MACM 316
TEST 1: SOLUTIONS

10:30 – 11:20am, June 9, 2003
Instructor: Adrian Lewis

Family name: ____________ **Initials:** ______
Student ID number: ____________

READ INSTRUCTIONS CAREFULLY:

- **Do not lift the cover page until instructed!**
- Fill out your name and ID in the space provided.
- You may use an approved graphing calculator. No other aids.
- Answer all questions, **explaining your answers** carefully in the space provided. If you run out of space, use the back of the preceding page.
- This exam consists of 4 questions and 7 pages (including this one).

Question	1	2	3	4	Total
Grade	/4	/10	/7	/4	/25

1. **[4 marks]** Suppose x is a large number. Explain why computing

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

directly in floating point arithmetic may give an inaccurate answer, and suggest how to calculate a more accurate answer.

Solution:

$$\begin{aligned}
\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}} &= \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x}\sqrt{x+1}} \\
&= \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\
&= \frac{1}{(x+1)\sqrt{x}+x\sqrt{x+1}} \\
&= O(x^{-3/2}).
\end{aligned}$$

So, in original form, we're subtracting two numbers of size $O(x^{-1/2})$ whose difference is much smaller ($O(x^{-3/2})$). So, roundoff error will cause considerable loss of accuracy.

Calculating

$$\frac{1}{(x+1)\sqrt{x}+x\sqrt{x+1}}$$

instead will cause no loss of accuracy.

2. Suppose we apply the bisection method to solve the equation

$$x^2 - 3 = 0$$

on the initial interval $[a_0, b_0] = [1, 2]$, producing the sequence of intervals $[a_n, b_n]$ and estimates $p_n = \frac{1}{2}(a_n + b_n)$.

(i) **[2 marks]** Calculate p_2.

$$p_0 = \frac{3}{2}, \quad \left(\frac{3}{2}\right)^2 - 3 < 0 < 2^2 - 3,$$

so $a_1 = 3/2$ and $b_1 = 2$.

$$p_1 = \frac{7}{4}, \quad \left(\frac{7}{4}\right)^2 - 3 > 0 > \left(\frac{3}{2}\right)^2 - 3,$$

so $a_2 = 3/2$ and $b_2 = 7/4$, so $p_2 = 13/8$.

(ii) **[2 marks]** How large must n be to ensure

$$|p_n - \sqrt{3}| < 10^{-6}?$$

We need

$$2^{-n} = b_n - a_n < 2 \cdot 10^{-6}$$

so $-(n+1)\ln 2 < -6 \ln 10$, or

$$n > \frac{6 \ln 10}{\ln 2} - 1 = 18.9316\ldots$$

So we need n at least 19.

(iii) **[2 marks]** Define *order of convergence*.

"$q_n \to q$ with order of convergence α" means

$$\lim_n \frac{|q_{n+1} - q|}{|q_n - q|^\alpha}$$

exists.

(iv) **[1 mark]** What is the order of convergence of $\{b_n - a_n\}$?

Since $b_n - a_n = 2^{-n}$ and

$$\lim_n \frac{2^{-(n+1)}}{2^{-n}} = \frac{1}{2},$$

the convergence is order one.

(v) **[3 marks]** Explain why the fixed point method for the function

$$g(x) = x + \frac{3 - x^2}{6}$$

converges from any starting point in the interval $[1, 2]$.

$g'(x) = 1 - x/3 > 0$ for all $x \in [1, 2]$, so g is increasing on $[1, 2]$. Furthermore, $g(1) = 4/3 \in [1, 2]$ and $g(2) = 11/6 \in [1, 2]$. Hence g maps the interval $[1, 2]$ into itself. Notice g is continuous on $[1, 2]$ with $|g'(x)| \leq 2/3 < 1$. Hence the Fixed Point Theorem implies the fixed point iteration converges.

3. Consider the function $f(x) = \frac{x}{1+x^2}$. Suppose we apply Newton's method to solve the equation $f(x) = 0$, generating a sequence $\{p_n\}$.

(i) **[1 mark]** Verify $f'(x) = \frac{1-x^2}{(1+x^2)^2}$.

By the quotient rule

$$f'(x) = \frac{1 \cdot (1+x^2) - (2x) \cdot x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

(ii) **[2 marks]** Express p_{n+1} in terms of p_n.

$$\begin{aligned} p_{n+1} &= p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{\frac{p_n}{1+p_n^2}}{\frac{1-p_n^2}{(1+p_n^2)^2}} \\ &= p_n\left(1 - \frac{1+p_n^2}{1-p_n^2}\right) = \frac{-2p_n^3}{1-p_n^2}. \end{aligned}$$

(iii) **[1 mark]** If $p_0 = \frac{2}{3}$, calculate p_1.

$$p_1 = \frac{-2 \cdot \left(\frac{2}{3}\right)^3}{1 - \left(\frac{2}{3}\right)^2} = -\frac{16}{15}.$$

(iv) **[2 marks]** If p_0 is small, does $\{p_n\}$ converge quadratically?

Since f is a rational function with nonzero denominator, it is twice continuously differentiable near zero. Furthermore, $f(0) = 0$ and $f'(0) = 1 \neq 0$. Hence for initial points close to zero, Newton's method will converge at least quadratically.

(v) **[1 mark]** If p_0 is large, does $\{p_n\}$ converge?

For large p_n, we have $p_{n+1} \approx 2p_n$, so the sequence $\{p_n\}$ will diverge to $+\infty$.