Assignment #4

Macm 316
#4 homework sol^n.

$\boxed{2.1-14}$

$$|p_n - p| \leq \frac{4-1}{2^n} \leq 10^{-3} \Rightarrow \frac{3}{2^n} \leq 10^{-3} \Rightarrow 2^n \geq 3000$$

$$\Rightarrow n \geq \frac{\log 3000}{\log 2} \simeq 11.55$$

$$\Rightarrow \boxed{n \geq 12} \quad \boxed{p_{12} = 1.3787}$$

$\boxed{2.1-16}$

- for $n > 1$ $\quad |f(p_n)| = |(1 + \frac{1}{n} - 1)^{10}| = (\frac{1}{n})^{10} \leq (\frac{1}{2})^{10} < 10^{-3}$

$$\Rightarrow |f(p_n)| < 10^{-3}$$

- $|p - p_n| = \frac{1}{n} < 10^{-3} \Longleftrightarrow \boxed{n > 1000}$

$\boxed{2.1-18}$

$$\begin{matrix} -1 < a < 0 \\ 2 < b < 3 \end{matrix} \Rightarrow 1 < a+b < 3 \quad \text{OR} \quad \frac{1}{2} < \frac{a+b}{2} < \frac{3}{2}$$

$f(x) < 0$ for $-1 < x < 0$ and $1 < x < 2$

$f(x) > 0$ for $0 < x < 1$ and $2 < x < 3$

$a_1 = a$, $f(a_1) < 0$ $\quad b_1 = b$ and $f(b_1) > 0$

(a) Since $a+b < 2$, we have $p_1 = \frac{a+b}{2}$ and $\frac{1}{2} < p_1 < 1$. [redacted]

$f(p_1) > 0 \Rightarrow \left.\begin{matrix} a_2 = a_1 = a \\ b_2 = p_1 \end{matrix}\right\}$ The only zero of f in $[a_2, b_2]$ is $p = 0$, so convergence will be to 0

(b) Since $a+b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < \frac{3}{2}$.

$f(p_1) < 0 \Rightarrow \left.\begin{matrix} b_2 = b_1 = b \\ a_2 = p \end{matrix}\right\}$ The only zero of f in $[a_2, b_2]$ is $p = 2$, so convergence will be to 2.

(c) Since $a+b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. thus, a zero of f has been found on the first iteration.

2.2-4

~~a)~~ a and b diverge

c converges faster than d.

to check it, write a small piece of code.

3

$\boxed{2.4\text{-}6}$

a) $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

we have linear convergence.

To have $|p_n - p| < 5\times 10^{-2}$ $\quad |1/n| < 0.05 \Rightarrow \boxed{n \geq 20}$

b) $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)^2}{\left(\frac{1}{n}\right)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

we have linear convergence.

$$|p_n - p| < 5\times10^{-2} \Rightarrow \underline{\left(\frac{1}{n}\right)^2 < 5\times 10^{-2}} \Rightarrow n^2 > 20 \Rightarrow \boxed{n \geq 5}$$

$\boxed{2.4\text{-}8}$

a) $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \lim_{n\to\infty} \frac{|10^{-2^{n+1}} - 0|}{|10^{-2^n} - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{\left(10^{-2^n}\right)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

b) $$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}}$$

$$= \lim_{n\to\infty} 10^{2n^k - (n+1)^k}$$

$$= \lim_{n\to\infty} 10^{n^k\left(2 - \left(\frac{n+1}{n}\right)^k\right)} = \infty$$

$p_n = 10^{-n^k}$ does not converge quadratically.

$\boxed{2.4\_12}$

If f has a zero of multiplicity m at p, then f can be written as

$$f(x) = (x-p)^m q(x)$$

for $x \neq p$, where $\lim_{x \to p} q(x) \neq 0$

Thus

- $f^{(1)}(x) = m(x-p)^{m-1} q(x) + (x-p)^m q'(x)$ and $f^{(1)}(p) = 0$
- $f^{(2)}(x) = m(m-1)(x-p)^{m-2} q(x) + 2m(x-p)^{m-1} q'(x) + (x-p)^m q''(x)$ and $f^{(2)}(p) = 0$

In general for $k \leq m$

$$f^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j (x-p)^m}{dx^j} q^{(k-j)}(x)$$

$$= \sum_{j=0}^{k} \binom{k}{j} m(m-1)\cdots(m-j+1)(x-p)^{m-j} q^{(k-j)}(x).$$

For $0 \leq k \leq m-1$ we have $f^{(k)}(p) = 0$ but

$f^{(m)}(p) = m! \lim_{x \to p} q(x) \neq 0$

Conversely, suppose that $f(p) = f^{(1)}(p) \cdots = f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$.

Consider the $(m-1)$th Taylor polynomial of f expanded about p:

$$f(x) = f(p) + f'(p)(x-p) + \cdots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!}$$

$$= (x-p)^m \frac{f^{(m)}(\xi(x))}{m!}$$

$\boxed{\text{2.4-12 contd.}}$ 5/11

where $\xi(x)$ is between x and p. Since $f^{(n)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}$$

then $f(x) = (x-p)^m q(x)$ and $\lim_{x \to p} q(x) \neq \emptyset$ □

$\boxed{2.}$

p is a root of $f(x)=0$

f, f', f'' are continuous

$f'(x) \neq 0$

if $\{x_n\}_{n=1}^{\infty} \to p \Rightarrow \lim_{n\to\infty} \frac{|f(x_{n+1})|}{|f(x_n)|^2} = c$ (3)

* $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$ (1)

* $f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n) + \frac{(x_{n+1}-x_n)^2}{2} * f''(\xi(x_{n+1}))$ (2)

by replacing (1) into (2)

$$f(x_{n+1}) = \cancel{f(x_n) + \left(-\frac{f(x_n)}{f'(\cancel{x_n})} * f'(x_n)\right)} + \frac{f(x_n)^2}{2 \cdot f'(x_n)^2} f''(\xi(x_{n+1}))$$

$$\Rightarrow f(x_{n+1}) = \frac{f(x_n)^2}{2 f'(x_n)^2} f''(\xi(x_{n+1}))$$

$$\Rightarrow \lim_{n\to\infty} \frac{|f(x_{n+1})|}{|f(x_n)|^2} = \frac{\cancel{f(x_n)^2} |f''(\xi(x_{n+1}))|}{2 f'(x_n)^2 \cancel{f(x_n)^2}} = \frac{|f''(\xi(x_{n+1}))|}{2 f'(x_n)^2}$$

using 3 and $x_n < \xi(x_{n+1}) < x_{n+1}$ we have $\frac{|f''(p)|}{2 f'(p)} = c$