Macm 316
2005

4 1a) Evaluate $x - \sqrt{x^2 - 1}$ for $x = 100$ using 2-digit chopping arithmetic. Calculate the relative error to at least 2 digits of accuracy. Explain the poor result.

6 b) Rearrange $x - \sqrt{x^2 - 1}$ to construct a formula that will reduce round-off error. Show that your new formula produces better results for large values of x by calculating the relative error for $x = 100$ to at least 2 digits of accuracy.

9 3a) Consider the construction of a clamped cubic spline that interpolates the function $f(x) = \ln x$ at the nodes $x = 1, 2, 3$. Give all of the equations needed to fully determine this cubic spline. Simplify your set of equations to a system of 2 equations in 2 unknowns, **but do not solve.**

b) What is the clamped cubic spline approximation to $x^3 + 3x^2 + 2x + 1$ on the interval $[0, 1]$?

2 4a) When can a matrix be factored into the form LDL^T (D is a diagonal matrix, L is a lower triangular matrix with 1's on the diagonal)?

5 b) Consider the matrix below. Is this matrix diagonally dominant? Show that this matrix satisfies the criteria from part **a)**. Find the LDL^T for this matrix.

$$A = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

6 c) Use your factorization found in part **b)** to calculate the solution to $Ax = [10, 10, 9]^T$. Explain carefully, in terms of the number of operations, the advantage of solving a system in its factorized form, versus doing a complete Gaussian Elimination.

d) True or False. If $||Ax_c - b||$ is very small, then it necessarily follows that $||x_c - x||$ is very small, where x is the exact solution of the system $Ax = b$. Carefully explain your answer.

5. Consider the linear system $Ax = b$, where

$$A = \begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}, \qquad b = \begin{bmatrix} 9 \\ 9 \end{bmatrix}.$$

The exact solution to this system is $x = [1, 1]^T$.

[4] a) Construct the matrix T and the vector c in the iterative solution procedure $x^{(k)} = Tx^{(k-1)} + c$ that would be used with Jacobi iteration.

[6] b) Starting with an initial approximation of $x^{(0)} = [0, 0]^T$, construct the first two iterations for Jacobi iteration $x^{(1)}$ and $x^{(2)}$. What is the absolute error of $x^{(2)}$?

[4] c) What is the spectral radius for your matrix T found in part a) ? How is the spectral radius related to the rate of convergence of Jacobi iteration?

[7] 6a) Using Taylor's theorem and the three function values $f(x)$, $f(x+h)$, $f(x+3h)$, find a finite difference approximation to $f'(x)$ that has a truncation error that is at worst $O(h^2)$

[3] b) Explain what is meant by the statement "numerical differentiation is unstable".

[4] 7a) Determine the number of subintervals n that would be required to approximate

$$\int_0^2 \frac{1}{x+4}\,dx$$

to within 10^{-6} using the composite Simpson's rule.

8. Consider the following initial value problem:

$$\begin{aligned} \frac{dy}{dt} &= \frac{y}{t^{\frac{1}{3}}}, \qquad 1 \le t \le 2 \\ y(1) &= e^{\frac{3}{2}}. \end{aligned}$$

[2] a) Verify that $y = e^{(3/2)t^{2/3}}$ is a solution to the above initial value problem. Is it unique?

[5] b) Write out a detailed, clear algorithm that uses Taylor's method of order 2 and with a step size h to approximate the solution to the above initial value problem (all derivatives should be evaluated).

MACM 316 Final Exam: December 13, 1999

Instructor: R. Russell

Points

Question 1

4 each

Are the statements below TRUE or FALSE? Justify your answers in one or two sentences, indicating briefly how to correct the FALSE statements.

(a) Rolle's Theorem can be considered as a special case of the Mean Value Theorem.

(b) A Taylor expansion for $f(x) = \cos(x)$ about the point $a = 0$ is

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} sin(\xi) \tag{0.1}$$

where $0 < \xi < x$.

(c) For small h, $\cos(h) = 1 + O(h^2)$ and $\cos(h) = 1 + O(h)$.

(d) If $\bar{p}$ is an approximation to $p \neq 0$ with relative error $\frac{|p-\bar{p}|}{|p|} \leq 10^{-k}$, then $\bar{p}$ has at least k significant digits as an approximation to p *regardless* of the size of p.

(e) If $g(x)$ is a continuous function which satisfies $g(x) \in [a, b]$ for all $x \in [a, b]$, then $g(x)$ has a fixed point satisfying $p \in [a, b]$ and for an initial approximation sufficiently close to p, fixed point iteration using $g(x)$ converges to p.

(f) If $f(x)$ is a polynomial with a root p such that $f'(p) \neq 0$, then when sufficiently close to p, Newton's method converges to p for any initial approximation, and the convergence rate is quadratic.

(g) If $f(x)$ is a polynomial of degree k and $x_0, x_1, ...x_n$ are $n + 1$ distinct, unequally spaced points with $n > k$, then the divided difference table using the values $f(x_0), f(x_1), ..., f(x_n)$ consists of only zeros in the columns corresponding to the lth divided differences for all $l > k$.

(h) Given a smooth function $f(x)$, since a natural interpolatory spline $s(x)$ satisfies the minimum curvature property, $s'(x)$ is not a good function to use to approximate $f'(x)$.

(i) Gaussian elimination without pivoting for solving a general system of equations $Ax = b$ is not a stable algorithm, and while Gaussian with partial pivoting *is* generally stable, it is considerably less efficient.

(j) If one solves $Ax = b$ with Gaussian elimination in the case where A is a positive definite matrix, it is not necessary to pivot, and it is more efficient to use a factorization of the form $A = L\,L^T$ where L is a lower triangular matrix.

(k) If A is a (square) nonsingular matrix with an eigenvalue λ, then $\frac{1}{\lambda}$ must be an eigenvalue of A^{-1}.

Points

(l) The simple approximation $\frac{f(a+h)-f(a-h)}{2h}$ to the derivative $f'(a)$ of a smooth function $f(x)$ is a stable algorithm as $h \to 0$.

(m) Even if the IVP (initial value problem) $y'(t) = f(t, y(t)),\ t \geq a$ with $y(a) = \alpha$ is ill-posed, a stable numerical method to solve for $y(t)$ will control the growth of roundoff errors (versus truncation errors).

(n) The Runge Kutta and Adams methods for solving the IVP above do not require taking derivatives of $f(t, y)$, unlike the Taylor methods.

Question 2

(a) Given a sequence of approximations $\{p_n\}$ to a *known* value p, show how the rate of convergence can be approximated. 5

(b) Given a *linearly* convergent sequence $\{\hat{p}_n\}$ to p and a *quadratically* convergent sequence $\{\bar{p}_n\}$ to p, with $\hat{p}_0$ and $\bar{p}_0$ both within 10^{-1} of p, show that, making reasonable assumptions, the number of iterates necessary to obtain accuracy 10^{-n} is proportional to n and $\log n$, respectively. 6

(c) Is it possible that the accuracy of the approximations $\{\hat{p}_n\}$ can be improved using extrapolation? If not, why not? If so, explain roughly what determines whether or not extrapolation will work. 6

Question 3

(a) Find the cubic Hermite polynomial $H(x)$ which interpolates the function $f(x) = \cos(x)$ and its derivative at $\frac{\pi}{2}$ and π. 8

(b) Find a bound for the error $|H(x) - f(x)|$ for $\frac{\pi}{2} \leq x \leq \pi$. 6

Question 4

(a) What does it mean to say that a (square) matrix A is strictly diagonally dominant? For a square matrix T, how is the matrix norm $||T||_\infty$ defined? 4

(b) Prove that if A is strictly diagonally dominant, then solving the system of equations $Ax = b$, the Jacobi iterative method converges for any initial guess. 6

(c) Describe iterative refinement for solving $Ax = b$ and explain in detail what determines how accurate it is at each iteration when $Ax = b$ is solved using Gaussian elimination with pivoting. 6

Question 5 Points

(a) Simpson's rule to approximate an integral is of the form 4

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]. \quad (0.2)$$

What is the error for this quadrature method?

(b) Using this, derive the error formula for the composite Simpson's rule. 6

(c) Explain what happens to both the truncation error and the roundoff error for the composite Simpson's rule as the number of subintervals becomes large. 5

Question 6

(a) Euler's method for solving the initial value problem 5

$$y' = f(t, y), \; t > a, \;\; y(a) = \alpha \quad (0.3)$$

finds the approximations $w_i \approx y(ih)$ by computing

$$w_{i+1} = w_i + hf(t_i, w_i), \; i > 0, \;\; w_0 = \alpha. \quad (0.4)$$

Find the local truncation error for Euler's method.

(b) Outline the proof that Euler's method converges. 7

(c) Show how Euler's method can be used to solve the initial value problem 6

$$u"(t) = t^2 u(t) + u'(t), \;\; t > 0, \;\; u(0) = 2, u'(0) = 3. \quad (0.5)$$

If you wish, you can show this by example, using $h = .1$ and taking two Euler steps – without necessarily carrying out all of the arithmetic.

SFU Macm 316

Final Exam: Final Exam, December 6, 2001

Instructions: Answer all questions. Closed book.

Time: 3 Hours Max. Marks 55

1. (8 marks) Recall that iterative methods for solving the following 3x3 linear system of equations

$$\begin{aligned} 2x - 3y \quad &= -1 \\ -x + y - z &= -1 \\ -3y + z &= -2 \end{aligned}$$

can be written in the form

$$x^{(n)} = M\, x^{(n-1)} + C$$

for some initial guess $x^{(0)}$.

(a) Find M for the Jacobi's method (i.e find M_J)

(b) Write down the first two iterations of Jacobi's method for the above system with $x^{(0)} = (0,0,0)$. Does the Jacobi's method for the above system converge ? Justify.

(c) Would the Gauss- Seidel be an appropriate iteration technique to solve the above system ? Justify your answer.

2. (8 marks) Some Chapter 1 stuff

(a) Find a way to calculate accurate values near zero for the function

$$f(x) = (e^{\sin x} - e^x)/ x^3$$

(Hint: Expand $e^{\sin x}$ by Taylor Series up to appropriate number of terms)

(b) Determine $\lim_{x\to 0} f(x)$.

(c) What is the rate of convergence of $f(h)$ as $h\to 0$?

3. (6 marks) Some questions on direct methods for solving linear systems follow:

(a) Solve the equations

$$x + y + z = 6$$
$$3x + (3+\varepsilon)y + 4z = 20$$
$$2x + y + 3z = 13$$

using Gauss elimination method without pivoting, where ε is a small number such that $1 \pm \varepsilon^2 \approx 1$. Show that the solution may be very inaccurate if ε is of order of the round off error.

(b) Use the Crout's method to solve the following system:

$$x + y + z = 1$$
$$4x + 3y - z = 6$$
$$3x + 5y + 3z = 4$$

4. (8 marks) Some questions on numerical integration:

(a) Give a geometric interpretation of the composite trapezoidal rule for approximating $\int_a^b f(x)dx$.

(b) Obtain the generalised trapezoidal formula of the form

$$\int_a^b f(x)dx = (h/2)\,(\,f(a) + f(b)\,) + ph^2(\,f'(a) - f'(b)\,)$$

by determining the constant p and the error term. (Hint: you may assume a=0 and b=1).

(c) The mid-point rule for approximating $\int_{-1}^{1} f(x)dx$ gives the value 12, the composite mid point rule with $n = 2$ gives 5 and composite trapezoidal rule with $n = 4$ gives 6. Use the fact that $f(-1) = f(1)$ and $f(-0.5) = f(0.5) - 1$ to determine $f(-1)$, $f(-0.5)$, $f(0)$, $f(0.5)$ and $f(1)$.

5. (8 marks) Some questions on Interpolation and Approximation:

(a) Find the continuous least squares trigonometric polynomial $S_2(x)$ for $f(x) = x^2$ on $[-\pi, \pi]$.

(b) The equation $x - 9^{-x} = 0$ has a solution in $[0, 1]$. Find the interpolation polynomial on $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ for the function on the left side of the equation. By setting the interpolation polynomial equal to zero and solving the equation, find an approximate solution to the equation.

(c) The polynomial $p(x) = 2 - (x + 1) + x(x + 1) - 2x(x + 1)(x - 1)$ interpolates the first four points in the table :

x	-1	0	1	2	3
y	2	1	2	-7	10

By adding one additional term to p find a polynomial that interpolates the whole table.

6. (9 marks) Initial Value Problems

(a) Show that the initial value problem

$$y' = (t + \sin y)^2, \quad 0 \le t \le 1, \quad y(0) = 3$$

has a unique solution.

(b) Approximate $y(1)$ using the Euler method with $h = 0.5$

(c) Find the region of absolute stability of Modified Euler's method.

7. (6 marks) Some problems on root finding techniques

(a) Let $g(x) = (5/x^2) + 2$ for $x \in [2.5, 3]$. Show that for any number $p_0 \in [2.5, 3]$, the sequence defined by $p_n = g(p_{n-1}),\ n \ge 1$, converges to the unique fixed point p in $[2.5, 3]$. Starting with the initial value $p_0 = 2.5$ perform the first two iterations, i.e., calculate p_1 and p_2.

(b) Derive the secant method for finding the roots of the equation $f(x) = 0$ and describe it graphically. How does this method differ from Regula-Falsi method ?

8. (2 marks) Find all values of α so that the matrix

$$A = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

is positive definite.