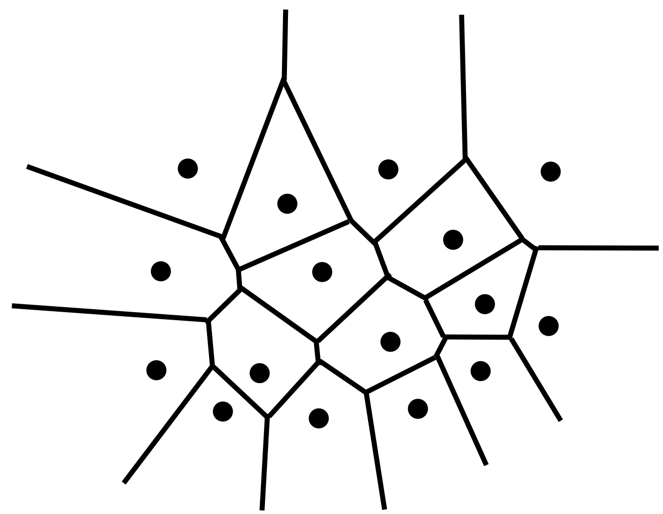
Voronoi Diagrams in the Plane

Chapter 5 of O'Rourke text

Chapter 7 and 9 of course text

Voronoi Diagrams



- As important as convex hulls
- Captures the neighborhood (proximity) information of geometric objects
- Concept has been known since 1850 (Dirichlet)
- First published in 1908 by Voronoi

Voronoi Diagrams Properties

- Each Voronoi region $V(p_i)$ is convex. Could be bounded or unbounded.
- Edges of V(P) are called Voronoi edges and the vertices are called Voronoi vertices
- A point in the interior of a Voronoi edge has two nearest sites
- A Voronoi vertex has at least three nearest neighbors

Voronoi Diagrams Applications

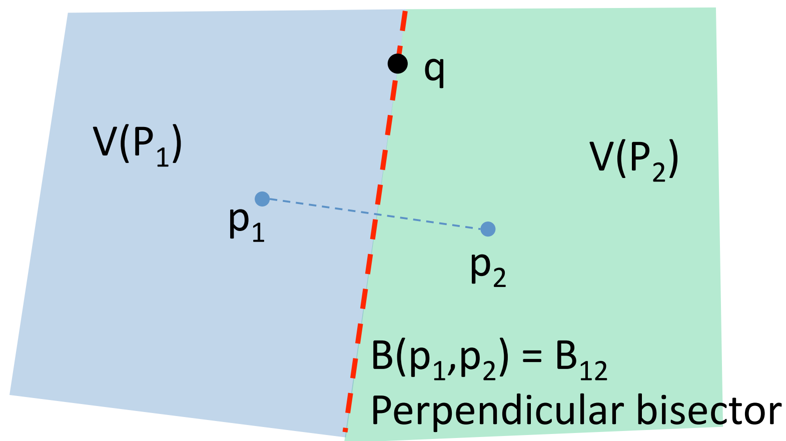
- **Geometric Modeling:** finding good triangulation of a 3D surface
- **Marketing:** Where could I place a Burger King outlet in a market dominated by McDonalds
- **Meteorology:** Estimate regional rainfall averages given data at discrete rain gauges (Thissen polygons)
- **Pattern Recognition:** Find simple descriptors of shapes that extract 1D characterizations from 2D shapes
- **Robotics:** Path planning in the presence of obstacles

Voronoi Diagrams Applications (continued)

- **Statistics and Data Analysis:** Analyze statistical clustering (“Natural Neighbours”) interpolation
- **Zoology:** Model and analyze the territories of animals
- **Geometric Problems:**
  - **Post Office:** Given a set of locations for post offices how do you determine the closest post office to a given house?
  - **Closest Pair:** Find the two closest points of a given set
  - **Euclidean Minimum Spanning Tree**
  - **Largest Empty Circle:** Toxic waste dump problem

Voronoi Diagrams

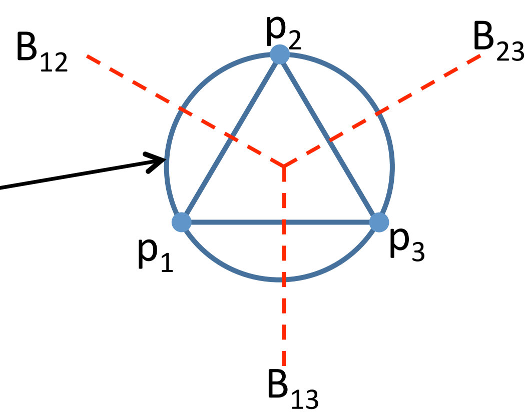
- **Two Sites:**



$q \in V(p_1)$ and $q \in V(p_2)$

- **Three Sites:**

Circumcircle of P_1, P_2, p_3



Voronoi Diagrams Definition and Basic Properties

- P = { p_1, p_2, …, p_n } a set of n points in the Euclidean plane
- | p_i – p_j | denotes the Euclidean distance between p_i and p_j

$$|p_i - p_j| = \sqrt{\left(p_i(x) - p_j(x)\right)^2 + \left(p_i(y) - p_j(y)\right)^2}$$

$$= d(p_i, p_j)$$

- Define $V(p_i)$, the Voronoi region of p_i, to be the set of points x in the plane that are at least as close to pi as to any other site:

$$V(p_i) = \{ x : d(p_i, x) \leq d(p_j, x)\ \forall\ j \neq i \}$$

Half-planes

- We can define $V(p_i)$ in terms of the intersection of half-planes
- $H(p_i,p_j)$: closed half-plane with boundary B_{ij} and containing p_i

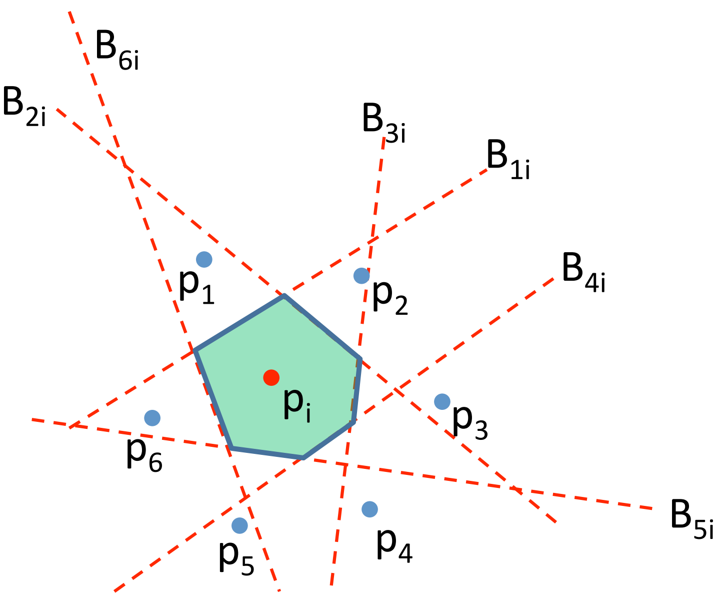
$$= \{ x \mid d(p_i, x) \leq d(p_j, x) \}$$

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j)$$

↑

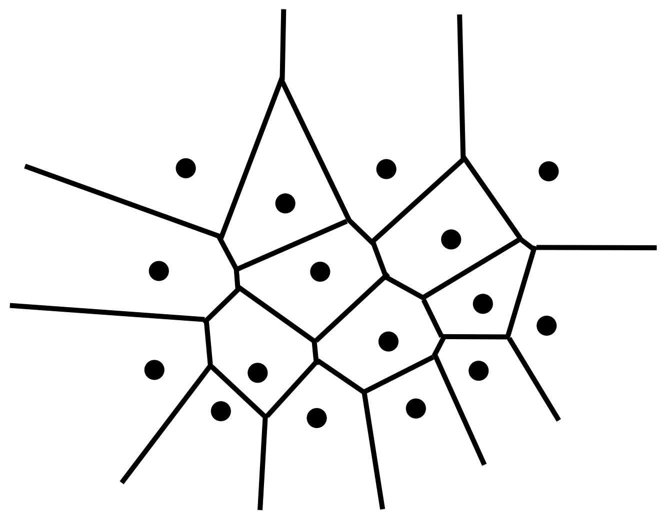
Intersection is to be taken over all j, $j \neq i$

Half-planes



$V(p_i) = H(p_i, p_1) \cap H(p_i, p_2) \cap H(p_i, p_3) \cap H(p_i, p_4) \cap H(p_i, p_5) \cap H(p_i, p_6)$

Voronoi Diagram



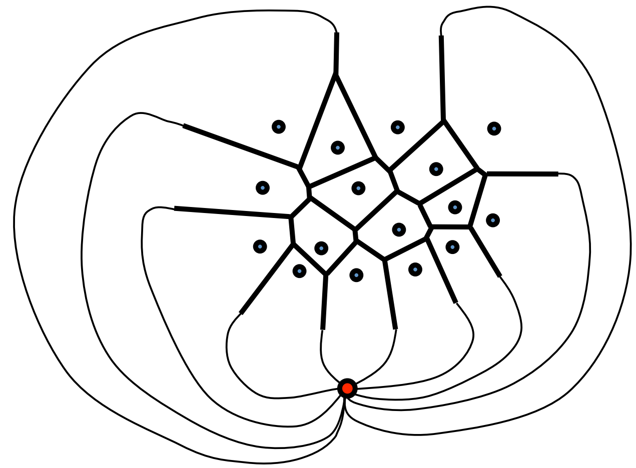
- The Voronoi Diagram of P, denoted by V(P) is the union of Voronoi cells of points of P.
- Bounding lines of half-planes (bisectors) are the building blocks of Voronoi diagrams

Size of Voronoi Diagram of n Points

n_f = number of faces
(**Voronoi regions**)

n_e = number of edges
(**Voronoi edges**)

n_v = number of vertices
(**Voronoi vertices**)



- **Euler's Formula (Planar Graph)**

$$\mathbf{n_v - n_e + n_f = 2}$$

- **Here:**
  - $n_f = n$
  - Each vertex has three edges, and each edge is shared by two vertices, except the boundary edges.
    - $3n_v \le 2n_e$

    $\Rightarrow 0 = \mathbf{n_v - n_e + n_f - 2 \le 2/3\, n_e - n_e + n_f - 2 \Rightarrow n_e \le 3n_f - 6}$

    $\mathbf{\Rightarrow n_v \le 2/3\, n_e \le 2n_f - 4.}$

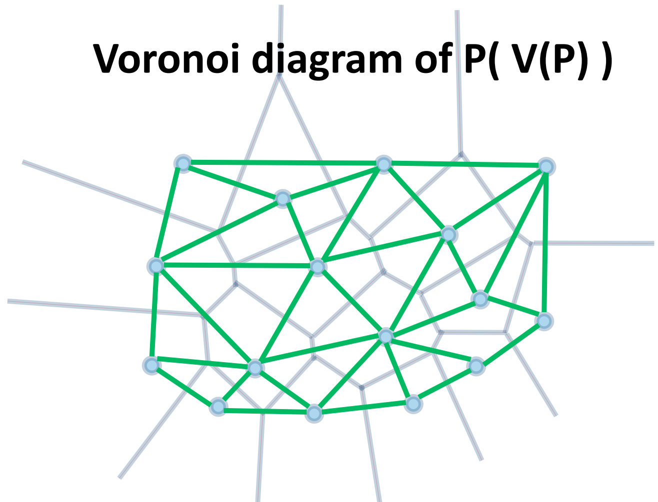
Delaunay Triangulation (DT)

- Assume no four points are co-circular. This implies that every Voronoi vertex is of degree three
- P = set of n points in the plane
- V(P) = Voronoi diagram of P
- G = dual graph of V(P)
  - The nodes of G are points (sites) of V(P)
  - Two nodes are connected by an edge if the corresponding Voronoi polygons share a Voronoi edge (share a positive length edge).

- Delaunay (1934) proved that when the dual graph is drawn with straight lines, it produces a planar triangulation of the Voronoi sites p_i
  - No four sites are co-circular

Delaunay Triangulation (DT)

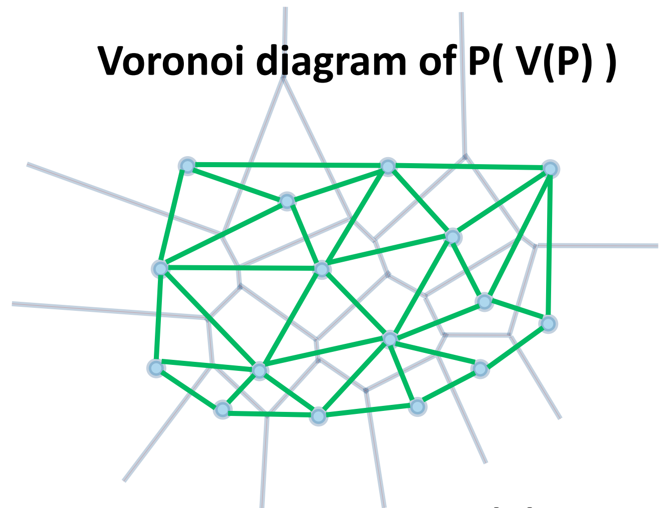
Voronoi diagram of P(V(P))



Delaunay Triangulation DT(P)
Planar Graph (edges do not intersect)

- **Properties:**
  - **D1:** DT(P) is a straight line dual of V(P)
  - **D2:** DT(P) is a triangulation if no four points are co-circular
  - **D3:** Each face of DT(P) corresponds to a Voronoi vertex of V(P)
  - **D4:** Each edge of DT(P) corresponds to an edge of V(P)

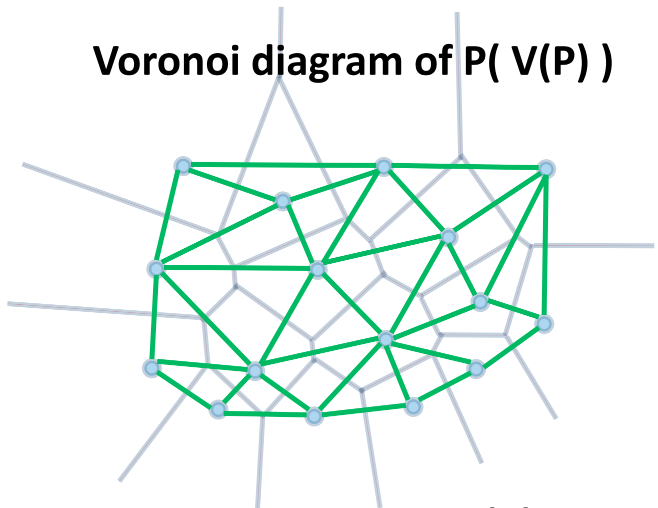
Delaunay Triangulation (DT)



Delaunay Triangulation DT(P)
Planar Graph (edges do not intersect)

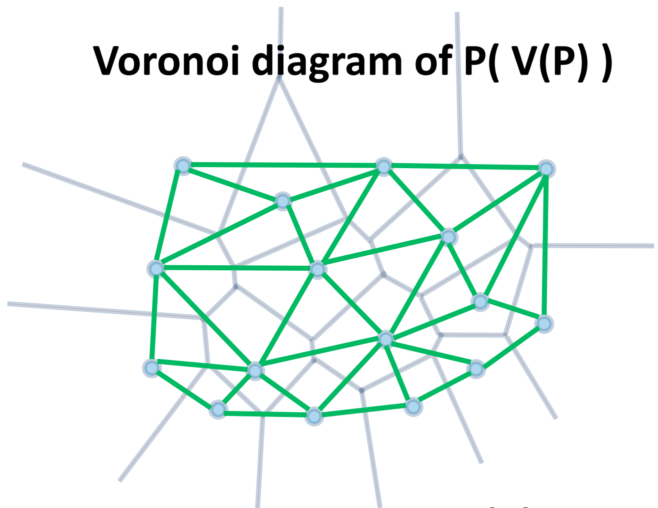
- **<u>Properties:</u>**
  - **D5:** Each node of DT(P) corresponds to a region of V(P)
  - **D6:** The boundary of DT(P) is the convex hull of its sites (points)
  - **D7:** The interior of each face of DT(P) contains no sites

Delaunay Triangulation (DT)



- **<u>Properties of Voronoi Diagrams:</u>**
  - **V1:** Each Voronoi region V(P) is convex
  - **V2:** $V(p_i)$ is unbounded if and only if p_i is on the convex hull of the point set
  - **V3:** If v is a Voronoi vertex at the junction of $V(p_1)$, $V(p_2)$, and $V(p_3)$, then v is the center of the circle C(v) determined by p_1, p_2, and p_3

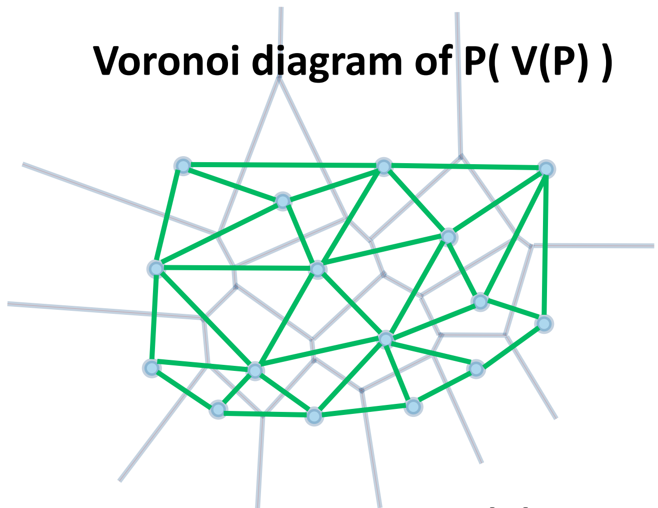
Delaunay Triangulation (DT)



Delaunay Triangulation DT(P)
Planar Graph (edges do not intersect)

- **<u>Properties of Voronoi Diagrams:</u>**
  - **V4:** C(v) is the circumcircle of the Delaunay triangle corresponding to v
  - **V5:** The interior of C(v) contains no sites
  - **V6:** If p_j is nearest neighbor to P_i, (p_i, p_j) is an edge of DT(P)

Delaunay Triangulation (DT)



- **<u>Properties of Voronoi Diagrams:</u>**
  - **V7:** If there is some circle through Pi and Pj that contains no other sites, then (pi, pj) is an edge of DT(P). The reverse is also true. For every Delaunay edge there is some empty circle