimes B_0)(t) = \int B_{l-1}(s) B_0(t-s)\,ds$$

- $B_0(t)$, degree-0 B-spline, is the **box function** at $t = 0$

Convolution

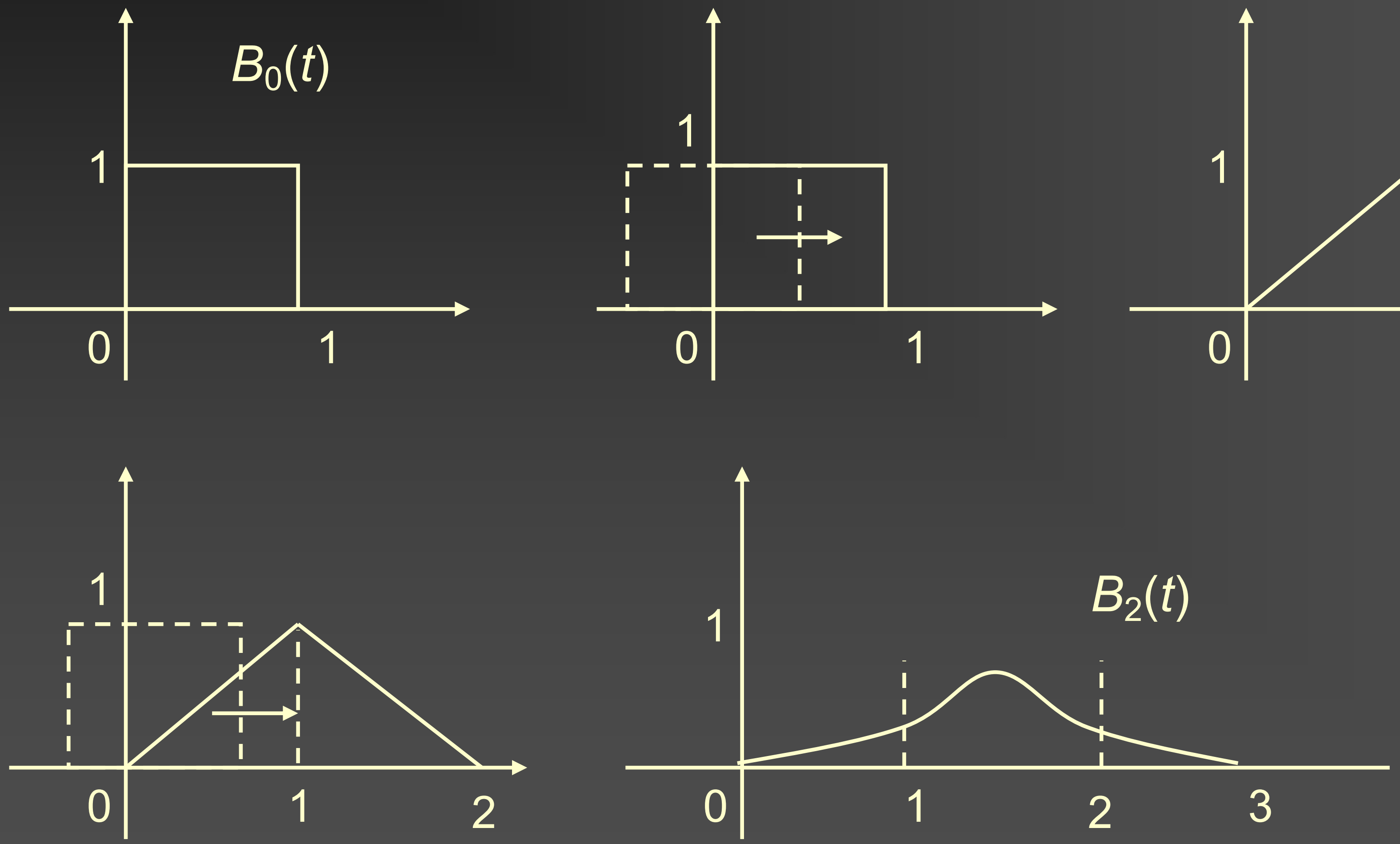
- An integral that computes a “running weighted average”

$$(f \otimes g)(t) = \int f(s)g(t-s)ds$$



- Kernel/weighting function *g* is often **symmetric** about 0

B-splines via convolution



……

A few words on convolution (aside)

- Function g first **reversed**: differ from **cross correlation**
  - To ensure **commutativity**: $f \otimes g = g \otimes f$
  - Convolution is also **associative**: $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
  - And **distributive over addition**: $f \otimes (g + h) = f \otimes g + f \otimes h$
- Discrete convolution in 1D: **serial products**
  - $\{f_0, f_1, \ldots, f_{m-1}\} \otimes \{g_0, g_1, \ldots, g_{n-1}\} = \{f_0 g_0, f_0 g_1 + f_1 g_0, \ldots, f_{m-1} g_{n-1}\}$
  - Length of resulting sequence: $n + m - 1$
  - Matrix formulation: multiplication by a **Toeplitz matrix**
  - **Circular convolution** defined by a **circulant matrices**, i.e., $C_{ij} = C_{kl}$ if and only if $i - j \equiv k - l \pmod{n}$

Important properties of subdivision

- **Convergence:**
    - Sequence of control polygons/meshes approach some continuous limit curve/surface
- **Interpolation – only for some subdivision schemes**
    - Possible with interpolating subdivision schemes, e.g., Butterfly (next)
- **Local control:**
    - Allows local change to a shape, e.g., through lifting of a single vertex
    - Local change does not influence the shape globally
    - This is a result of having **local subdivision rules**, i.e., geometric results only depend on information in a small local neighborhood

Important properties (continued)

- **Affine invariance:**
  - To transform a shape, it is sufficient to explicitly transform its (compact) set of control points
  - New shape is reconstructed (via subdivision) in transformed domain
  - This is related to the **row sum** of the subdivision matrix
- **Smoothness:**
  - The limit curve/surface should be smooth: a local property
  - Related to **eigenvalues** of the subdivision matrix

Subdivision matrix is key

- Subdivision matrix *S* characterizes the scheme
- Most relevant properties are derived from the subdivision matrix, e.g., local control (sparseness), convergence, smoothness, etc.
- First example, consider **affine invariance**
  - Requires $S\mathbf{1} = \mathbf{1}$, i.e., $[1\ 1\ \ldots\ 1]^T$ is an eigenvector of *S* with eigenvalue 1
  - Equivalently, *S* needs to have **unit row sum**
  - Proof?

Affine invariance

- Original vector of m points in dimension k: $u \in \mathrm{R}^{m \times k}$
- Vector of n points after subdivision: $v = Su \in \mathrm{R}^{n \times k}$, $n > m$
- Subdivision matrix $S \in \mathrm{R}^{n \times m}$
- Affine transformation of a point $p \in \mathrm{R}^{k \times 1}$ in dimension k: $p \to Ap + b$

Affine transform of subdivided points v:

$$v \to (Av^{\mathrm{T}} + b\mathbf{1}_n^{\mathrm{T}})^{\mathrm{T}} = [A(Su)^{\mathrm{T}} + b1_n^{\mathrm{T}}]^{\mathrm{T}}$$

$$= SuA^{\mathrm{T}} + \mathbf{1}_n b^{\mathrm{T}}$$

Subdivide affine transformed points u:

$$u \to S(Au^{\mathrm{T}} + b\mathbf{1}_m^{\mathrm{T}})^{\mathrm{T}}$$

$$= SuA^{\mathrm{T}} + S\mathbf{1}_m \mathrm{b}^{\mathrm{T}}$$

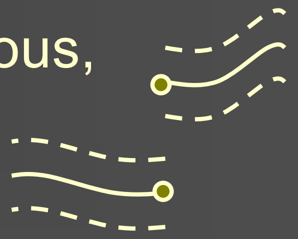
Results are equivalent if $S\mathbf{1}_m = \mathbf{1}_n$, implying **unit row sum** for S

Convergence proof (do no cover)

- To show: successively refined (piecewise linear) control polygons approach a continuous limit curve
- Aim for **uniform convergence**

A sequence of functions f_i defined on some interval $[a, b]$ converge uniformly to a limit function f if for all $\varepsilon > 0$ there exists an $n' > 0$ such that for all $n > n'$, $\max_{a \leq t \leq b} |f(t) - f_n(t)| = \| f(t) - f_n(t) \|_\infty < \varepsilon$

- Continuity of the f_i's + uniform convergence $\Rightarrow$ continuity of the limit function f
- Since our control polygons are **piecewise linear** but continuous, only need to prove uniform convergence



Proof using differences (do not cover)

- Expand a piecewise linear control polygon by **linear B-splines** B_1(■)'s

$$P^j(t) = B_1(2^j t)\,p^j, \quad p^j \text{ are control points at subdivision level } j$$

- Consider difference between consecutive points along the control polygon at level *j*

$$(\Delta p^j)_i = p^j_{i+1} - p^j_i$$

Lemma: If $\|\Delta \mathbf{p}^j\|_\infty < c\gamma^j$ for constant $c > 0$ and shrinkage factor $0 < \gamma < 1$ for all $j > j_0 \geq 0$ then $P^j(t)$ converges to a continuous limit $P^\infty(t)$

i.e., if the differences shrink fast enough, the limit curve will exist and be continuous (‖ ■ ‖$_\infty$ or simply, ‖ ■ ‖, is the max norm)

Proof of Lemma (do not cover)

S: any subdivision matrix in question

S_1: subdivision matrix for linear B-splines

Get matrix R such that $S - S_1 = R\Delta$, where Δ is the difference matrix, $\Delta_{ii} = -1$ and $\Delta_{i,\,i+1} = 1$ and 0 otherwise. Clearly, we can simply let $R_{ij} = -\Sigma_{\,k = i\,..\,j}\,(S - S_1)_{ik}$

Now we have

$$\|M\| = \max_{1\le i\le n} \sum_{k=1}^{n} | M_{ik} |$$

$$\begin{aligned}
\| P^{\,j+1}(t) - P^{\,j}(t) \| &= \| B_1(2^{\,j+1}t)\mathbf{p}^{\,j+1} - B_1(2^{\,j}t)\mathbf{p}^{\,j} \| \\
&= \| B_1(2^{\,j+1}t)S\mathbf{p}^{\,j} - B_1(2^{\,j+1}t)S_1\mathbf{p}^{\,j} \| \\
&= \| B_1(2^{\,j+1}t)(S - S_1)\mathbf{p}^{\,j} \| \\
&\le \| B_1(2^{\,j+1}t) \| \, \| R\Delta\mathbf{p}^{j} \|, \text{ note } \| M\mathbf{q} \| \le \| M \| \, \| \mathbf{q} \| \\
&\le \| R \| \, \| \Delta\mathbf{p}^{j} \| \le \| R \| c\gamma^{\,j}, \text{ for sufficiently large } j
\end{aligned}$$

Proof sketch continued (do not cover)

Then for any j , we have

$$\| P^{\infty}(t) - P^{j}(t) \|$$

$$= \| P^{j+1}(t) - P^{j}(t) + P^{j+2}(t) - P^{j+1}(t) + \dots \|$$

$$\le \| P^{j+1}(t) - P^{j}(t) \| + \| P^{j+2}(t) - P^{j+1}(t) \| + \dots$$

$$\le \| R \| c\gamma^{j} (1 + \gamma + \dots)$$

$$= \| R \| c\gamma^{j} / (1 - \gamma)$$

Therefore, if j is sufficiently large, we can make the above quantity less than a given ε, and proving uniform convergence of the sequence of continuous control polygons $\mathbf{p}^{j}$, $\mathbf{p}^{j+1}$, … This further implies the continuity of the limit function $P^{\infty}(t)$.

To ensure $\|\Delta p^j\|_\infty < c\gamma^j$? (do not cover)

- Derive a **subdivision matrix D for the differences** $\Delta \rightarrow D$ is related to the subdivision S by $\Delta S = D\Delta$. So

$$\Delta p^j = \Delta S^j p^0 = D^j \Delta p^0$$

- Let $c = \|\Delta p^0\|$. Then it is sufficient to make sure that

$$\|D\| = \gamma < 1 \quad \text{where} \quad \|D\| = \max_{1 \le i \le n} \sum_{k=1}^{n} |D_{ik}|$$

- But does D always exist?
  - For $\Delta S = D\Delta$, need $D_{i,j-1} - D_{i,j} = S_{i+1,j} - S_{i,j}$ for all i and j.
  - So **necessary** that $\Sigma_j S_{i,j} = \Sigma_j S_{i+1,j}$ for all i — affine invariance

Summary on convergence (do not cover)

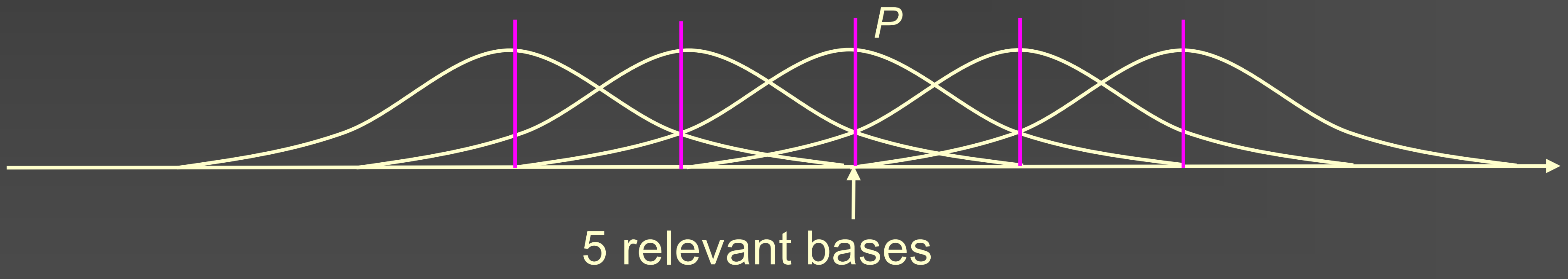
- Convergence for a subdivision curve results from
  - affine invariance and
  - certain condition on the max norm of the subdivision matrix *D* for the differences
- The proofs are not specific to B-spline subdivision
- Linear B-splines only used as bases of control polygons
- No general, systematic method to find subdivision rules to ensure convergence — this is the difficult part

Smoothness of limit curve/surface

- Analyze the behavior of a subdivision scheme **on or near a particular control point**
- To study smoothness, we care not only about point locations, but also **existence of tangent** line/plane at the point in question, etc.
- So far, we have assumed subdivision matrix is bi-infinite
- To obtain a finite subdivision matrix, need to decide which control points influence the **neighborhood** of the point of interest
- Typically, the neighborhood structure does not change through subdivision — **invariant neighborhood**

Invariant neighborhood

- Consider spline curves represented by spline basis functions
- To decide which control points influence the behavior of the spline curve near a particular point *P* …
- Look at how many spline bases influence *P*'s neighborhood
- As an example, consider cubic B-splines



Invariant neighborhood in subdivision

- Let us look at subdivision …
- Generally, and without a picture to help, note that

Final curve, i.e., polygonal curve joining control points after *j*-th level subdivision

Linear B-spline basis at refinement level *j* (i.e., the hat function)

control points obtained after *j*-th level subdivision

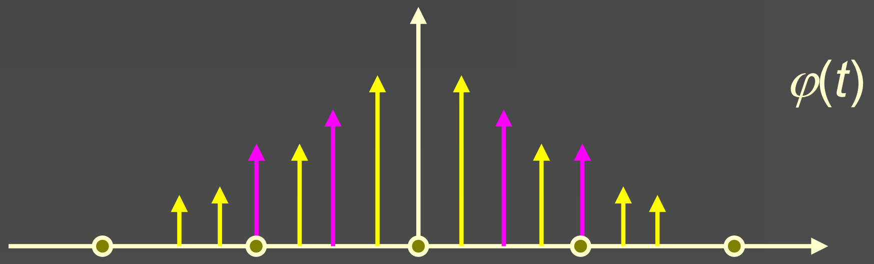
$$p^j(t) = B_1(2^j t)\,p^j = B_1(2^j t)\,S^j p^0$$
$$= B_1(2^j t)\,S^j\left[\sum_i p_i^0 \mathbf{e}_i\right] = \sum_i p_i^0\left[B_1(2^j t)\,S^j \mathbf{e}_i\right] = \sum_i p_i^0 \varphi_i^j(t)$$

Canonical basis vectors (or impulse vectors)

Invariant neighborhood in subdivision

$$p^j(t) = \sum_i p_i^0 \varphi_i^j(t) \Rightarrow p^\infty(t) = \sum_i p_i^0 \varphi_i(t), \quad \varphi_i(t) = \lim_{j\to\infty} \varphi_i^j(t)$$

- Each $\varphi_i(t)$ is the result of subdividing an **impulse**
- For stationary subdivision, $\varphi_i(t) = \varphi_0(t - i)$, i.e., they are all the same, just translates of each other
- $\varphi(t)$: the **fundamental solution** of the subdivision
- To determine size of invariant neighborhood, look at the influence of the fundamental solution
- E.g., for cubic B-spline subdivision, influence is 4 unit intervals, so **5 nearby control points influence the center point**



Local subdivision matrix

- Subdivision matrix is $n \times n$ if invariant neighborhood size is n

Cubic B-spline subdivision:

$$\begin{bmatrix} p_{-2}^{j+1} \\ p_{-1}^{j+1} \\ p_{0}^{j+1} \\ p_{1}^{j+1} \\ p_{2}^{j+1} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{bmatrix} \begin{bmatrix} p_{-2}^{j} \\ p_{-1}^{j} \\ p_{0}^{j} \\ p_{1}^{j} \\ p_{2}^{j} \end{bmatrix}$$

- E.g., **local subdivision matrix** for cubic B-spline is 5 × 5
- Let us use **eigenanalysis** of subdivision matrix S to determine limit behavior about the point p_0^∞

Eigenvalues and eigenvectors

- Cubic B-spines (see Matlab Demo)
  - Eigenvalues

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8}\end{bmatrix}$$

  - Complete set of eigenvectors

$$\begin{bmatrix}1 & -1 & 1 & 1 & 0 \\ 1 & -1/2 & 2/11 & 0 & 0 \\ 1 & 0 & -1/11 & 0 & 0 \\ 1 & 1/2 & 2/11 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1\end{bmatrix}$$

Eigen-analysis

- For eigenanalysis to apply, eigenvectors of *S* need to form a basis, i.e., **linear independence**
- Not all subdivision schemes satisfy this (e.g., four-point scheme)
- Assume set of eigenvectors x_i's are linearly independent, write the vector of (2D or 3D) control points as

$$p = \sum_{j=0}^{n-1} x_i a_i = X\mathbf{a}$$

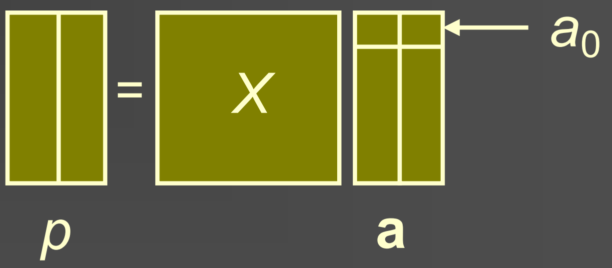
- Subdivision and repeated subdivision:

$$Sp^0 = S\sum_{i=0}^{n-1} x_i a_i = \sum_{i=0}^{n-1} \lambda_i x_i a_i$$

$$p^m = S^m p^0 = \sum_{i=0}^{n-1} \lambda_i{}^m x_i a_i$$

Eigenanalysis: convergence

- Assume that $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$, just an order …
- Affine invariance requires 1 to be an eigenvalue
- If $\lambda_0 > 1$, then divergence. So $\lambda_0 = 1$
- It can be shown that only one eigenvalue = 1 [Warren 95]
- If one and only one eigenvalue is 1, the limit point is a_0 (How to compute? Note that $a = X^{-1}p$, $X = [x_0, \ldots, x_{n\text{-}1}]$)



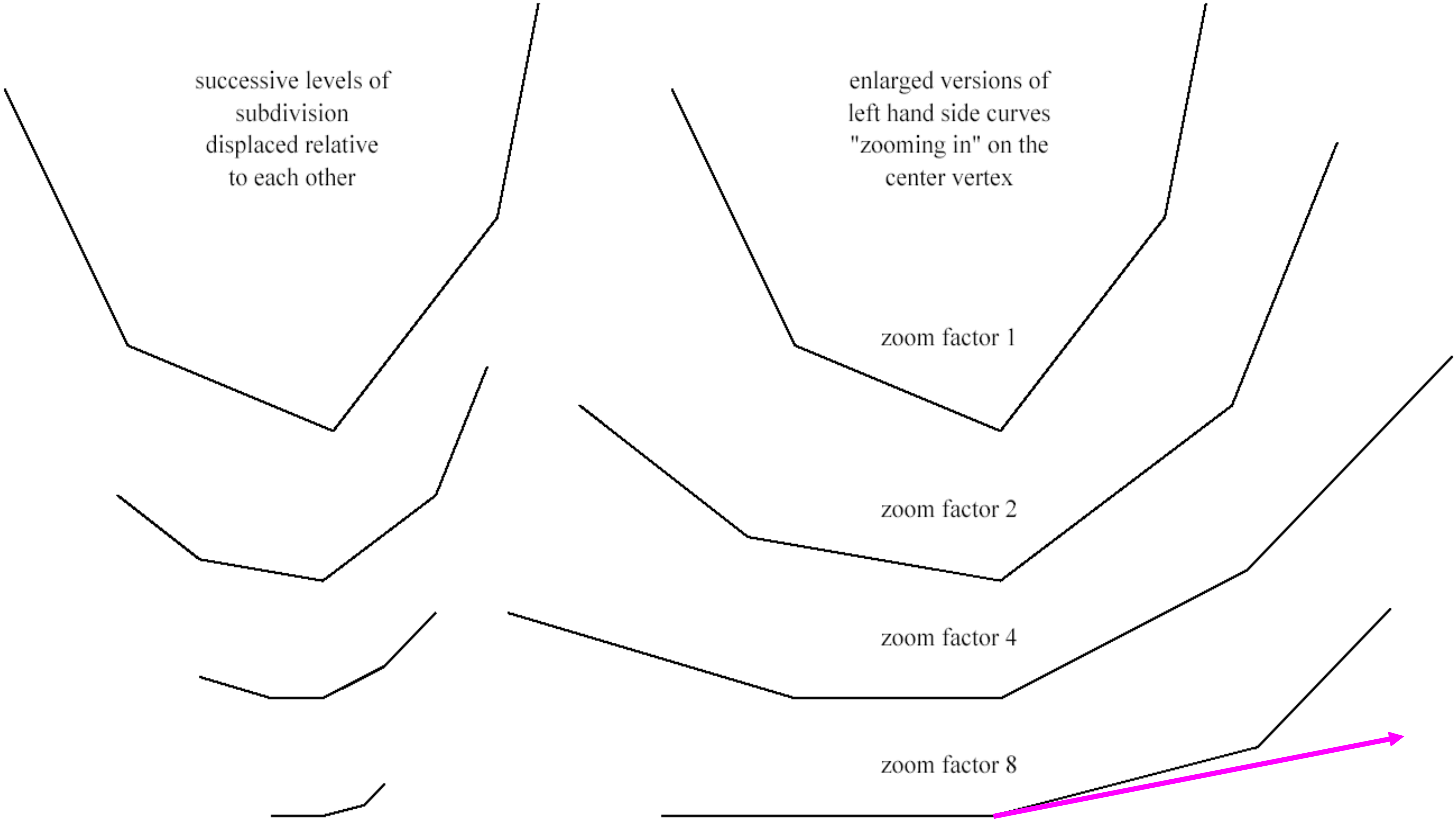
- How about tangent at limit point? – **think 2D: a_i's are 2D vectors**

Eigenanalysis: tangent

- Choose coordinate system so that a_0 is the origin

$$p^j = \sum_{i=1}^{n-1} \lambda_i^j x_i a_i \quad \text{and} \quad \frac{p^j}{\lambda_1^j} = x_1 a_1 + \sum_{i=2}^{n-1} \left(\frac{\lambda_i}{\lambda_1} \right)^j x_i a_i$$

- If λ_1, the subdominant eigenvalue, is unique, then there exists a tangent line, aligned with *vector* a_1, at p^∞
- How to compute the tangent? – Again, need to get the inverse of the eigenvector matrix X



Progressively aligned with tangent vector [pp. 44, Zorin 00]

Example: cubic B-splines

$$X = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -1/2 & 2/11 & 0 & 0 \\ 1 & 0 & -1/11 & 0 & 0 \\ 1 & 1/2 & 2/11 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1.8333 & -3.6667 & 1.8333 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}$$

Limit behavior

- $p_0^\infty = p_{-1}^0/6 + 2p_0^0/3 + p_{+1}^0/6$
- Tangent at p_0^∞ is $p_{+1}^0 - p_{-1}^0$

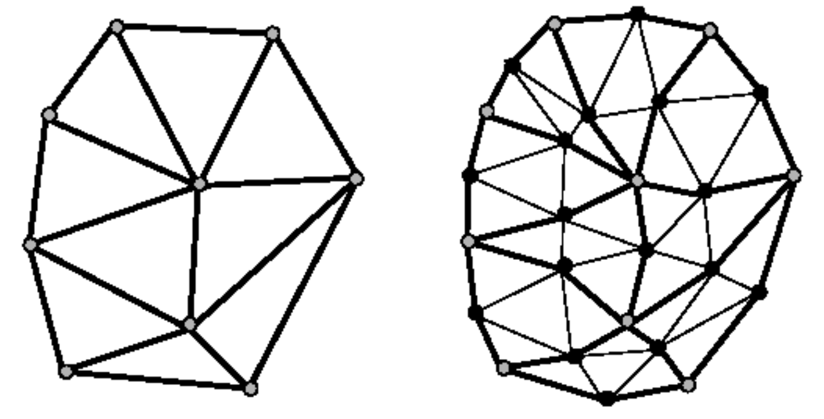
Summary of desirables

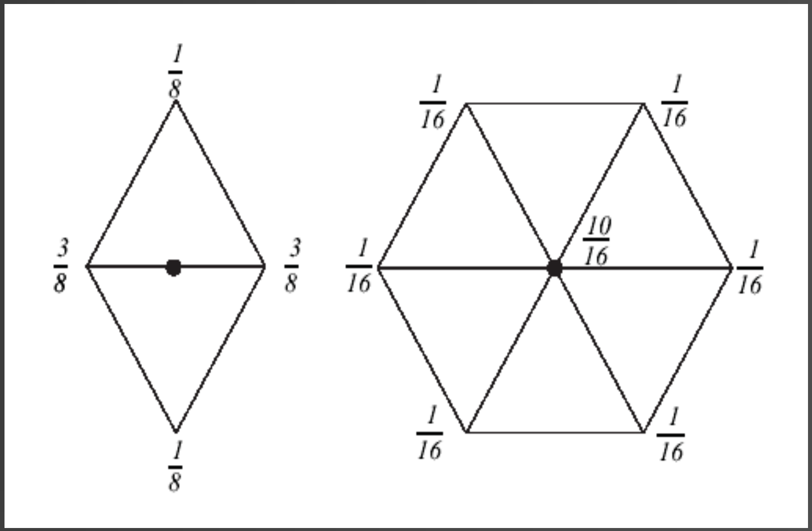
- Eigenvectors form a basis, i.e., complete set
- Largest eigenvalue is 1 – affine invariance and convergence
- The subdominant eigenvalue is less than 1 – convergence
- All the other eigenvalues are less than the subdominant eigenvalue – existence of tangent, but does not say about $\mathbf{C}^1$…
- Note: most of these are **sufficient** conditions

Eigen-analysis of subdivision surfaces

- Local control – same as for curves
- Affine invariance – same – need row sum of subdivision matrix to be 1
- Sufficient conditions for tangent existence a bit different
- There may be **extraordinary vertices**
    - Subdivision rules are often different there in order to ensure nice properties at and near these vertices
    - One fundamental solution per extraordinary case

Example: Loop Scheme





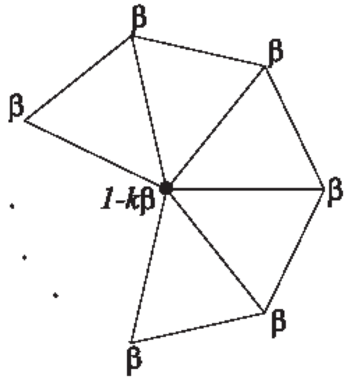


Figure 3.3: *Loop scheme: coefficients for extraordinary vertices. The choice of* β *is not unique; Loop [16] suggests* $\frac{1}{k}(5/8-(\frac{3}{8}+\frac{1}{4}\cos\frac{2\pi}{k})^2)$.

[pp. 48-50, Zorin 00]

Analysis

- Similar to the case for curves, however …
- There will be at least **one subdivision matrix for each valence** (can also change between levels – non-stationary)
- Notion of invariant neighborhoods still applies

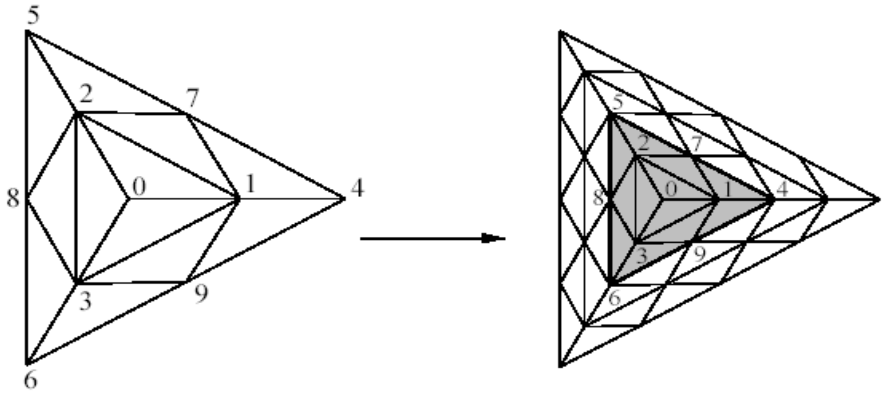


Figure 3.6: *The Loop subdivision scheme near a vertex of degree 3. Note that* $3 \times 3 + 1 = 10$ *points in two rings are required.*

Eigenanalysis

- Express control vector as linear sum of the eigenvectors of the subdivision matrix *S*, assuming linear independence

$$p = \sum_{j=0}^{n-1} x_i a_i$$

- Subdivision and repeated subdivision

$$Sp^0 = S\sum_{i=0}^{n-1} x_i a_i = \sum_{i=0}^{n-1} \lambda_i x_i a_i$$

$$p^m = S^m p^0 = \sum_{i=0}^{n-1} \lambda_i{}^m x_i a_i$$

- Note that a_i's are now 3D points

Eigenanalysis

- Again, assume that $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$
- For affine invariance and convergence, require $\lambda_0 = 1$ and be unique
- For existence of **tangent plane**, note that

$$\frac{p^j}{\lambda^j} = x_1 a_1 + x_2 a_2 + \left(\frac{\lambda_3}{\lambda}\right)^j x_3 a_3 + \ldots$$

if origin is at $a_0 = \mathbf{0}$, and

$$\lambda = \lambda_1 = \lambda_2 > \lambda_3$$

- The tangent plane will be spanned by vectors a_1 and a_2

Smoothness of subdivision surfaces

- Two notions: **C^1-continuous** vs. **tangent plane continuous**
  - Technical definition of **C**^{1} continuity of surface [pp. 56, Zorin 00]
  - Tangent-plane continuity (weaker) requires the limit of normals exist
  - Tangent-plane continuity + one-to-one projection between surface and tangent plane ⇒ **C**^{1} continuity
- Essential/pioneering work for subdivision surfaces near extraordinary vertices:
  - Reif 's sufficient conditions for subdivision surfaces to be **C**^{1} — [Section 3.5, Zorin 00] as further reading