MACM 101

Discrete Mathematics I

Date: October 28, 2011

Problem Set 8 (Sections 2.5 and 4.1 of the text)

(Not to be handed in)

Practice problems for the in-class quiz 3 (November 2, 2011)

1. Problems (Exercise 2.5) $6, 12, 15$

2. Prove by contradiction the following

   (a) For all positive integer n, n^2 is even implies n is even.

   (b) There doesn't exist a largest integer.

3. Problems (Exercise 4.1) $19, 23, 24$

4. A cashier is to disburse payments to vendors, where the payment amounts would be positive integers greater than equal to 8. The cashier has at her disposal only notes of denomination \$3 and \$5, but in abundance. Show that the cashier can make any payment that is a positive integer greater than equal to 8 using the notes of the said denomination.

5. Show that for any positive integer n, $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

6. Let S be a set of positive integers such that:

   (a) Positive integer n_0 is in S.

   (b) If integers $n_0, n_0 + 1, n_0 + 2, ..., k$, the natural number $k + 1$ is also in S.

   Show that S is the set of all integers greater than or equal to n_0.

7. Mr. X claims that he is one-third Indian. When asked how this is posiible, his answer was, "My father was a one-third Indian and my mother was a one-third Indian." Is this a correct proof by induction?

8. There are N students in the class. The teacher comes in with N hats some of which are white and the rest are black. The students do not know how many white or black hats are there. The teacher asks the students to close their eyes and then places a hat on the head of each student. The students can now look at others but cannot talk to each other. A student is also not allowed to take off her own hat and see its colour. The teacher now goes away and comes back after every 5 minutes. Assume that 5 minutes are enough to look around to be sure of the colour of the hat worn by others. The teacher now demands that on his i^{th} arrival $(1 \leq i \leq N)$, if there are only i students wearing white hats, they should be able to come up and report. Can the students meet this demand of the teacher? Use induction to prove this result.

   (Hint: Inductive hypothesis: Suppose there are k students wearing white hats and they have figured out that they were wearing white hats and informed the teacher on the kth interval. Now prove the case that $k + 1$ students are wearing white hats and they figured this out on the (k+1)th interval.)

9. A jigsaw puzzle consists of a number of pieces. Two or more pieces with matched boundaries can be put together to form a "big" piece. To be more precise, we use the term block to refer to either a single piece or a number of pieces with matched boundaries that are put together to form a "big" piece. Thus, we can simply say that blocks with matched boundaries can be put together to form another block. Finally, when all pieces are put together as one single block, the jigsaw puzzle is solved. Putting 2 blocks together with matched boundaries is called one move. Prove (using strong induction) that for a jigsaw puzzle of n pieces always takes n-1 moves to solve.

10. Here are some formulas for sequences that are great for practicing proofs by induction.

    - $1 + 2 + 3 + ... + n = n(n + 1)/2$
    - $1 + 3 + 5 + ... + (2n - 1) = n^2$
    - $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$
    - $1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$
    - $1^3 + 2^3 + 3^3 + ... + n^3 = [n(n + 1)/2]^2$

- $1 * 2 + 2 * 3 + ... + n(n+1) = n(n+1)(n+2)/3$
- $1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$