Solutions to midterm test

1 (a) False ; .00001 is the absolute error.

b) False ; the exponent influences the difference.

(c) False ; Clearly chopping 1.999 to 1.0 would ~~[illegible]~~ be incorrect. if using fixed point arithmetic.

(d) False ; $\begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix}$ is ~~not~~ strictly diagonally dominant but is not positive definite

(e) False ; the matrix has to be positive definite & symmetric.

(f) True ; follows from the definition.

(g) False ; If $f(x) \notin C^2[a,b]$, $f'(x)$ for some x ~~[illegible]~~ does not exist.

(h) False scaled partial pivoting makes Gaussian algorithm stable.

i) False Since the complexity of Gaussian algorithm is $O(n^3)$, the computation time will be increased by $3^3 = 27$ times i.e. the total time needed is 27×4 secs $= 108$ secs.

Q2 for large x $\sqrt{x} \approx \sqrt{x+1}$ so the formula will involve subtraction of nearly equal #'s.

suggested more accurate form:

$$\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x(x+1)}}$$

$$= \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x(x+1)}} \left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right)$$

$$= \frac{\cancel{x}+1 - \cancel{x}}{\sqrt{x(x+1)}\sqrt{x+1} + \sqrt{x(x+1)}\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}(x+1) + x\sqrt{x+1}}$$

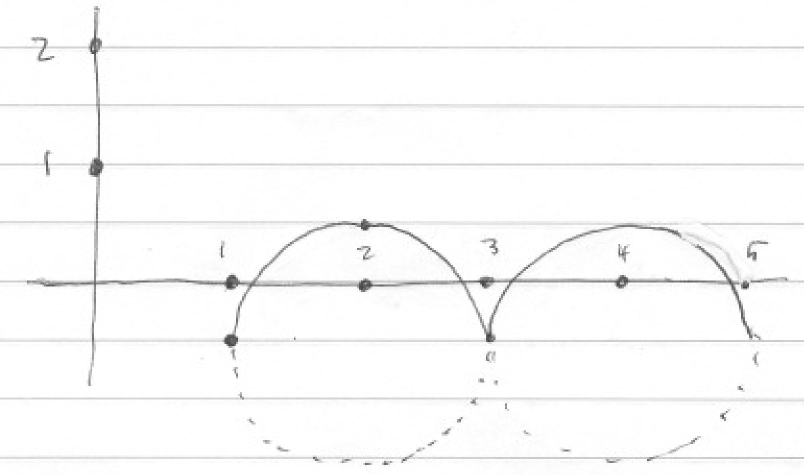
$$= \frac{1}{\text{"big"} + \text{"big"}} = \frac{1}{\text{"big"}}$$

Q3 a) quadratic convergence vs linear convergence.

b) guarantee of convergence vs might diverge

c) $\lim_{n\to\infty} \frac{|p^{n+1} - p|}{|p^n - p|^\alpha} = \lambda$, with $\alpha = 2$ and $0 < \lambda < \infty$

Q4 a)



$[a, b] = [1, 2]$ $\quad f(a) = -.5000, \quad f(b) = .5000$

iter #1
$p_1 = \frac{a+b}{2} = 1.5 \Rightarrow f(1.5) = .8660\cancel{254} - 0.5$
$\approx .3660 > 0$

so $a_2 = a_1 = 1$
$b_2 = p_1 = 1.5$

iter #2
$p_2 = \frac{a_2 + b_2}{2} = 1.25 \Rightarrow f(1.25) =$ [illegible]
$=$ [illegible]
$= \sqrt{.4375} - 0.5$
$= .6614\cancel{378} - 0.5$
$= .1614 > 0.$

so $a_3 = a_2 = 1$
$b_3 = p_2 = 1.25$

b) see text.

c) f' is not defined at $x = 1$, $x = 3$, $x = 5$ because the function has cusps (corners) at those points. Also $f'(2) = f'(4) = 0$

5a, Let the search interval be (a, b)

$$4^{-N}(b-a) < 10^{-4}$$

$$-N\log 4 + \log(b-a) < -4$$

$$N > \frac{4+\log(b-a)}{\log 4}$$

b, For quadsection

$$\lim_{n\to\infty} \frac{|P_{n+1}-P|}{|P_n - P|} = \lim_{n\to\infty} \frac{\frac{1}{4^{n+1}}}{\frac{1}{4^n}} = \frac{1}{4}$$

For bisection

$$\lim_{n\to\infty} \frac{|P_{n+1}-P|}{|P_n-P|} = \frac{1}{2}$$

Both methods converge linearly

6a, Symmetric

b, Since $6 > 2+2$. $5 > 2+2$, the matrix is strictly diagonally dominant

$$A = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix} \xrightarrow[E_3 - \frac{1}{3}E_1 \to E_3]{E_2 - \frac{1}{3}E_1 \to E_2} \begin{bmatrix} 6 & 2 & 2 \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & \frac{13}{3} \end{bmatrix} \xrightarrow{E_3 - \frac{1}{4}E_2 \to E_3}$$

$$\begin{bmatrix} 6 & 2 & 2 \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & 0 & 4 \end{bmatrix}$$

$\therefore A = LDL^T$ with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix};\ D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{16}{3} & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

6c. $LDL^T x = [10, 10, 9]^T$

Solve $Lz = [10, 10, 9]^T$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{cases} z_1 = 10 \\ z_2 = 10 - \frac{1}{3}(10) = \frac{20}{3} \\ z_3 = 9 - \frac{1}{4}\left(\frac{20}{3}\right) - \frac{1}{3}(10) = 4 \end{cases}$$

Solve $Dy = z$

$$\Rightarrow \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{16}{3} & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{20}{3} \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = \frac{5}{3}. \quad y_2 = \frac{5}{4}. \quad y_3 = 1$$

Solve $L^T x = y$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{4} \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_3 = 1 \\ x_2 = \frac{5}{4} - \frac{1}{4}(1) = 1 \\ x_1 = \frac{5}{3} - \frac{1}{3}(1) - \frac{1}{3}(1) = 1 \end{cases}$$

$\therefore x = [1, 1, 1]^T$

(7) Scaled pivoting, when applied, results the following system

$$\begin{bmatrix} 2.00 & 1.00 \\ 4.00 & 40.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.00 \\ 60.0 \end{bmatrix}$$

$\Downarrow$ (2,1) element is reduced to zero

$$\begin{bmatrix} 2.00 & 1.00 \\ 0.00 & 38.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.00 \\ 56.0 \end{bmatrix}$$

$$\therefore \quad x_2 = \frac{38.0}{56.0} = 1.47 \quad \text{(3 decimal digit)}$$

$$\therefore x_1 = (2.00 - 1.47)/2$$

$$= 0.53/2 = 0.265$$

8 a) page 455 in text

8b) small r cannot guarantee

accurate sol/n:

$\|x-\bar{x}\| \leq \|r\| \cdot \|A^{-1}\|$

answer depends on $\|r\|$ and $\|A^{-1}\|$

$\|A^{-1}\|$ could be very big thus

cannot guarantee accurate sol.

accurate sol cannot guarantee

small r

$\underbrace{\|x-\bar{x}\|}_{\text{small}} \leq \|r\| \cdot \|A^{-1}\|$

accurate sol just gives a lower bound

and $\|A^{-1}\|$ could be very small

and therefore $\|r\|$ should be big.