MACM 316 D01.00
INSTRUCTOR: Michael Lunney

Midterm Examination

FRIDAY, JUNE 16, 2000
10:30-11:20 AQ3182

This examination has 5 questions, worth a total of 25 points. The numbers in the margin describe the marking scheme.

LAST NAME (please print): ______________________

First name: ______________________

Student number: ______________________

Signature: ______________________

Instructions

1. **Time:** 50 minutes.
2. Fill in the information above now.
3. Do not open the examination booklet until you are told to do so.
4. Attempt all questions. Make sure there are 6 pages and 5 questions.
   You may use the back of the pages for rough work.
5. Please no books, no notes and no programmable or graphing calculators. Scientific calculators are permitted.

Question	Marks	Max
1		/5
2		/3
3		/7
4		/5
5		/5
Total		/25

3. [7 points] A mathematical model for a one-dimensional heat flow problem leads to the following $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} 2+\frac{1}{n^2} & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2+\frac{1}{n^2} & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2+\frac{1}{n^2} & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -1 & 2+\frac{1}{n^2} & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2+\frac{1}{n^2} & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+\frac{1}{n^2} \end{bmatrix}.$$

[3] (a) Find a Cholesky factorization for $\mathbf{A}$ when $n = 2$.

[4] (b) The $n \times n$ matrix $\mathbf{A}$ is known to have an $\mathbf{LL}^T$ factorization in which $\mathbf{L}$ has only $2n-1$ non-zero entries, given by

$$l_{i,i} \quad \text{for } i = 1, \ldots, n, \qquad \text{and} \qquad l_{i,i-1} \quad \text{for } i = 2, \ldots, n.$$

Assuming these entries have already been calculated, write the pseudocode for an algorithm to solve the $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x} = [x_1, \ldots x_n]^T$ in $\mathcal{O}(n)$ floating point operations.

[3] 2. After 2 stages of Gaussian elimination, a 4×4 matrix $\mathbf{A}$ has been row-reduced to the matrix

$$\mathbf{A}^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 12 & 12 \end{bmatrix}.$$

The sequence of row operations to obtain $\mathbf{A}^{(3)}$ from $\mathbf{A}$ is given by

$$(E_2 - E_1) \rightarrow (E_2), \quad (E_3 - E_1) \rightarrow (E_3), \quad (E_4 - E_1) \rightarrow (E_4)$$

in the first stage, and

$$(E_3 + 2E_2) \rightarrow (E_3), \quad (E_4 + 3E_2) \rightarrow (E_4)$$

in the second stage.

Suppose $\mathbf{A} = \mathbf{LU}$ with $\mathbf{L}$ lower-triangular and $\mathbf{U}$ upper-triangular. Find $\mathbf{L}, \mathbf{U}$ and $\mathbf{A}$.

4. 5 points Consider an indirect method for solving the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is the $n \times n$ matrix from Question 3.

1 (a) Find the Jacobi iteration matrix $\mathbf{T}$ where

$$\mathbf{x}^{(k)} = \mathbf{T}\mathbf{x}^{(k-1)} + \mathbf{c}.$$

2 (b) Calculate $\|\mathbf{T}\|_\infty$ and show that $\|\mathbf{T}\|_\infty \to 1$ as $n \to \infty$.

2 (c) Let $n = 2$. Show that after $k = 4$ iterations of the Jacobi method we have

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq B_n \|\mathbf{x}^{(0)} - \mathbf{x}\|_\infty$$

with $B_2 \approx 5/8$.

(d) 2 points BONUS: Show that for arbitrary n, after $k = dn^2$ iterations, the above error bound holds with B_n satisfying

$$\lim_{n \to \infty} B_n = e^{-d/2}.$$

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5. [5 points] Let the real number $x = 0.d_1 d_2 \ldots d_{10} d_{11} \cdots \times 10^e$ have floating point number representation

$$fl\{x\} = 0.d_1 d_2 \ldots d_{10} \times 10^e, \qquad \text{where } d_i \in \{0, 1, \ldots, 9\},\ d_1 \neq 0, \quad -100 \leq e \leq 100.$$

Consider the function $\boxed{f(h) = \sqrt{1+h} - 1}$.
Let $\boxed{\varepsilon = 0.5 \times 10^{-10}}$. Then $fl\{f(\varepsilon)\} = 0.2499999999 \times 10^{-10}$.

[2] (a) Consider $g(h) = fl\Big\{ fl\{\sqrt{fl\{1+h\}}\} - 1 \Big\}$ as an implementation of $f(h)$ in floating point arithmetic. Estimate the relative error in approximating $f(\varepsilon)$ by $g(\varepsilon)$.

[2] (b) Find a Taylor polynomial approximation for f, such that

$$f(h) = P(h) + \mathcal{O}(h^2) \qquad \text{as } h \to 0.$$

[1] (c) Give a floating point implementation of a function which approximates $f(h)$ to 10 significant digits at $h = \varepsilon$.

(d) [1 point] BONUS: Show that your function from (c) still accurately represents $f(h)$ for any h in $\varepsilon_0 < h < \varepsilon$ for some $0 < \varepsilon_0 < \varepsilon$.