§6.4

8) The system $A\bar{x} = \bar{b}$ has infinite sol^{n} when A is singular i.e. when $\det A = 0$. If we apply Gaussian we get

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 2 \\ -2 & \alpha & 3 \end{bmatrix} \xrightarrow[\text{stage}]{\text{1st}} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & \alpha-1 & 6 \end{bmatrix} \xrightarrow[\text{stage}]{\text{2nd}} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 0 & 6+4\frac{(\alpha-1)}{4} \end{bmatrix}$$

$\therefore$ The matrix is singular if $6 + \frac{4(\alpha-1)}{4} = 0$ i.e. $\alpha = -5$.

1/4

−5

(10) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $\det A = a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{21}a_{12})$

$\det \tilde{A} = a_{31} \begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} - a_{32} \begin{vmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{vmatrix} + a_{33} \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$

$= -a_{31}(a_{12}a_{23} - a_{22}a_{13}) + a_{32}(a_{11}a_{23} - a_{21}a_{13}) - a_{33}(a_{11}a_{22} - a_{21}a_{12})$

$= -\det A$

The other two cases are similar.

§ 6.5

10 a) $O(n^3)$ for multiplications/divisions

$O(n^3)$ for additions/subtractions

b) Each time we perform a simple row interchange on P to P^*, then $\det P^* = -\det P$. So if P^* is obtained by k interchanges we have $\det P^* = (-1)^k \det P = (-1)^k$

§ 6.6

6 b) $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \sqrt{5}/5 & \sqrt{95}/5 \end{bmatrix}$

d) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{11}}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{11}}{22} & \frac{\sqrt{209}}{11} & 0 \\ \frac{1}{2} & -\frac{5\sqrt{11}}{22} & \frac{7\sqrt{209}}{209} & \frac{2\sqrt{266}}{19} \end{bmatrix}$

16. $\alpha > \frac{8}{7}$ (Check all three leading principal submatrices have determinant > 0)

20. a) Yes (b) not necessarily. Consider $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

c) not necessarily; consider $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

d) not necessarily; consider $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

e) not necessarily; consider $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

22. a) $\alpha = 2$; (b) A can not be strictly diagonally dominant for any α.

c) Symmetric for any α. d) A is positive definite for $\alpha > 2$ (why?)

---

32) a)

$$D^{1/2} D^{1/2} = \begin{bmatrix} \sqrt{d_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} \sqrt{d_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_{nn}} \end{bmatrix} = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix}$$

$$= D$$

b)

$$(\hat{L} D^{1/2})(\hat{L} D^{1/2})^t = \hat{L} D^{1/2} D^{1/2^t} L^t$$

$$= \hat{L} D^{1/2} D^{1/2} L^t$$

$$= \hat{L} D L^t$$

$$= A$$

But since $L L^t = A \Rightarrow L = \hat{L} D^{1/2}$

§7.1

8) $\|A\|_{①} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$

(i),(ii) Since $\|A\|_{①} \geq 0$ and $\|A\|_{①} = 0$ iff all entries of $A = 0$

$\Rightarrow$ properties (i) & (ii) hold (in defn 7.1)

(iii) $\|\alpha A\|_{①} = \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha a_{ij}| = |\alpha| \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| = |\alpha| \, \|A\|_{①}$

(iv) $\|A+B\|_{①} = \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij} + b_{ij}|) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|)$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|$$

$$= \|A\|_{①} + \|B\|_{①}$$

(v) $$\|AB\|_{①} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{ik} b_{kj}\right|$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}||b_{kj}|$$

$$= \sum_{i=1}^{n}\left[\sum_{k=1}^{n} |a_{ik}| \sum_{j=1}^{n} |b_{kj}|\right]$$

$$\leq \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}| \left(\sum_{k=1}^{n}\sum_{j=1}^{n} |b_{kj}|\right)$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}| \; \|B\|_{①}$$

$$= \|A\|_{①} \|B\|_{①}$$

The $\|\cdot\|_{①}$ norms for metrices in Q4 are

a) 26 b) 26 c) 10 and d) = 28

10. $$\|Ax\|_2^2 = \sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij} x_j\right|^2 \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n} |a_{ij}||x_j|\right)^2$$

$$\leq \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}\left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}\right]^2$$

(by Cauchy - Buniakowsky - Schwarz inequality)

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)\|x\|_2^2$$

$$= \|A\|_F^2 \|x\|_2^2$$

Q2. For $i = 1$ to $\frac{n-1}{2}$ (1 to 3)

do forward substitution

$$\left(\begin{aligned} b_1 &= \frac{b_1}{d_1} \\ b_i &= \frac{b_i - a_{i-1} b_{i-1}}{d_i} \end{aligned} \right)$$

For $i = n$ to $\frac{n+3}{2}$ (4 to 7)

do backward substitution

$$\left(\begin{aligned} b_n &= \frac{b_n}{d_n} \\ b_{i-1} &= \frac{b_{i-1} - a_{i-1} b_i}{d_{i-1}} \end{aligned} \right)$$

For $i = \frac{n+1}{2}$ ($i = 4$)

do $$b_i = \frac{b_i - b_{i-1}a_{i-1} - b_{i+1}a_{i+1}}{d_i}$$ (solve for b_i)

3, For $i = 2$ to n

do (Gaussian eliminations)

$$d_i = d_i - \frac{c_{i-1} a_{i-1}}{d_{i-1}}$$

$$b_i = b_i - \frac{b_{i-1} a_{i-1}}{d_{i-1}}$$

For $i = n$ to 1

do (B substitution)

$$b_n = \frac{b_n}{d_n}$$

$$b_{i-1} = \frac{b_{i-1} - c_{i-1} b_i}{d_{i-1}}$$

The running time should be of order $O(n)$