MACM 101

Discrete Mathematics I
Quiz 1
Date September 23, 2011
Solution

Answer all four questions.

1. (10 points) Consider selecting 3 objects from the set $A = \{a, b, c, d, e, f\}$. Evaluate the answers to each of the following questions.

   (a) How many ordered sequences can be chosen from A without repetition?

   (ordered; no repetition) $P(6, 3)$

   (b) How many ordered sequences with repetition can be chosen from A?

   (ordered; repetition) 6^3

   (c) How many unordered sequences without repetition can be chosen from A?

   (not ordered; no repetition) $C(6, 3)$

   (d) How many unordered sequences with repetition can be chosen from A?

   (not ordered; repetition) $C(3 + 6 - 1, 6 - 1) = C(8, 5)$

2. (10 points)

   (a) In how many ways can the letters in UNUSUAL be arranged?

   (ordered; no repetition; non-distinct object) $\frac{P(7,7)}{3!}$

   (b) For the arrangements in part (a), how many have all three U's together?

   (ordered; no repetition; distinct object) $P(5, 5)$

   (c) How many of the arrangements in part (a) have no consecutive U's.

   The number of arragements with 4 letters (ignoring U's) is $4!$**. Given any arrangement, xxxx, there are five places –x–x–x–x– where three U's can go. Total arrangements are** $C(5, 3)4!$

3. (10 points) Five rooms of a house are to be painted in such a way that rooms with an interconnecting door have different colors. If there are n colors available, how many different color schemes are possible when the rooms in the house are arranged in the following way?

   (a) Connected rooms form a linear order with one door interconnecting two adjacent rooms.

   The first room can be colored in n different ways, the second room can be colred in n-1 dfferent ways, the third room can be colored in n-1 different ways and the last room can be colored in n-1 different ways. Total number of ways is $n(n-1)^4$**.**

   (b) Connected rooms form a linear order with one door interconnecting two adjacent rooms. The first and last rooms must be colored differently.

   If we are not careful, we probably would answer $n(n-1)(n-1)(n-1)(n-2)$**. The number of colors available for the last room is not always** $n-2$**. If room 4 is painted with the same color as that of room 1, room 5 can be colored in n-1 ways. Now the question is how many ways can you color the first four rooms such that first room and the last room are colored the same. If the first and the fourth room are colored the same, observe that the second and the third rooms cannot be colored with the color of the first room. Therefore, the number of ways to color the first four rooms such that the first and the last rooms have the same color is** $A = n.(n-1)(n-2)$**. Therefore, the number of ways to color the first four rooms such that the first and the last rooms have different colors** $B = n(n-1)(n-1)(n-1) - n(n-1)(n-2)$**. Thus, the answer to question (b) is** $C = A.(n-1) + B.(n-2)$**.**

   (c) Connected rooms form a circular order with one door interconnecting two adjacent rooms.

   Each valid circular arrangement will appear five times in the arrangements valid for question (b). Therefore, the

total number is $\frac{C}{5}$.

4. (15 points) **(i)** Find the number of integral solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20$ subject to the conditions

   (a) $x_i \geq 1, i = 1, 2, 3, 4, 5, 6$

   This is same as finding the number of integral solutions to $x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 14$ where $x'_i \geq 0, i = 1, 2, 3, 4, 5, 6$.

   (b) $x_1 \geq 5$, $x_i \geq 1, i = 2, 3, 4, 5, 6$

   This is same as finding the number of integral solutions to $x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 10$ where $x'_i \geq 0, i = 1, 2, 3, 4, 5, 6$.

   (c) $0 \leq x_1 \leq 4$, $x_i \geq 1, i = 2, 3, 4, 5, 6$

   This is obtained by eliminating the illegal solutions from part (a) solutions. The number of such illegal solutions is given by part (b).

   (ii) Describe the questions 1(c) and 1(d) using the formulation-type in part (i) of this question.

   1(c) This is equal to the number of integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$ where $0 \leq x_i \leq 1, i = 1, 2, 3, 4, 5, 6$.
   1(d) This is equal to the number of integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$ where $x_i \geq 0, i = 1, 2, 3, 4, 5, 6$