A-1

1. (3 marks) Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0}\left[\exp(h^2) - h\sin(h)\right] = 1$$

Show your steps.

3. (3 marks) Are the following true or false? GIVE YOUR REASONING.

   (a) Let $f(x) = (x+2)(x+1)^2x(x-1)$. Newton's method gives quadratic convergence to the root -1 using the initial guess -1.02.

   (b) Suppose $f(\cdot)$ is smooth. A computer approximation of the first derivative using the two-point forward difference formula

   $$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

   produces an error which tends to zero as h tends to zero.

   (c) The function $\sin(x)$ has precisely two fixed points over the set of real numbers.

5. (3 marks) Construct an efficient algorithm for finding the cube root of a number N. Use your algorithm to find $\sqrt[3]{5}$ to 4 significant digits.

8. (6 marks) Some questions on numerical differentiation and integration.

   (a) Use differencing formulas to determine the three missing entries in the table. Show your work. In each case use the most accurate formula possible.

x	$f(x)$	$f'(x)$
0.5	0.479	
0.6	0.564	
0.7	0.644	

   (b) Using the data above, apply Simpson's Rule to determine an approximation for

   $$\int_{0.5}^{0.7} f(x)dx$$

   (c) Derive the degree of precision of Trapezoid Rule using the definition of degree of precision.

9. (6 marks) Some questions on interpolation. If more than one solution is possible, choose the best alternative.

   (a) A natural cubic spline S on $[0,2]$ is defined by

$$S = \begin{cases} S_0(x) = 1 + 2x - x^3 & \text{on } [0,1) \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{on } [1,2] \end{cases}$$

   Find b, c, d.

   (b) Suppose we are given the following:

$$\cos(0.698) = 0.7661, \cos(0.733) = 0.7432, \cos(0.768) = 0.7193, \cos(0.803) = 0.6946$$

   Using a Lagrange intepolating polynomial of degree 2, approximate $\cos(0.775)$.

10. (4 marks) Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to 0 of order α with asymptotic error constant λ. Also suppose $\{\tilde{p}_n\}_{n=0}^{\infty}$ is a sequence that converges to 0 of order $\tilde{\alpha}$ with asymptotic error constant $\tilde{\lambda}$.

    The first few values of the sequences are given below:

n	p_n	$\tilde{p}_n$
1	4.400×10^{-1}	2.500×10^{-1}
2	1.620×10^{-1}	3.906×10^{-3}
3	6.401×10^{-2}	1.490×10^{-8}
4	2.560×10^{-2}	8.272×10^{-25}
5	1.024×10^{-2}	1.415×10^{-73}
6	4.096×10^{-3}	7.082×10^{-220}
7	1.638×10^{-3}	0

    Estimate $\alpha, \tilde{\alpha}, \lambda$, and $\tilde{\lambda}$. Explain why the last entry in the second column is exact assuming the sequence was a computer output.

1a) [5 marks] Let

$$f(h) = \frac{e^h - (1+h)}{h}.$$

Find the largest integer p such that $f(h) = O(h^p)$ as $h \to 0$.

1b) [3 marks] Suppose we used Newton's method to approximate the 20−th root of 2, i.e. a number p such that $p^{20} = 2$. This method would be fixed point iteration for some function $g(x)$. What is the function $g(x)$?

3a) [7 marks] Let $P_3(x)$ be the Langrange interpolating polynomial of degree 3 for the data: $(0,0), (0.5, y), (1,3), (2.2)$. Suppose we know that the coefficient of x^3 in $P_3(x)$ is 6. Find y.

3b) [8 marks] **A natural cubic spline** S on $[0,2]$ is defined by

$$S = \begin{cases} S_0(x) = 1 + 2x - x^2 & \text{on } [0,1) \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{on } [1,2] \end{cases}$$

Find b, c, and d.

4a) [6 marks] Use the most accurate three-point formula to approximate the four missing entries in the table. Show your work.

x	$f(x)$	$f'(x)$
1.1	9.025	
1.2	11.023	
1.3	13.464	
1.4	16.445	

4b) [9 marks] Consider the integral

$$I = \int_{-1}^{1} (10x^2 + 4)(5x^2 - 2) = 4$$

Find a numerical quadrature formula Q of the form

$$Q = \sum_{i=0}^{n} a_i f(x_i),$$

and a specific value of n, such that $|Q - I| < 10^{-4}$. You will have a lot of flexibility in your choices. In the end you will have to choose an n and justify your choice by either evaluating Q and comparing it with $I = 4$ **or** by finding an a priori error bound on $|Q - I|$.

B-2

5a) [**7 marks**] Let

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}.$$

Find an LU decomposition for this matrix such that all the diagonal entries of L are 1.

5b) [**3 marks**] Use the LU decomposition of part a) to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1, 1, 1)$.

5c) [**5 marks**] If one is performing Gaussian elimination on a matrix, what does it mean to use **scaled partial pivoting**? What would be a reason for using it?

7) All parts of question 7 pertain to the following initial value problem: Find $y(t)$ for $t \in [0, 5]$ such that

$$\frac{dy}{dt} = (t^2 + 2)y + \sqrt{t+1} \qquad y(0) = 3.$$

b) [**5 marks**] Perform two steps of Euler's method with time step size $h = 0.5$. You will end up with a number. What does this number have to do with the initial value problem, i.e. what is it an approximation to?

c) [**5 marks**] Perform one step of the second-order Taylor method with time step size $h = 0.5$. What is the advantage of using this method over Euler's method?

C-1

1. (6 marks) Suppose

$$A = \begin{bmatrix} 10 & 1 & 0 \\ -1 & 5 & 1 \\ 0 & -1 & 4 \end{bmatrix}$$

and $b^t = (2, 0, 0)$.

(a) Calculate $||A||_\infty$.

(b) Find the first two iterations of Jacobi's method starting with the zero vector as $x^{(0)}$.

(c) Will Gauss-Seidel converge for this problem? Explain.

(d) Would Choleski's method be a good choice for this problem? Explain.

2. (5 marks) Some Chapter 1 stuff

(a) Suppose -0.125 is rounded to 2 decimal digits. What is the relative error in the result?

(b) Find the rate of convergence of the following sequence:

$$\lim_{h \to 0} [\ln(1+h) - \cos(h) - h] = -1$$

Show your steps.

3 (a) **Briefly** describe under what conditions you would use the following methods instead of Newton's method, i.e., finish the following sentences: (2pts)

i. I would use the secant method when

ii. I would use the modified Newton's method when

4. (3 marks) Are the following true or false? GIVE YOUR REASONING.

(a) Let $f(x) = (x+2)(x+1)^2x(x-1)$. Newton's method gives quadratic convergence to the root -1 using the initial guess -1.02.

(b) Suppose $f(\cdot)$ is smooth. A computer approximation of the first derivative using the two-point forward difference formula

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

produces an error which tends to zero as h tends to zero.

C-2

6. (6 marks) Some questions on direct methods for solving linear systems follow. Suppose

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & 9 \\ 2 & 4 & 5 \end{bmatrix}$$

   (a) Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1's on its diagonal and U is upper triangular. Do not find L or U.

   (b) We wish to solve $Ax = b$. Express x as a product of the matrices $L, L^{-1}, U, U^{-1}, P, P^{-1}$ and the vector b. Do not *solve* for x.

   (c) Could we solve $Ax = b$ using Crout factorization? Briefly explain why/why not.

   (d) Factor B into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

(5pts)

(a) Determine the values of n and h required to approximate

$$\int_0^2 \frac{1}{x+4} dx$$

to within 10^{-4} using the composite trapezoidal rule.

D-1

1 (10 pts).

Suppose 4-digit rounding arithmetic is used to solve the quadratic equation $1.01x^2 + 100x - 1.01 = 0$. Use appropriate rationalization procedure to reduce the effect of roundoff error.

3. (15 pts)

a) Derive a three-point formula of truncation error $O(h^2)$ to approximate $f''(x_0)$ that uses $f(x_0 - h), f(x_0), f(x_0 + h)$.

x	1.28	1.29	1.30	1.31	1.32
$f(x)$	13.5244	13.7818	14.0428	14.3074	14.5758

Use the formula derived in a) to calculate $f''(1.30)$ with $h = 0.2$ and $h = 0.1$

c) Use the above results and one step of appropriate Richardson's extrapolation to calculate $f''(1.30)$

6. (15 pts)

Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

a) Is A strictly diagonally dominant? Why? If yes, find its LU factorization so that all diagonal entries of L is 1.

b) Is A positive definite? Why? If yes, find its Cholesky factorization LL^t.