MACM 101 (Discrete Mathematics I)

Quiz 4: November 18, 2011

Answer all questions.

1. (10 points) Let $A = \{6{:}00, 6{:}30, 7{:}00, \ldots, 9{:}30, 10{:}00\}$ denote the set of 9 half-hour periods in the evening. Let $B = \{3, 12, 15, 17\}$ denote the set of four local television channels. Let R_1 and R_2 be two relations from A to B. Explain the meaning of the relations $R_1, R_2, R_1 \cup R_2, R_1 - R_2$.

$R_1 = \{(6{:}00, 12), (6{:}00, 17), (9{:}30, 15)\}$

$R_2 = \{(a, 12), (a, 15), \forall a \in A\}$

$R_1 \cup R_2 = R_2 \cup \{(6{:}00, 17)\}$

$R_1 - R_2 = \{(6{:}00, 17)\}$

2. (10 points)Let I be the set of integers from 1 to 7.

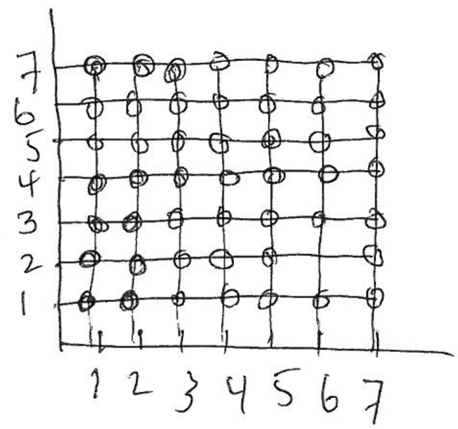
(a) Is there a natural way to interpret the ordered pairs in $I \times I$ as points in the plane?

(b) What are the elements of the relation $R' = \{(x, y) | x \leq y\}$? Give the boolean matrix representation of R'.

(c) Let R be a binary relation on $I \times I$, i.e $R \subseteq (I \times I) \times (I \times I)$.

- Write an element of $R \subseteq (I \times I) \times (I \times I)$.
- What is the cardinality of the set $(I \times I) \times (I \times I)$?

(a)



Marked gridpoints are element of

$R = \{(i, j) \mid 1 \leq i \leq 7,\ 1 \leq j \leq 7$, i & j are integers$\}$

(b) Boolean matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) First Part : $R = \{((1,2),(1,3)), ((1,4),(7,2)), ((2,6),(6,2))\}$

Second Part : $|I| = 7 \rightarrow |I \times I| = 49$

$|(I \times I)| = 49 \rightarrow |(I \times I) \times (I \times I)| = 49 \times 49$

3. (10 points) Suppose A and B are two sets containing n and 2 elements respectively. In this question we will consider the functions: $f : A \to B$ and $g : B \to A$.

(a) How many different functions f are there from A to B?

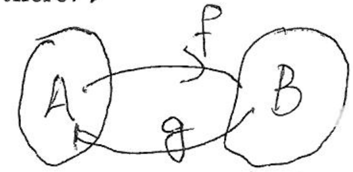
(b) Describe the functions from $A \to B$ that are not onto.

(c) How many onto functions $f : A \to B$ are there?

(d) How many different functions g are there from B to A?

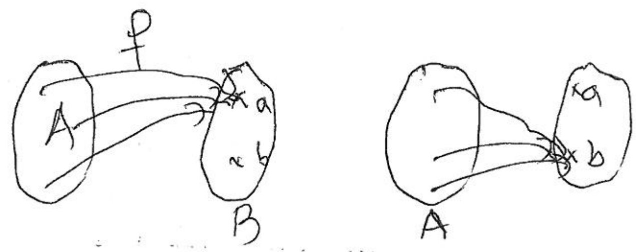
(e) How many one-to-one functions $g : B \to A$ are there?

a) $\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$



$|A| = n$ $|B| = 2$

b)



are the only two non-onto function $f: A \to B$.

(c) Total # of functions $f:A\to B = 2^n$

Total # of non-onto functions $f: A \to B = 2$

$\therefore$ total # of onto functions $f: A \to B = 2^n - 2$

(d) $\underbrace{n \times n}_{2 \text{ times}} = n^2$

(e) $n(n-1) = P(n,2)$

4. (10 points) Consider two functions $f, g : \mathbf{Z} \times \mathbf{Z}$ where $f(x) = 2x$ and $g(x) = \lfloor \frac{x}{2} \rfloor$. Determine whether these functions are one-to-one, onto functions. Determine the ranges of the functions. Is g the inverse of f?

- f is one-to-one but not onto.
  - one-to-one: Let if possible f is not one-to-one
    $\therefore \exists\ x_1, x_2, x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$ i.e. $2x_1 = 2x_2$
    i.e. $x_1 = x_2$ $\therefore f(x_1) = f(x_2) \to x_1 = x_2$, a contradic
  - not onto: $\exists$ any x s.t. $f(x) = 3$.
- g is not one-to-one but is onto
  - not one-to-one: $g(2) = g(3)$
  - ~~onto~~
  - is onto: For any y, $g(2y) = y$. Therefore y is the image of $2y$.
- $g \circ f(x) = x$ but $f \circ g(x) \neq x$ $\therefore g$ is not the ~~...~~ inverse of f.

5. (10 points) Show that the function $f : R \to R$ where $f(x) = x^3$ is invertible. Determine the inverse function $f^{-1} : R \to R$.

- f is one-to-one (Proof is ~~...~~ similar as in 4)
- f is onto " " " " " " "
- $f^{-1} : R \to R$

  Let $f(x) = x^3$ $\therefore f^{-1}(x_3) = x$

  Let $y = x^3$ $\therefore x = y^{1/3}$

  $\therefore f^{-1}(y) = y^{1/3}$ is the inverse function of f.