MACM 316-2
TEST 1 – SOLUTIONS

10:30 – 11:20am, June 5, 2002

Instructor: Adrian Lewis

Family name: ____________ **Initials:** ______

Student ID number: ____________

READ INSTRUCTIONS CAREFULLY:

- **Do not lift the cover page until instructed!**
- Fill out your name and ID in the space provided.
- You may use an approved graphing calculator. No other aids.
- Answer all questions, **explaining your answers** carefully in the space provided. If you run out of space, use the back of the preceding page.
- This exam consists of 4 questions and 7 pages (including this one).

Question	1	2	3	4	Total
Grade	/9	/4	/5	/7	/25

1. Consider the equation

$$e^x - 3x = 0.$$

(a) **[2 marks]** Explain why the equation must have at least one solution in the interval $[0, 1]$.

The function $f(x) = e^x - 3x$ is continuous.
It satisfies $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$.
The intermediate value theorem now shows there is an $x \in (0, 1)$ satisfying $f(x) = 0$.

(b) **[2 marks]** Explain why the equation has a unique solution in the interval $[0, 1]$.

Part (a) shows it has at least one solution.
Since $f'(x) = e^x - 3 < 0$ for all $x \in (0, 1)$, the function f is strictly decreasing on $[0, 1]$.
Hence f can have at most one zero in $[0, 1]$.

Question 1 (continued)

(c) **[3 marks]** Starting with the interval $[0, 1]$, how many steps of the bisection method would you need to find a number guaranteed to be within 0.001 of the exact solution of the equation?

After n steps, current interval has length 2^{-n}.
On the $(n+1)$th step, we calculate the midpoint, which must be within 2^{-n-1} of the exact solution.
Providing $2^{-n-1} \leq .001$, we are then done.
This is equivalent to $(-n-1)\ln 2 \leq \ln(.001)$, or

$$n + 1 \geq 9.96\ldots.$$

Hence we need $n \geq 9$, so we need 10 steps of bisection.

(d) **[2 marks]** Starting close to the solution, what would the order of convergence of Newton's method be?

$f \in C^2[0, 1]$.
$f'(x) = e^x - 3 \neq 0$ for all $x \in [0, 1]$.
Hence Newton's method converges quadratically to the solution, if we begin nearby.

2. [4 **marks**] Suppose δ is a small number. Explain why calculating

$$\sqrt{1+\delta} - \sqrt{1-\delta}$$

directly may give an inaccurate answer. Suggest how to calculate a more accurate answer.

If δ is small, $\sqrt{1+\delta}$ and $\sqrt{1-\delta}$ are nearly equal, so subtracting them will typically cause roundoff errors.

By rationalizing the numerator, we have

$$\sqrt{1+\delta} - \sqrt{1-\delta} =$$

$$\frac{(\sqrt{1+\delta} - \sqrt{1-\delta})(\sqrt{1+\delta} + \sqrt{1-\delta})}{\sqrt{1+\delta} + \sqrt{1-\delta}}$$

$$= \frac{2\delta}{\sqrt{1+\delta} + \sqrt{1-\delta}}.$$

In this way we avoid the roundoff errors caused by subtracting nearly equal numbers.

3. Consider the function

$$h(x) = 1 - \frac{x^3}{4}$$

on the interval $[0, 1]$.

(a) **[3 marks]** Use the Fixed Point Theorem to prove h has a unique fixed point in the interval $[0, 1]$.

h is continuous on $[0, 1]$.
$h'(x) = -3x^2/4 < 0$ for all $x \in (0, 1)$.
Hence h is decreasing on $[0, 1]$.
$h(0) = 1$ and $h(1) = 3/4$, so

$$h(x) \in \left[\frac{3}{4}, 1\right] \subset [0, 1] \quad \text{for all } x \in [0, 1].$$

Hence h has a fixed point.
Since $|h'(x)| < 1$ for all $x \in (0, 1)$, the fixed point p is unique.

(b) **[2 marks]** What is the order of convergence of the fixed-point iteration for this function?

In addition to the above conditions, we have

$$|h'(x)| \leq \frac{3}{4} < 1 \quad \text{for all } x \in (0, 1).$$

Hence the fixed point iteration converges to p.
Clearly $p \neq 0$.
Hence $h'(p) \neq 0$, so the convergence is only linear.

4. (a) **[2 marks]** Define linear convergence.

If $p_n \to p$ and $p_n \neq p$ for all n and

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p}$$

exists, then p_n converges linearly to p.

Now suppose you apply Newton's method to the equation

$$|x|^{3/2} = 0,$$

starting with the initial iterate $x_0 = 1$.

(b) **[1 mark]** Find the next iterate x_1.

If $f(x) = |x|^{3/2}$, then $f(x) = x^{3/2}$ for $x > 0$.
Hence

$$f'(x) = \frac{3}{2}x^{1/2}$$

for $x > 0$.
Hence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{3/2}}{3x^{1/2}/2} = x_n - \frac{2x_n}{3} = \frac{x_n}{3}.$$

Hence

$$x_1 = \frac{1}{3}.$$

Question 4 (continued)

(c) **[2 marks]** Find the order of convergence of the sequence $\{x_n\}$ generated by Newton's method.

By the calculations in part (b), $x_n = 3^{-n}$.
Hence $x_n \to 0$ and $x_n \neq 0$ for all n, and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{3}.$$

Hence $x_n \to 0$ linearly (ie. with order one).
But for any $\alpha > 1$,

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n^\alpha} = \lim_{n\to\infty} \frac{3^{-n-1}}{3^{\alpha n}} = \lim_{n\to\infty} 3^{(\alpha-1)n-1} = \infty.$$

Hence x_n does not converge with any order higher than one.

(d) **[2 marks]** How large must n be before x_n approximates the solution with absolute error less than 5×10^{-6}.

We need

$$x_n = 3^{-n} < 5 \times 10^{-6}$$

or

$$-n \ln 3 < \ln 5 - 6 \ln 10.$$

Hence we need $n > 11.11\ldots$, so n must be at least 12.