1. A function table is given by

x	f(x)
-0.3	2.50
0	3.25
0.5	4.50

a) Taylor series around 0 is given by

$f(h)= f(0) + h\, f'(0) + (h^2/2)f''(0) + (h^3/6)f'''(0) + \ldots$

Based on this, write expressions for f(-3h) and f(5h).

(1) $f(-3h) = f(0) - 3h f'(0) + \frac{9h^2}{2} f''(0) - \frac{27h^3}{6} f'''(0) + o(h^4)$

(2) $f(5h) = f(0) + 5h f'(0) + \frac{25h^2}{2} f''(0) + \frac{125h^3}{6} f'''(0) + o(h^4)$

b) Derive the best approximation to calculate f'(0) with data given above.

To eliminate f'' term, we consider $h = 0.1$ then:

$$\left.\begin{aligned}(1) &\rightarrow f(-0.3) = f(0) - 0.3 f'(0) + \frac{0.09}{2} f''(0) - \frac{0.027}{6} f'''(0) \\ (2) &\rightarrow f(0.5) = f(0) + 0.5 f'(0) + \frac{0.25}{2} f''(0) + \frac{0.125}{6} f'''(0)\end{aligned}\right\} \Rightarrow$$

$0.25 \times (1) - 0.09 \times (2) \Rightarrow 0.25 f(-0.3) - 0.09 f(0.5) = 0.16 f(0) - 0.12 f'(0) - \frac{0.018}{6} f'''(0)$

$\Rightarrow f'(0) = \frac{1}{0.12}\left[-0.25 f(-0.3) + 0.09 f(0.5) + 0.16 f(0) - \frac{0.018}{6} f'''(0)\right]$

c) What is the error term for your approximation?

$err = \frac{-0.018}{0.12 \times 6} f'''(0) = -0.025 f'''(0)$

d) Calculate f'(0) by the formula you derived.

$f'(0) = \frac{1}{0.12}\left[-0.25(\underbrace{2.5}_{f(-0.3)}) + 0.09(\underbrace{4.5}_{f(0.5)}) + 0.16(\underbrace{3.25}_{f(0)})\right] = 2.5$

2. a) Calculate $I = \int_2^6 e^{x/3}\, dx$ by using the composite Midpoint rule for 4 subintervals.

$m=3$, $n=2m=6$, $a=2$, $b=6$, $h=\frac{b-a}{n+2}=\frac{1}{2}$, $x_j = 2+0.5(j+1)$
$j=-1,0,\ldots,7$

$$I = 2h\sum_{j=0}^{3} f(x_{2j}) = 2(0.5)\left[f(x_0)+f(x_2)+f(x_4)+f(x_6)\right]$$
$$=\left[f(2.5)+f[3.5]+f(4.5)+f(5.5)\right]$$
$$= e^{2.5/3} + e^{3.5/3} + e^{4.5/3} + e^{5.5/3} = 16.24864$$

b) Find the error of the method and the round off error in finding the above integral.

$$e_1 = \left| \frac{b-a}{6} h^2 f''(\mu) \right| \qquad 2 \le \mu \le 6$$

$$f(x) = e^{x/3}, \quad f'(x) = \frac{e^{x/3}}{3}, \quad f''(x) = \frac{1}{9} e^{x/3} \Rightarrow \underset{2\le\mu\le6}{\text{Max}}\ f''(\mu) = \frac{1}{9} e^2$$

$$\Rightarrow e_1 \le \left| \frac{4}{6} \left(\frac{1}{2}\right)^2 \frac{1}{9} e^2 \right| = \frac{e^2}{54} = 0.136834$$

Round off error : $e_2 \le 4 \times 10^{-5} \leftrightarrow$ 5 digits have been rounded.

Total error $= e_1 + e_2 = 0.136844$

c) How many subintervals are necessary so that the error for integral in question a) is at most 10^{-5}? (consider only the error of the method, and not the round off error).

$b-a=4$ and $f''(x)=\frac{e^2}{9}$ $\underline{2\leq x\leq 6}$

$$err=\left|\frac{4}{6}h^2\frac{e^2}{9}\right|\leq 10^{-5} \Rightarrow h^2\leq\frac{27}{2e^2}\times 10^{-5} \Rightarrow h\leq 0.0042744$$

$$n=\frac{b-a}{h}-2\geq\frac{4}{0.0042744}-2\simeq 933.8 \rightarrow m=\frac{n}{2}\geq 466.9$$

$$\Rightarrow \boxed{m=467}$$

d) Let the round off error be 10^{-5}. For what value of subinterval size h in integral in question a) are round off error and the error of the method approximately equal?

$$\left.\begin{aligned} &\text{We found error} = \frac{2}{27}e^2h^2 \\ &\text{round off error} = 2h\,(4\times 10^{-5}) \end{aligned}\right\} \Rightarrow \frac{2}{27}e^2h^2 = 2h\,(4\times 10^{-5})$$

$$\Rightarrow h = 0.00014616$$

3. Use Heun's method with h=0.1, to approximate the solution of the following initial-value problem:

$y' = -(t-1)^2 + y, \ y(2)=3, \ 2 \le t \le 2.3.$

$w_{i+1} = w_i + h(f(t_i,w_i) + 3\,f(t_i + 2h/3,\ w_i + 2h\,f(t_i,w_i)/3))/4.$

$$\left.\begin{array}{l} t_0 = 2 \\ t_1 = 2.1 \\ t_2 = 2.2 \\ t_3 = 2.3 \end{array}\right\} \Rightarrow \text{ we construct a table with } \left[\begin{array}{l} A = f(t_i, \omega_i) \\ B = t_i + \frac{2}{3}h \\ C = \omega_i + \frac{2}{3}hA \\ D = 3f(B,C) \end{array}\right. \Rightarrow$$

So : $\omega_{i+1} = \omega_i + \frac{0.1}{4}\left[A + D\right]$

i	t_i	ω_i	A	B	C	D
0	2	3	2	2.06667	3.13333	5.98664
1	2.1	3.19967	1.98967	2.16667	3.33231	5.91357
2	2.2	3.39725	1.95725	2.26667	3.52773	5.76983
3	2.3	3.59043				

So:

$(t_0, \omega_0) = (2, 3)$

$(t_1, \omega_1) = (2.1, 3.19967)$

$(t_2, \omega_2) = (2.2, 3.39725)$

$(t_3, \omega_3) = (2.3, 3.59043)$

4. Consider the following system of linear equations:

$$8.888\,x + 3333.33\,y + 15.002\,z = 3200.12$$
$$3007\,x + 2008\,y + 0.005\,z = 2753.001$$
$$1001.2\,x + 2013\,y + 20.425\,z = 2043.403.$$

With rounding-off after third significant digit (rounf off the coefficients of the system and all intermediate results), solve the system using Gaussian elimination with the scale column pivoting.

$$\left[\begin{array}{ccc|c} 8.888 & 3333.33 & 15.002 & 3200.12 \\ 3007 & 2008 & 0.005 & 2753.001 \\ 1001.2 & 2013 & 20.425 & 2043.403 \end{array}\right] \longrightarrow$$

$$\begin{array}{l} E_1/3333.33 \\ E_2/3007 \\ E_3/2013 \end{array} \left[\begin{array}{ccc|c} 0.00267 & 1 & 0.0045 & 0.960 \\ 1 & 0.668 & 0.00000166 & 0.916 \\ 0.497 & 1 & 0.0101 & 1.015 \end{array}\right]$$

$$E_1 \leftrightarrow E_2 \quad \left[\begin{array}{ccc|c} 1 & 0.668 & 0.00000166 & 0.916 \\ 0.00267 & 1 & 0.0045 & 0.960 \\ 0.497 & 1 & 0.0101 & 1.015 \end{array}\right]$$

$$\begin{array}{l} E_2 - 0.00267(E_1) \to E_2 \\ E_3 - 0.497(E_1) \to E_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0.668 & 0.00000166 & 0.916 \\ 0 & 0.998 & 0.0045 & 0.958 \\ 0 & 0.668 & 0.0101 & 0.560 \end{array}\right]$$

$$\begin{array}{l} E_2/0.998 \\ E_3/0.668 \end{array} \left[\begin{array}{ccc|c} 1 & 0.668 & 0.00000166 & 0.916 \\ 0 & 1 & 0.00451 & 0.960 \\ 0 & 1 & 0.0151 & 0.838 \end{array}\right]$$

$$E_3 - E_2 \longrightarrow E_3 \quad \left[\begin{array}{ccc|c} 1 & 0.668 & 0.00000166 & 0.916 \\ 0 & 1 & 0.00451 & 0.960 \\ 0 & 0 & 0.0106 & -0.122 \end{array}\right]$$

$$\Rightarrow x_3 = \frac{-0.122}{0.0106} = -11.5 \Rightarrow x_2 = 0.960 - 0.00451\,x_3 \Rightarrow x_2 = 1.01 \Rightarrow x_1 = 0.241$$

$$X = (0.241,\ 1.01,\ -11.5)$$

5. Consider the following system of linear equations:

$10x_1 + 2x_2 - x_3 = 7$
$x_1 + x_2 + 8x_3 = 10$
$x_1 - 7x_2 + 5x_3 = -1$

a) Write down the corresponding system of equation that will converge, and state the reason why it will converge.
b) Apply Jacobi's algorithm on your equivalent system, beginning with initial solution (0,0,0), until tolerance 10^{-1} is achieved, or until maximum of 4 iterations.

$$x_1 = \frac{-x_2}{5} + \frac{x_3}{10} + \frac{7}{10}$$
$$x_2 = \frac{x_1}{7} + \frac{5}{7}x_3 + \frac{1}{7}$$
$$x_3 = \frac{-x_1}{8} - \frac{x_2}{8} + \frac{5}{4}$$

$$\Rightarrow T = \begin{bmatrix} 0 & \frac{-1}{5} & \frac{1}{10} \\ \frac{1}{7} & 0 & \frac{5}{7} \\ -\frac{1}{8} & -\frac{1}{8} & 0 \end{bmatrix} \quad c = \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix}$$

$$\Downarrow$$

$$\|T\|_\infty = \max\left\{\frac{3}{10}, \frac{6}{7}, \frac{1}{4}\right\} < 1$$

$\rightarrow$ will converge

& $\rho(T) \leq \|T\|_\infty < 1$

In addition, the linear system is:

$Ax = b$ where $A = \begin{bmatrix} 10 & 2 & -3 \\ 1 & -7 & 5 \\ 1 & 1 & 8 \end{bmatrix}$ which is strictly diagonally dominant

$\Rightarrow$ By this we have convergence

b) $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = X^{(0)} = (0,0,0)$

$$X^{(1)} = \begin{bmatrix} 0 & -1/5 & 1/10 \\ 1/7 & 0 & 5/7 \\ -1/8 & -1/8 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix} = \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix} \Rightarrow \frac{\|X^{(1)} - X^{(0)}\|_\infty}{\|X^{(1)}\|_\infty} = 1$$

$$X^{(2)} = \begin{bmatrix} 0 & -1/5 & 1/10 \\ 1/7 & 0 & 5/7 \\ -1/8 & -1/8 & 0 \end{bmatrix} \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix} + \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix} = \begin{bmatrix} 0.79643 \\ 1.13571 \\ 1.19464 \end{bmatrix} \Rightarrow \frac{\|X^{(2)} - X^{(1)}\|_\infty}{\|X^{(2)}\|_\infty} =$$

$$\frac{139/140}{641/560} = 0.867$$

$$X^{(3)} = \begin{bmatrix} 0 & -1/5 & 1/10 \\ 1/7 & 0 & 5/7 \\ -1/8 & -1/8 & 0 \end{bmatrix} \begin{bmatrix} 0.79643 \\ 1.13571 \\ 1.19464 \end{bmatrix} + \begin{bmatrix} 7/10 \\ 1/7 \\ 5/4 \end{bmatrix} = \begin{bmatrix} 0.58732 \\ 1.07423 \\ 1.00848 \end{bmatrix} \Rightarrow \frac{\|X^{(3)} - X^{(2)}\|_\infty}{\|X^{(3)}\|_\infty} = \frac{0.20911}{1.00848}$$

$$= 0.20735$$

$$X^{(4)} = \begin{bmatrix} & // & \end{bmatrix} \begin{bmatrix} 0.58732 \\ 1.07423 \\ 1.00848 \end{bmatrix} + \begin{bmatrix} // \end{bmatrix} = \begin{bmatrix} 0.586002 \\ 0.94710 \\ 1.04231 \end{bmatrix} \Rightarrow \frac{\|X^{(4)} - X^{(3)}\|_\infty}{\|X^{(4)}\|_\infty} = \frac{0.12713}{1.04231}$$

$$= 0.121$$

we stop here at $X^{(4)}$.

c) Find a bound for the absolute error $\|x^{(4)} - x\|_\infty$ where x is the exact solution of the system, without finding the exact value of x.

We have: $\|x^{(4)} - x\|_\infty \leq \dfrac{\|T\|_\infty^4 \; \|x^{(1)} - x^{(0)}\|_\infty}{1 - \|T\|_\infty}$

$\|T\|_\infty = \frac{6}{7}$ (part a)

$\|x^{(1)} - x^{(0)}\|_\infty = \frac{5}{4}$

$\Rightarrow$

$$\|x^{(4)} - x\|_\infty \leq \frac{(6/7)^4}{1 - \frac{6}{7}} \left(\frac{5}{4}\right) = 4.72303$$

$$\|x^{(4)} - x\|_\infty \leq 4.72303$$