Taylor polynomials & their uses

~~$f(x) = f(x_0)$~~ Taylor's polynomial with error term

f: a reasonable function

Fix an input value x_0 (base point)

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$+ \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1} \qquad (1)$$

where $\xi(x)$ lies between x_0 & x.

Input into $f^{(n+1)}$ is not the basepoint x_0, but instead a mysterious c which we truly know nothing but it lies between x_0 & x.

The last term is the error term. The part of (1) without the error term is the degree n Taylor polynomial at x_0. The last term is the error term.

Linear Taylor

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi(x))}{2!}(x - x_0)^2$$

where $\xi(x)$ is between x & x_0.

$$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}.$$

difference formula

We can control the error by taking larger terms & making h smaller

Classic Taylor poly

Base point = 0

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Computational Tricks using Taylor polynomial

Infinite series of Taylor polynomial (around x_0)

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots\cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots\cdots$$

Finite form:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{3!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

$\xi(x)$ lies between x_0 & x.

Ex $\quad e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots$ (base point 0)

or $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^{\xi(x)}}{4!} \cdot x^4$

Where $\xi(x)$ lies between 0 & x

Question 1 Given

- a Taylor polynomial approximation of $f(x)$ at some basepoint
- a tolerance (error no more than the tolerance)

"How large an interval around the given point does Taylor polynomial achieve the tolerance?"

Ex. $f(x) = \sqrt{x}$

$$f(x) \approx \sqrt{25} + \frac{1}{2} \cdot \frac{1}{\sqrt{25}} (x-25)$$

base point $= 25$

tolerance $= .001$.

$$\text{Error} = \frac{1}{2!} \frac{1}{4} \cdot \frac{(x-25)^2}{(\xi(x))^{3/2}}$$ where $\xi(x)$ is between 25 & x.

We need to find the largest value of x for which

$$\left| \frac{1}{8} \frac{(x-25)^2}{\xi(x)^{3/2}} \right| < tol = 0.001.$$

$5^{2 \cdot 3/_2}$

Clearly, $25 \leq \xi_P(x) \leq x \Rightarrow 25^{3/2} \leq \xi_P(x)^{3/2} \leq x^{3/2}$

Now: $\left|\frac{1}{8}\frac{(x-25)^2}{\xi_P(x)^{3/2}}\right| < \left|\frac{1}{8}\frac{(x-25)^2}{25^{3/2}}\right| = \left|\frac{1}{8}\cdot\frac{1}{125}(x-25)^2\right|$

$\therefore$ We need to find x s.t.

$$\frac{1}{1000}(x-25)^2 = .001$$

i.e. $(x-25)^2 = 1$

i.e. $x = 26$

Hence within the interval [24, 26], Taylor polynomial achieve the tolerance.

Q2. Given

- Taylor polynomial approximation of $f(x)$ around a basepoint.
- an interval around ~~the~~ the basept.
- "Within what tolerance does the Taylor polynomial approximate the function in that interval?"

Ex

$$f(x) = \cos x$$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad (\text{basepoint} = 0)$$

$$x \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

$$\text{Error} = \left| \frac{f^{(6)}(\xi(x))}{6!} x^6 \right|$$

Now $f^{(6)}(\xi(x)) = -\cos \xi(x)$; but $|\cos \xi(x)| \leq 1$ $\forall x$.

$$\therefore \left| \frac{\cos(\xi(x))}{6!} \cdot x^6 \right| \leq \left| \frac{x^6}{6!} \right| \leq \frac{1}{2^6 6!} \quad \because |x| \leq \tfrac{1}{2}$$

$\therefore$ The error is less than $\frac{1}{2^6 6!}$.

Question 3:

- Given a function $f(x)$
- an interval around the basepoint
- a required tolerance.

"Determine how many terms must be used in the Taylor expansion?"

Suppose : $f(x) = e^x$
basepoint $= 0$
tol $= .001$
interval $= [-1/2, 1/2]$

We need to find large enough n s.t.

$$E = \left| \frac{e^{\xi(x)} x^{n+1}}{(n+1)!} \right| \leq .001$$

$$\text{Since } |e^{\xi(x)}| \leq e^{1/2} \leq 2$$

$$\text{Also } |x^{n+1}| \leq \frac{1}{2^{n+1}}$$

used information from the problem setting (B)

$$\therefore E = \left| \frac{e^{\xi(x)} x^{n+1}}{(n+1)!} \right| \leq \frac{2}{(n+1)!} \cdot \left(\frac{1}{2}\right)^{n+1} \leq 0.001 \quad (\text{using B})$$

$$\Rightarrow n = 4.$$

$$\therefore e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Integrating Taylor polynomials

Suppose $f(x) = \frac{1}{1-x}$.

$$\therefore \int_0^T f(x)\,dx = \left[-\log(1-x)\right]_0^T$$
$$= -\log(1-T)$$

On the other hand.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad (\text{basepoint} = 0$$

$$\therefore \int_0^T \left(1 + x + x^2 + \cdots + \right) dx = \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]_0^T$$
$$= T + \frac{T^2}{2} + \frac{T^3}{3} + \cdots$$

Suppose $\frac{1}{1-x} \simeq 1 + x + x^2 + \cdots + x^n + \frac{1}{(n+1)(1-\xi(x))^{n+1}} x^{n+1}$

where $\xi(x)$ is between 0 & x.

We take $\xi(x) < 1$ ~~because~~ otherwise the error term will be diverging.

Suppose $0 \leq T \leq \frac{1}{2}$

In general

The error term is

$$E_f = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

Where $\xi(x)$ lies between x_0 & x.

If we don't know $f^{(n+1)}(\xi(x))$, we will assume that $f^{(n+1)}(\xi(x))$ is bounded above by a constant M.

$$\therefore \; E_f \lessapprox \frac{M}{(n+1)!}(x-x_0)^{n+1}$$

If $h = x-x_0 \Rightarrow \text{error}(E_f) \in O(h^{n+1})$

$$\& \; \text{error}\left(\int_a^b E_f \, dx\right) \in O(h^{n+2}).$$

Differentiation: Compute $f'(x_0)$ for some $f(x)$.

2-point approximation $(x_0, f(x_0)), (x_0+h, f(x_0+h))$

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!} h + \frac{f''(\xi)}{2!} h^2$$

here ξ is fixed since x_0+h is fixed.

$$\therefore f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{f''(\xi)}{2!} h$$

The error term is $-\frac{f''(\xi)}{2!} h$ if $f'(x_0)$ is approximated by $\frac{f(x_0+h) - f(x_0)}{h}$.

3-point approximation Compute $f'(x_0)$

3 points : $(x_0-h, f(x_0-h)), (x_0, f(x_0))$ &

$(x_0+h, f(x_0+h))$.

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(\xi_1)}{3!}h^3, \quad x_0 \le \xi_1 \le x_0+h$$

$$f(x_0-h) = f(x_0) - \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(\xi_2)}{3!}h^3, \quad x_0-h \le \xi_2 \le x_0$$

$$\therefore f(x_0+h) - f(x_0-h) = 2hf'(x_0) + \frac{h^3}{6}\left[f'''(\xi_1) + f'''(\xi_2)\right]$$

$$\therefore f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{12}[2M]$$

where 2M reflects the total value of $f'''(\xi_1) + f'''(\xi_2)$

$$\therefore \text{Error} = \left|\frac{h^2}{6}M\right|$$

if $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0-h)}{2h}$.

Problem Compute $f''(x_0)$ of some $f(x)$

3point approximation

Given $(x_0-h, f(x_0-h)), (x_0, f(x_0))$ & $(x_0+h, f(x_0+h))$

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{(4)}(\xi_1)}{4!}h^4$$

$$f(x_0-h) = f(x_0) - \frac{f'(x_0)}{1!} + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f^{4}(\xi_2)}{4!}h^4$$

where $x_0-h \le \xi_2 \le x_0$

& $x_0 \le \xi_1 \le x_0+h$.

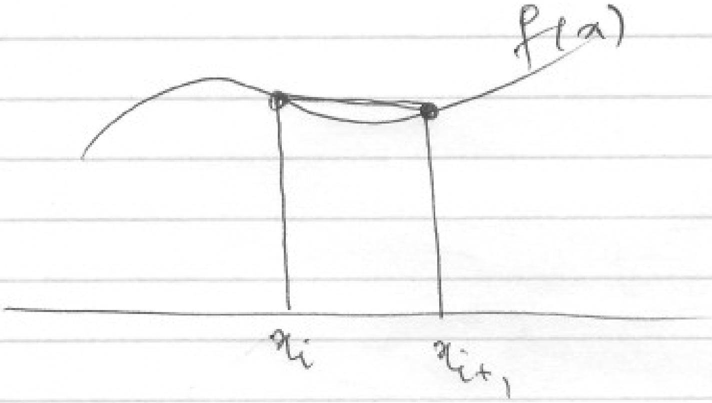
$$\therefore f(x_0+h) + f(x_0-h) = 2f(x_0) + f''(x_0)h^2 + \frac{h^4}{4!}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right]$$

$$\therefore f''(x_0) = \frac{1}{h^2}\left[f(x_0+h) - 2f(x_0) + f(x_0-h)\right] - \frac{h^2}{4!} \times 2M \quad \text{(like before)}$$

$\therefore$ Error $= \left|\frac{h^2}{12}M\right|$ if $f''(x_0) \approx \frac{1}{h^2}\left[f(x_0+h) - 2f(x_0) + f(x_0-h)\right]$

Integration

Trapezoidal rule



$$\therefore \quad f(x) = f(x_i) + \frac{f'(x_i)}{1!}(x-x_i) + \frac{f''(\xi_i(x))}{2!}(x-x_i)^2$$

Where $\xi_i(x)$ is between x_i & x_{i+1}.

$$\therefore \int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{x_i}^{x_{i+1}} \left[f(x_i) + \frac{f'(x_i)}{1!}(x-x_i) \right] dx$$

$$+ \int_{x_i}^{x_{i+1}} \frac{f''(\xi_i(x))}{2!}(x-x_i)^2\,dx.$$

The first part gives (assuming $h = x_{i+1} - x_i$)

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = f(x_i)\,h + f'(x_i)\cdot\frac{h^2}{2} + \frac{M}{2}\cdot\frac{h^3}{3} \quad (A)$$

But we know that

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{M}{2}h.$$

Replacing $f'(x_i)$ in (A) we get

$$\therefore \int_{x_i}^{x_{i+1}} f(x)\,dx = \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] + \frac{h^3}{2}\left[\frac{M}{3} - \frac{M}{2}\right]$$

$$= \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] - \frac{M}{12}h^3$$

Composite trapezoidal rule

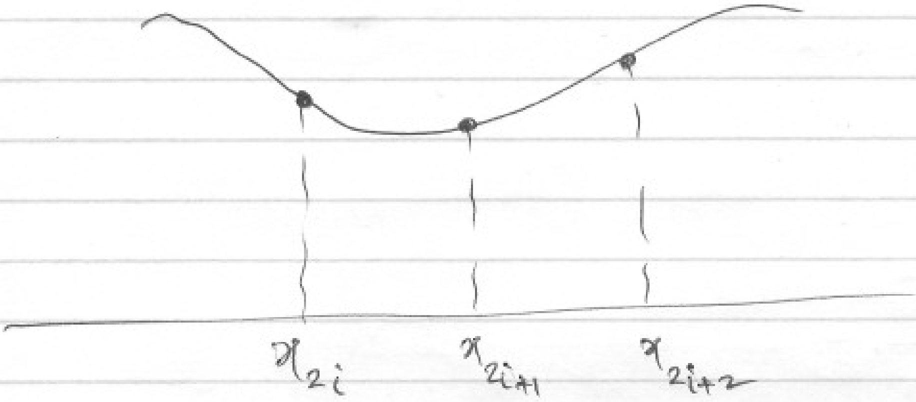
$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$= \sum_{i=0}^{n-1}\left[\frac{h}{2}\left[f(x_i) + f(x_{i+1})\right]\right] - \frac{M}{12}\cdot h^3\cdot n$$

But $h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h}$.

$\therefore$ Error is $\left|\frac{M}{12}h^3\cdot\frac{(b-a)}{h}\right| = \left|\frac{M h^2}{12}(b-a)\right|$.

Simpson's rule.



$$\therefore \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx = \int_{x_{2i}}^{x_{2i+2}} \Big[f(x_{2j+1}) + f'(x_{2j+1})(x - x_{2j+1})$$

$$+ \frac{f''(x_{2j+1})}{2!}(x - x_{2j+1})^2$$

$$+ \frac{f'''(x_{2j+1})}{3!}(x - x_{2j+1})^3 \Big] dx$$

$$+ \int_{x_{2i}}^{x_{2i+2}} \frac{f^{(4)}(\xi(x))}{4!}(x - x_{2j+1})^4\,dx.$$

Let $h = x_{2i+1} - x_{2i} = x_{2i+2} - x_{2i+1}$

$$\therefore \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx = 2h.\, f(x_{2i+1}) + 0 + \frac{f''(x_{2i+1})}{2!}\,\frac{2h^3}{3} + 0$$

$$+ \frac{M}{4!}\,\frac{2h^5}{5}$$

We know that

$$f''(x_{2i+1}) = \frac{f(x_{2i}) - 2f(x_{2i+1}) + f(x_{2i+2})}{h^2} - \frac{h^2 M}{12}$$

$$\therefore \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx = 2h\, f(x_{2i+1}) + \left[\frac{f(x_{2i}) - 2f(x_{2i+1}) + f(x_{2i+2})}{h^2}\right]\frac{h^3}{3}$$

$$+ M h^5 \left[\frac{1}{60} - \frac{1}{36}\right]$$

$$= \frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]$$

$$- \frac{M}{90} h^5.$$

Composite Simpsons

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n/2-1} \int_{x_{2i}}^{x_{2i+1}} f(x)\,dx$$

$$= \sum_{i=0}^{n/2-1} \left[\frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right] - \frac{M}{90} h^5 \right]$$

$$= \sum_{i=0}^{n/2-1} \left[h/3 \left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right] \right]$$

$$- \frac{M}{90} h^5 \times \frac{n}{2}$$

$$\text{Error} = \left| \frac{M}{90} h^5 \cdot \left(\frac{b-a}{2h}\right) \right| = \left| \frac{M h^4 (b-a)}{180} \right|.$$

Theorem Given $f \in C^2[a,b]$, $h = \frac{b-a}{n}$, $x_i = a + ih$. $i = 0, 1, \cdots, n$.

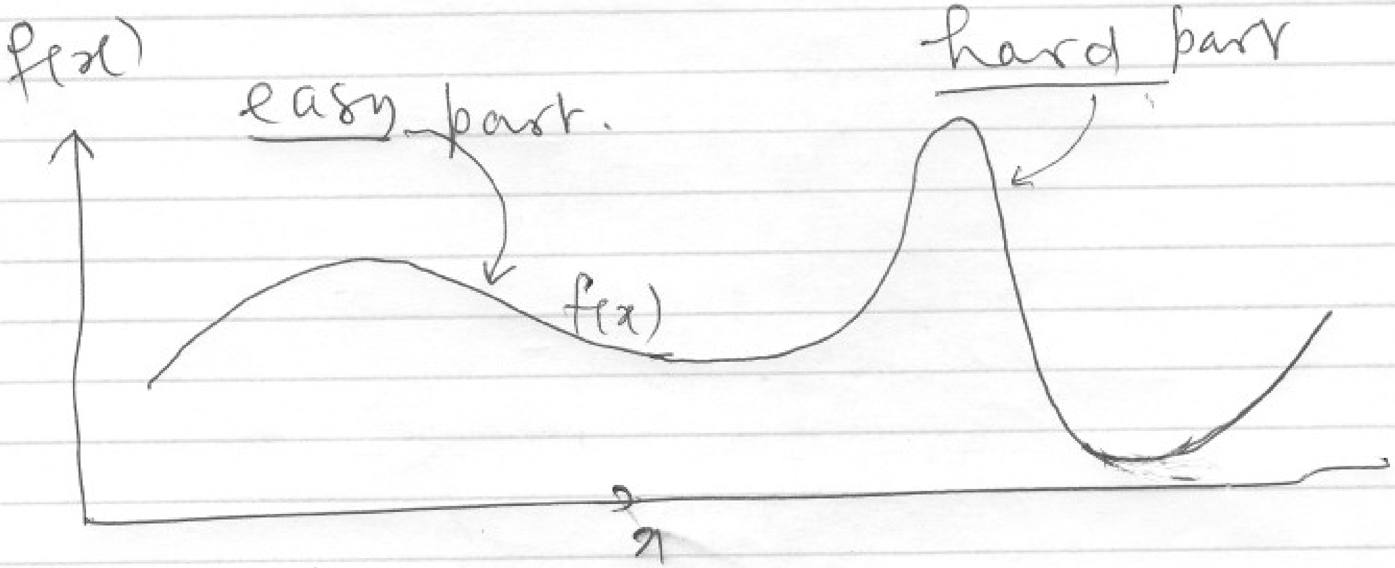
There exist a constant M for which the composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n/2-1} h/3 \left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]$$

$$- \frac{M(b-a)}{180} h^4.$$

Adaptive Simpson's method

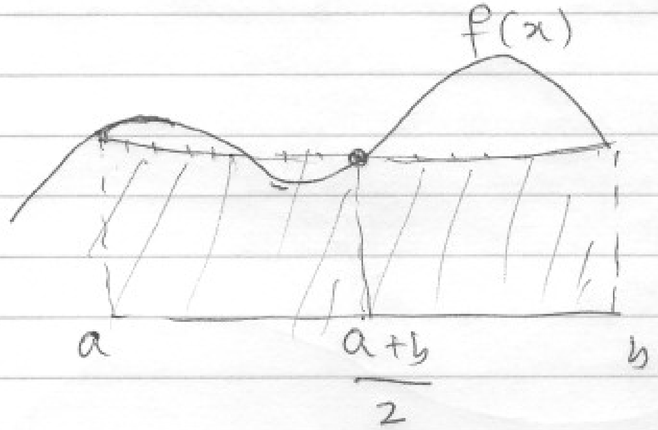
The composite formulas require the use of equally spaced nodes. This is not efficient since some part ~~is~~ of the function is easy to approximate & some ~~fun~~ part of the function hard to integrate.



Efficient approach would work hard on hard part (i.e. spacing would be finer) & work easy on easy part (i.e. spacing would be coarse).

These type of approaches are called adaptive approaches.

But it is not easy to know in advance on which part of the ~~interval~~ integral f varies rapidly. We can consider an adaptive integration method. The basic idea is we divide $[a,b]$ into two equal subintervals & then decide whether each one of them is to be divided into more subintervals. ~~This pro~~



$\underline{n=2}$

Let $S(a,b)$ be the simpson's estimate.

Then $\displaystyle\int_a^b f(x)\,dx = S(a,b) - \frac{M(b-a)}{180}\cdot(b-a)^4$ for some const. M.

$$= I_1 + E_1 \quad \text{, say.}$$

The next step is to determine an accuracy approximation that does not require M.

We apply the Composite Simpson's rule with $n=4$, & step size $\frac{(b-a)}{4} = \frac{h}{2}$, giving

$$\int_a^b f(x)\,dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{(b-a)M}{180} \times \left(\frac{b-a}{4}\right)^4$$

$$= I_2 + E_2 \quad \text{say.}$$

Since $\quad I = I_1 + E_1 = I_2 + E_2$

$$\therefore \; I_2 - I_1 = E_1 - E_2 = 16E_2 - E_2 = 15E_2$$

$$\therefore \; E_2 = \frac{1}{15}(I_2 - I_1).$$

Therefore if

$$\left| S(a,b) - \left(S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) \right) \right| \leq 15\epsilon \quad \text{— (A)}$$

then we expect

$$\left| \int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \leq \epsilon$$

~~Ex Look at ex 1 of page 215.~~

When condition (A) is not satisfied (i.e. differs more than 15ϵ) we apply the adaptive procedure to the subintervals $\left[a, \frac{a+b}{2}\right]$ & $\left[\frac{a+b}{2}, b\right]$. For each subproblem, the tolerance $= \frac{\epsilon}{2}$.

Pseudocode

function $I = \text{adapt}(a, b, \epsilon)$

- Compute $\int_a^b f(x)\,dx$ in two ways & call them I_1 & I_2
- Estimate the error in I_2 based on $|I_2 - I_1|$.
- If |estimated error| $\leq \epsilon$ then

  $I = I_2$

  else

  $c = \frac{a+b}{2}$

  $I = \text{adapt}(f, a, c, \epsilon/2) + \text{adapt}(f, c, b, \epsilon/2)$

  end.

This guarantees that

$$|I_{approx} - I_{true}| \leq \epsilon$$

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at $x'(t) = f(t, x)$.
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nt of data that may
a graphical monitor

n

ons and integration.

Copied from

"Numerical ~~methods~~ Mathematics & Computing"

by

Cheney & Kincaid.

Thompson Publishers.

(Visualising IVP for a first order differential equation.)

Vector Fields

Consider a generic first-order differential equation with prescribed initial condition:

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(a) = b \end{cases}$$

Before addressing the question of solving such an initial-value problem numerically, it is helpful to think about the intuitive meaning of the equation. The function f provides the slope of the solution function in the tx-plane. At every point where $f(t, x)$ is defined, we can imagine a short line segment being drawn through that point and having the prescribed slope. We cannot graph *all* of these short segments, but we can draw as many as we wish, in the hope of understanding how the solution function $x(t)$ traces its way through this forest of line segments, while keeping its slope at every point equal to the slope of the line segment drawn at that point. The diagram of line segments illustrates discretely the so-called **vector field** of the differential equation.

For example, let us consider the equation $x' = \sin(x + t^2)$ with initial value $x(0) = 0$. In the rectangle described by $-3 \leqq x \leqq 3$ and $0 \leqq t \leqq 8$, we can ask Matlab to furnish a picture of the vector field engendered by our differential equation. The command `dfield5` in the Matlab windows environment brings up a window with a default differential equation already inserted. This should be replaced by the equation $x' = \sin(x + t^2)$. Then the default rectangle can be changed, and finally one can *click* the label *proceed*. Behind the scenes, Matlab will then carry out the immense calculation to provide the vector field for this differential equation, and will display it in its own window, correctly labeled. To see the solution going through any point in the diagram, it is only necessary to use the mouse to position the pointer on such a point, and to click the left mouse button. Matlab will display the solution sought. By use of this tool, one can see immediately the effect of changing initial conditions. For the problem under consideration, several solution curves (corresponding to different initial values) are shown in Figure 10.1.

Another example, treated in the same way, is the differential equation $x' = x^2 - t$. (This is the default equation that appears when the command `dfield5` is issued.) Figure 10.2 shows a vector field for this equation and some of its solutions. Notice the phenomenon of many quite different curves all seeming to arise from the same initial condition. What is happening here? This is an extreme example of a differential equation whose solutions are exceedingly sensitive to the initial condition! One can expect trouble in solving this differential equation with an initial value prescribed at $t = 2$.

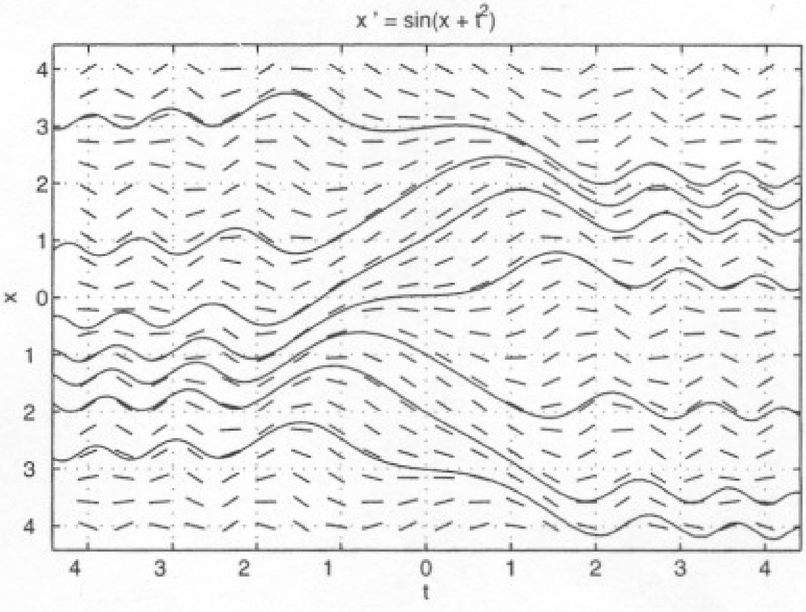


FIGURE 10.1
Vector field and some solution curves for $x' = \sin(x + t^2)$

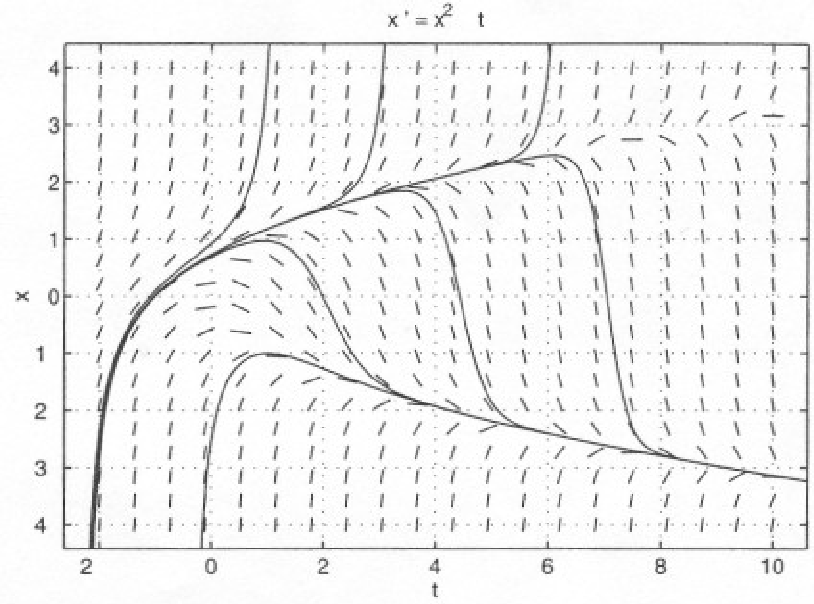


FIGURE 10.2
Vector field and some solution curves for $x' = x^2 - t$

How do we know that the differential equation $x' = x^2 - t$, together with an initial value, $x(t_0) = x_0$, has a unique solution? There are many theorems in the subject of differential equations that concern such existence and uniqueness questions. One of the easiest to use is as follows.

THEOREM 1

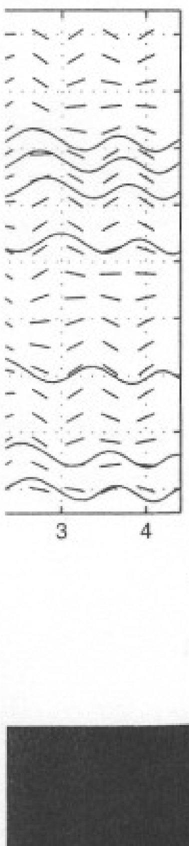
UNIQUENESS OF INITIAL-VALUE PROBLEMS

If f and $\partial f/\partial y$ are continuous in the rectangle defined by $|t - t_0| < \alpha$ and $|x - x_0| < \beta$, then the initial value problem $x' = f(t, x)$, $x(t_0) = x_0$ has a unique continuous solution in some interval $|t - t_0| < \epsilon$.

From the theorem just quoted, we cannot conclude that the solution in question is defined for $|t - t_0| < \beta$. However, the value of ϵ in the theorem is at least β/M, where M is an upper bound for $|f(t, x)|$ in the original rectangle.

10.1 Taylor Series Methods

The numerical method described in this section does not have the utmost generality, but it is natural and capable of high precision. Its principle is to represent the solution of a differential equation locally by a few terms of its Taylor series.



Ordinary differential equations (McGill)

- ODE is an equation that involves one or more derivatives of an unknown function.

- A solution of a differential equation is a specific function that satisfies the equation.

Eqs:

(1) $x' - x = e^t$

i.e. $\dfrac{dx}{dt} - x = e^t$; $x(t) = te^t + c.e^t$

$$\frac{dx}{dt} = e^t + te^t + ce^t$$

$$\frac{dx}{dt} - x = e^t$$

(2) $\dfrac{d^2x(t)}{dt^2} + 9x(t) = 0$; $x(t) = c_1 \sin 3t + c_2 \cos 3t$.

$x' + \dfrac{1}{2x} = 0 \Rightarrow x(t) = \sqrt{c - t}$.

In these three examples, the letter c denotes an arbitrary constant. The fact that such consts appear in the sol^n is an indication that a differential equation does not, in general, determine a unique solution function. Therefore, some auxiliary conditions are attached!!

like $\quad x' = f(x,t)$

$x(a)$ is given

$\therefore\ x' - x = e^t, \quad x(0) = 1 \Rightarrow$

$$x(t) = te^t + ce^t$$

$\therefore\ x(0) = 0 + c$

$\therefore\ c = 1$

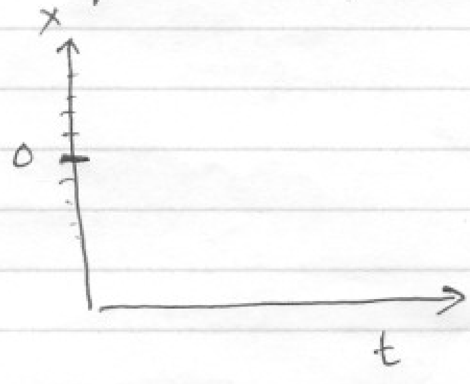
initial value problem

We are concerned, in this chapter, for a first order differential equation:

$$\begin{cases} x' = f(x,t) \\ x(a) \text{ is given} \end{cases}$$

It is understood that x is a function of t here.

- Closed form solution of IVP is complicated
- Want a numerical solⁿ
- Computer code for solving an ODE produces a sequence of lattice points (t_i, x_i), $i = 0, 1, \ldots$



Q. Suppose you have obtained these (t_i, x_i). Now you want to obtain an approximation value of $x(t)$ for some t which is within a given interval but not equal to any t_i, what can you do?

Question How do you know that the differential equation $x' = x^2 - t$ together with an initial value $x(t_0) = x_0$ has a unique sol^n?

There are many theorems on this subject. The easiest one is

Theorem If f & $\dfrac{\partial f}{\partial y}$ are continuous in the rectangle defined by $|t - t_0| < \alpha$ & $|x - x_0| \leq \beta$,

the IVP $x' = f(t, x)$, $x(t_0) = x_0$ has a unique continuous solution in some interval $|t - t_0| < \epsilon$.

Taylor series methods (Euler's method)

~~Our solution function x is~~

Problem Determine $x(t)$ at $a = t_0, t_1, \cdots, t_n = b$

where $t_{i+1} - t_i = h = \frac{b-a}{n}$ is the size step.

Suppose we have already obtained x_i, an approximation of $x(t_i)$. We would like to get x_{i+1}, an approximation of $x(t_{i+1})$.

$$\therefore \quad x(t_{i+1}) \approx x(t_i) + h \cdot x'(t_i)$$

$$= x(t_i) + h\, f(t_i, x(t_i))$$

Leads to Euler's method

$$x_{i+1} = x_i + h \cdot f(t_i, x_i).$$

Errors for Euler's method

Taylor polynomial gives

$$x(t_{i+1}) = x(t_i) + h\,\underbrace{f(t_i, x(t_i))}_{x'(t_i)} + \frac{h^2}{2!}\, x''(\xi_{i+1})$$

where $\xi_{i+1} \in [t_i, t_{i+1}]$.

Euler's method gives

$$x_{i+1} = x_i + h.f(t_i, x_i)$$

$$\therefore \underbrace{x(t_{i+1}) - x_{i+1}}_{\text{error at } t_{i+1}} = x(t_i) - x_i + h\left[f(t_i, x(t_i)) - f(t_i, x_i)\right] + \frac{1}{2}h^2 x''(\xi_{i+1})$$

The error at t_{i+1} arises from two sources:

0: The local truncation error: $\frac{1}{2}h^2 x''(\xi_{i+1})$.

1. Propagation error: $x(t_i) - x_i + h\left[f(t_i, x(t_i)) - f(t_i, x_i)\right]$

This is due to the accumulated effects of all local truncation errors at $t_1, t_2, \cdots, t_i$.

Plus Round-off errors

Trapezoid Euler method $\left.\begin{array}{l} x'(t) = f(t, x) \\ x(a) = 0 \end{array}\right\}$

$$\begin{cases} \hat{x}_{i+1} = x_i + h f(t_i, x_i) \\ x_{i+1} = x_i + \frac{1}{2} h \left[f(t_i, x_i) + f(t_{i+1}, \hat{x}_{i+1}) \right] \\ t_i = a + ih. \end{cases}$$

In the literature, this is also called the improved Euler's method.

Mid-point Euler method

$$\begin{cases} x_{i+1/2} = x_i + \frac{1}{2} h f(t_i, x_i) \\ x_{i+1} = x_i + \frac{1}{2} h f(t_i + 1/2 h, x_{i+1/2}) \\ t_i = a + ih. \end{cases}$$

General Taylor Series method

Taylor series expansion gives

$$x(t_{i+1}) \simeq x(t_i) + h x'(t_i) + \frac{h^2}{2!} x''(t_i) + \cdots + \frac{h^n}{n!} x^{(n)}(t_i)$$

From $x' = f(t,x)$, we can compute x'', x''', $\ldots$ $x^{(n)}$. Define x_i', x_i'', $\ldots$, $x_i^{(n)}$ as approximations to $x'(t_i)$, $x''(t_i)$, $\ldots$, $x^{(n)}(t_i)$ respectively. Then

$$x_{i+1} = x_i + h x_i' + \frac{h^2}{2!} x_i'' + \cdots + \frac{h^n}{n!} x_i^{(n)},$$

$$i = 0, 1, \ldots$$

x_0 is known from the initial conditions

Remarks

- Euler's method is a Taylor series method of order 1.
- If $f(t,x)$ is complicated, then high order Taylor series method may be very complicated

Runge - Kutta methods of order 2.

$$x_{i+1} = x_i + \left(1 - \frac{1}{2\alpha}\right) K_1 + \frac{1}{2\alpha} K_2$$

where

$$K_1 = h f(t_i, x_i)$$

$$K_2 = h f(t_i + \alpha h, x_i + \alpha K_1)$$

$$\alpha \neq 0$$

When $\alpha = 1$, we obtain the trapezoid Euler method, and when $\alpha = 1/2$, we obtain the midpoint Euler method.

Ruge-Kutta methods of order 4.

The Classical fourth order Ruge-kutta method

$$x_{i+1} = x_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Where

$$k_1 = h f(t_i, x_i)$$

$$k_2 = h f(t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}k_1)$$

$$k_3 = h f(t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}k_2)$$

$$k_4 = h f(t_i + h, x_i + k_3)$$

This method is in common use ~~in~~ for solving Initial Value Problems.