Trapezoidal Maps

Notes taken from CG lecture notes of Mount (pages 60-69)

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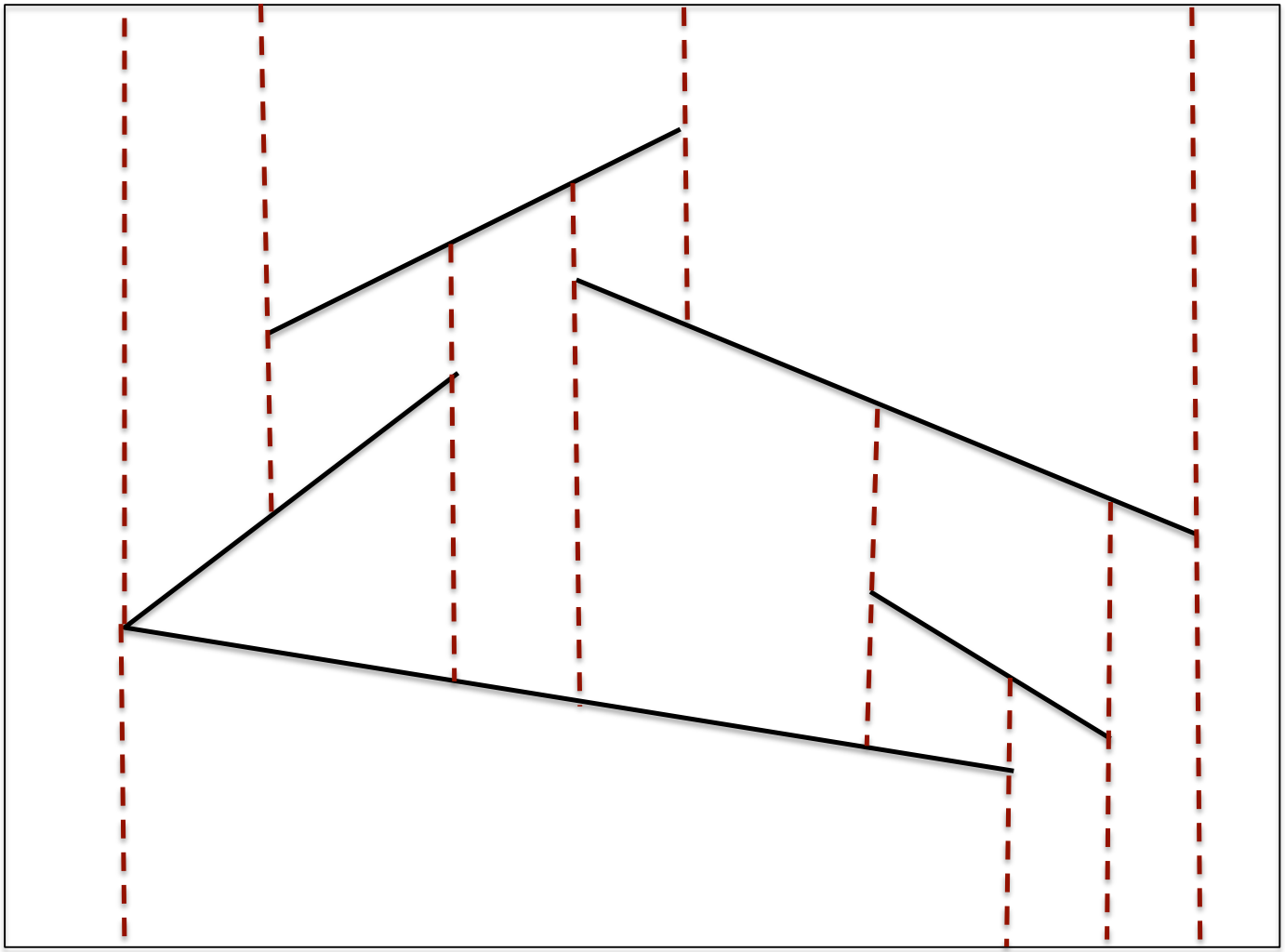
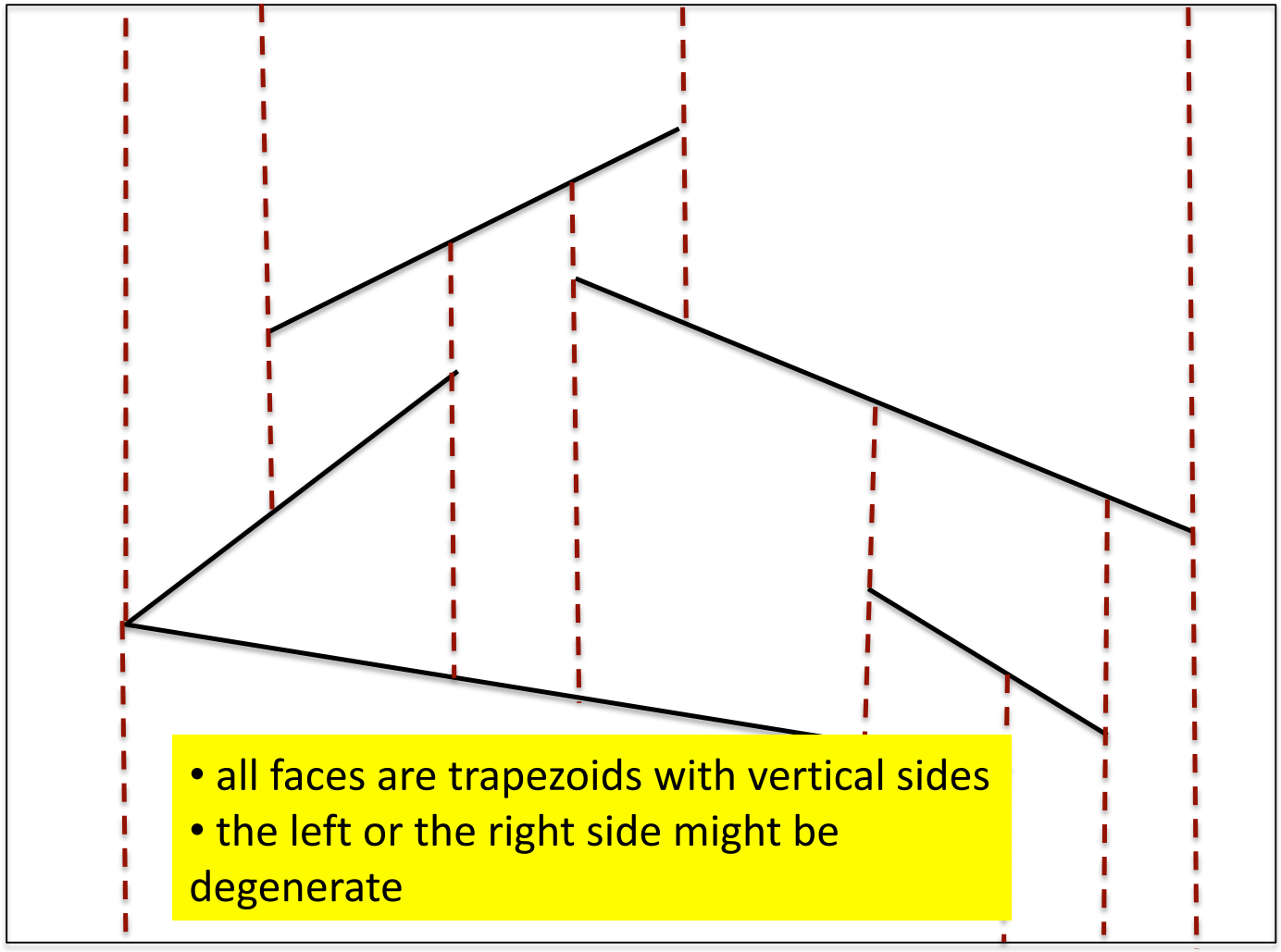
Trapezoidal Maps

- $S=\{s_1,s_2, ..., s_n\}$ is the set of line segments
  - segments don't intersect, but can touch
  - assuming that if the endpoints are not touching, the x-coordinates of their segments have distinct endpoints
- Trapezoidal map results when a bullet is shot upwards and downwards from each vertex until it hits another segment segment of S.
  - to avoid infinite bullet paths, we assume that S is contained within a large bounding box.




• all faces are trapezoids with vertical sides
• the left or the right side might be degenerate

Size of the map of n line segments

Claim: Given a polygonal subdivision with n segments, the resulting trapezoidal map has at most 6n+4 vertices and 3n+1 trapezoids.

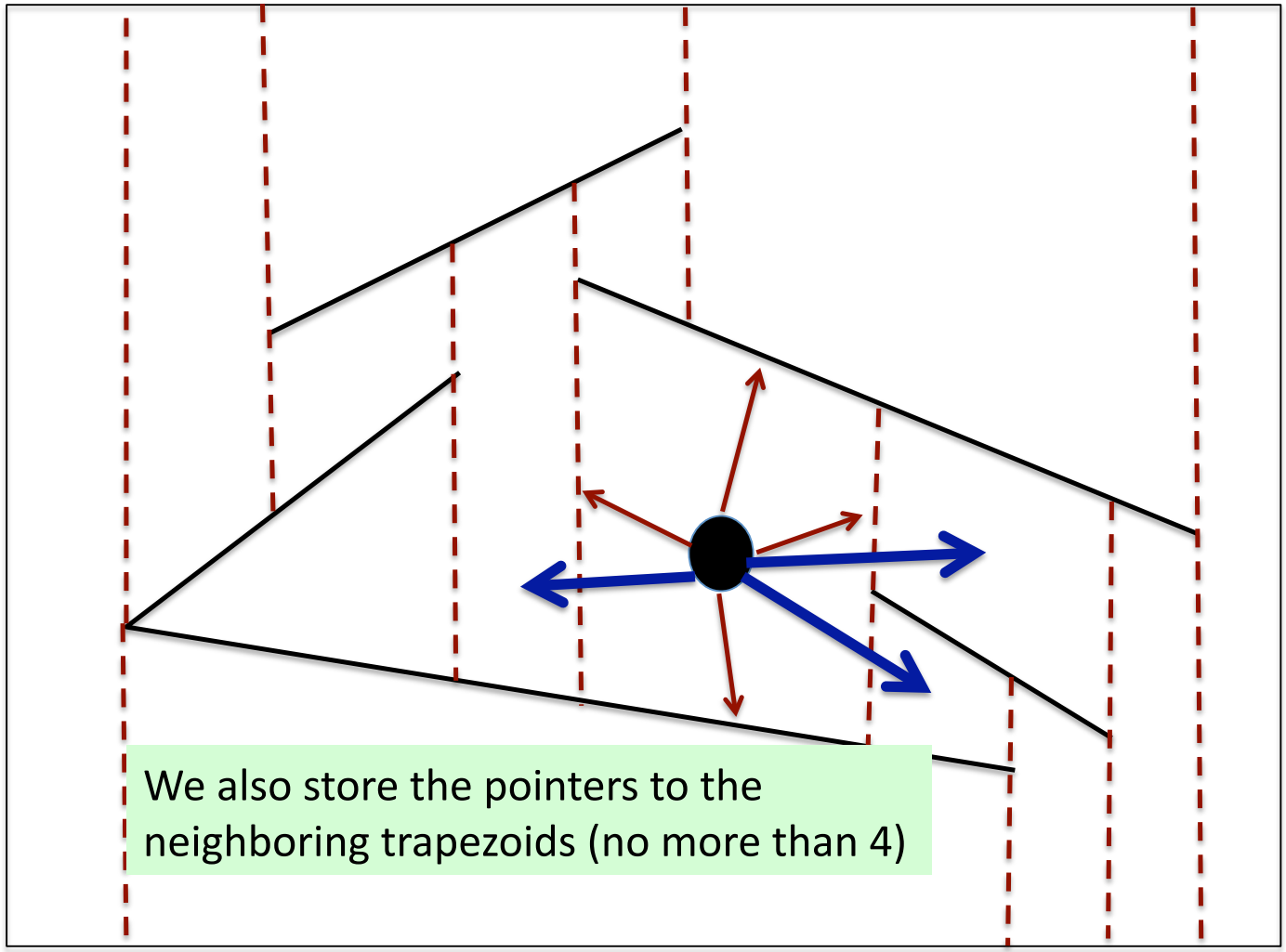
- bound on the vertices
  - each vertex shoots two bullets, thus creating two vertices; each segment has two endpoints; the bounding rectangle has 4 vertices
  - Total number of vertices: 2n (endpoints)+2.2n (bullet points) + 4 = 6n+4

Size of the maps of n line segments

Claim: Given a polygonal subdivision with n segments, the resulting trapezoidal map has at most 6n+4 vertices and 3n+1 trapezoids.

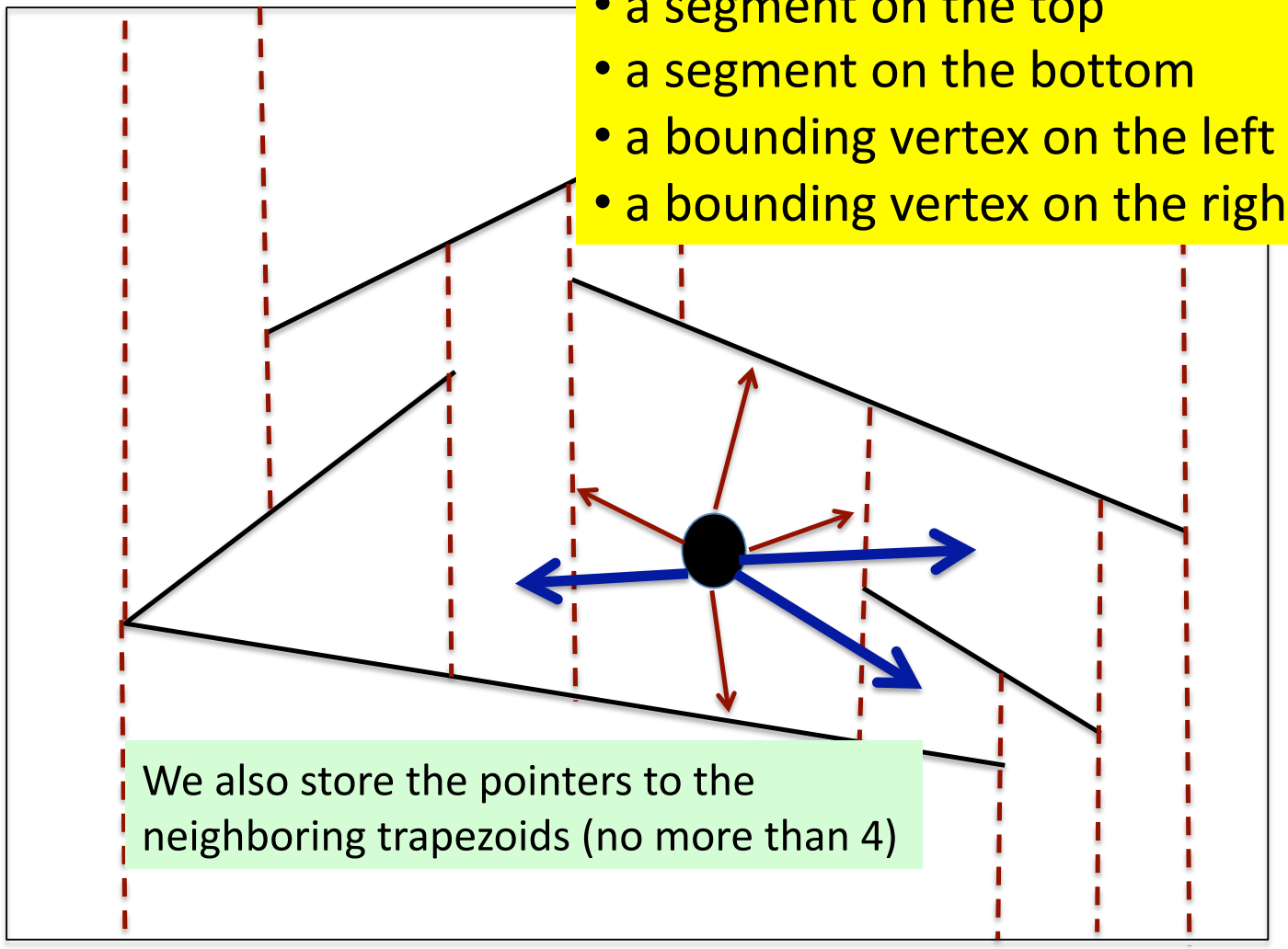
- bound on the trapezoids :
  - each segment realizes the following
    - left endpoint of s supports two trapezoids from the left.
    - right endpoint of s supports one trapezoid from the left
    - total is 3
  - leftmost trapezoid is not bounded by any point from the left
  - number of trapezoids is at most 3n+1

Data structure for each trapezoid


We also store the pointers to the
neighboring trapezoids (no more than 4)

Each trapezoid is determined by 4 entities

- a segment on the top
- a segment on the bottom
- a bounding vertex on the left
- a bounding vertex on the right



We also store the pointers to the neighboring trapezoids (no more than 4)

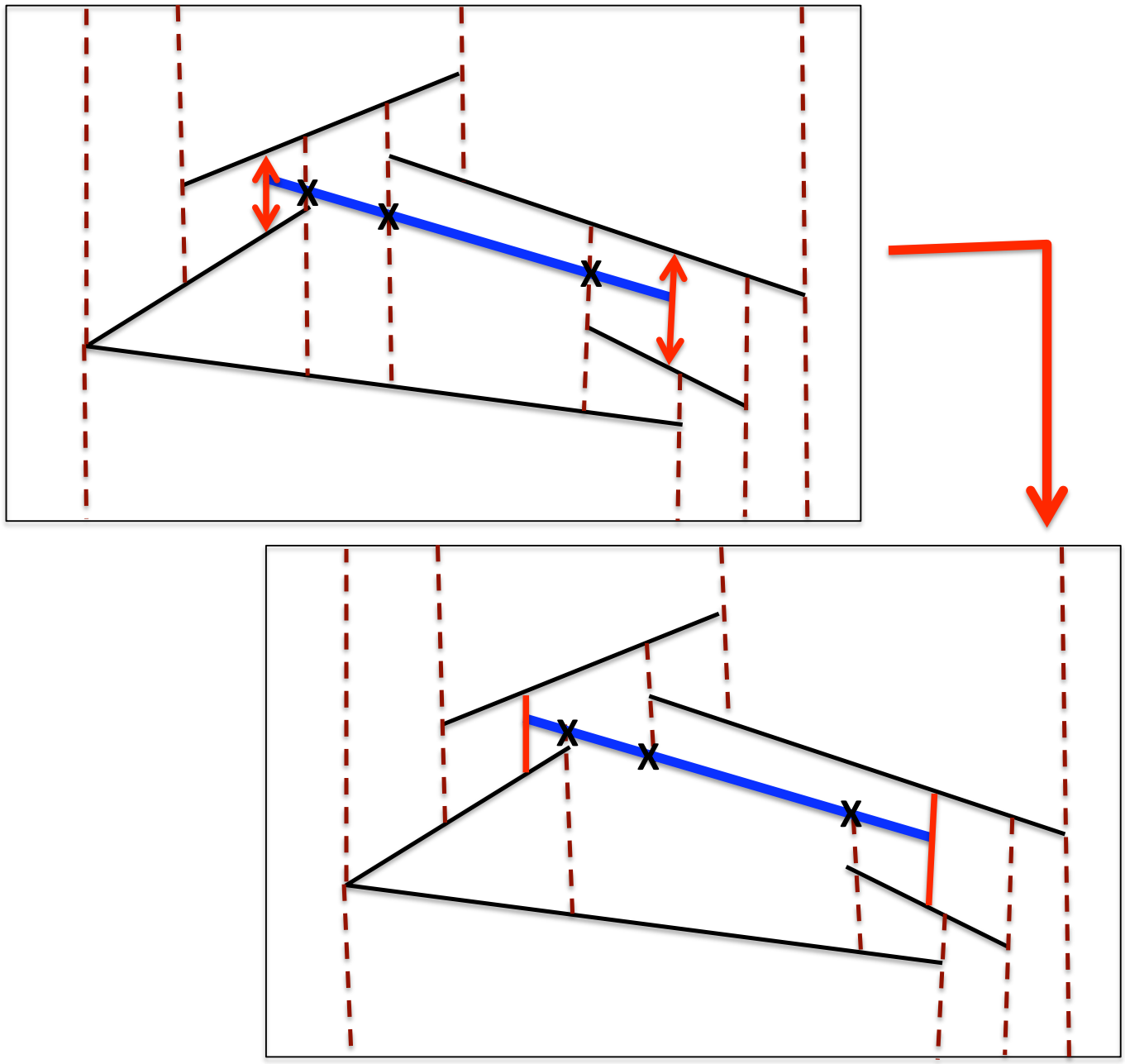
The data structure allows ...

... tracing a polygonal chain C through the trapezoidal map of S in $O(|C| + k)$ where k is the number of trapezoids intersected by C, provides C does not properly intersect any segment of S. (We are assuming that the starting trapezoid is given)

Incremental algorithm

- Start with the bounding rectangle (starting trapezoid)
- We then add the segments in random order one at a time. As each segment is added, update the trapezoidal map.
- S_i = set of first i random segments

  T_i = the resulting trapezoidal map
- When a new segment s_i is added, we perform the following operations on T_{i-1}.
  - Find the trapezoids of T_{i-1} that contain the left and the right endpoint of s_i.
  - Trace the line segment s_i from left to right, determining which trapezoids it intersects.
  - Go back to these trapezoids and fix them.
  - The left and the right endpoint of s_i need to have bullets fixed from them.
  - One of the earlier bullet path might hit this line segment. We need to trim this bullet path.



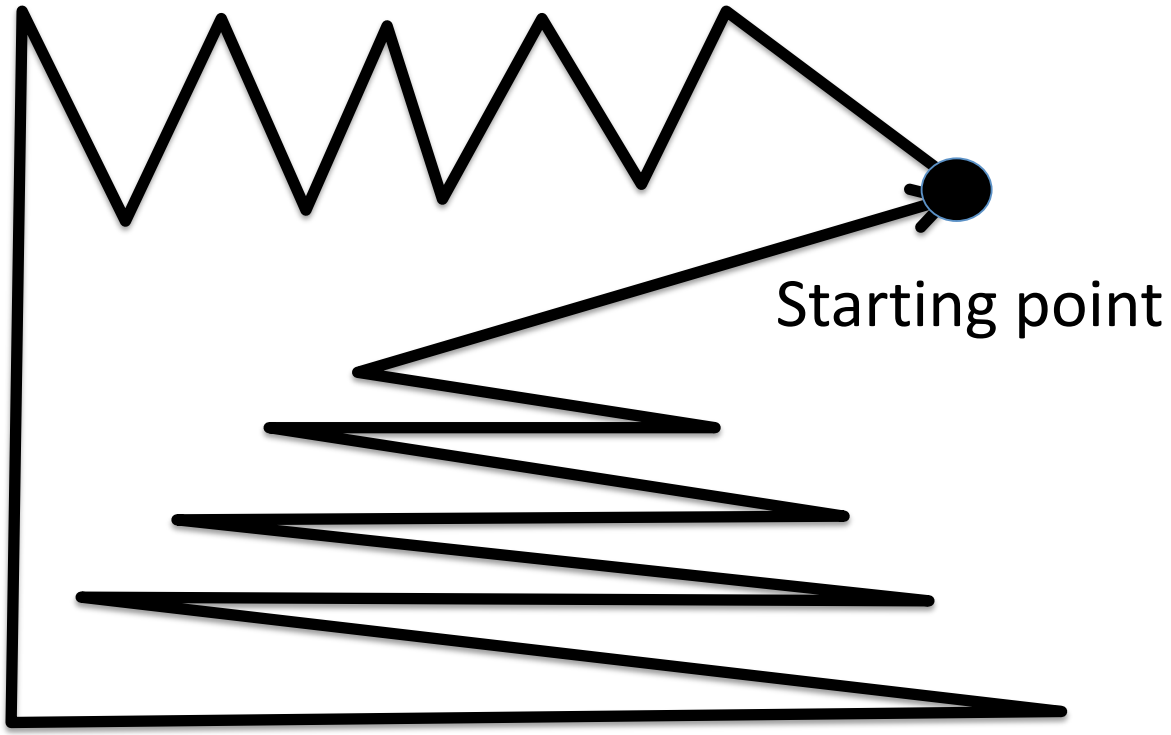
- Observe that the structure does not depend on the order in which the segments are added.
- Ignoring the time spent on locating the endpoint s_i, the time it takes to insert s_i is $O(k_i)$ where k_i is the number of created trapezoids.

Expected time to build T_n

- Later we will argue that O(log n) time is needed on an average to locate the trapezoid containing the left endpoint of each new segment.
- We will show that the expected value of k_i, $E(k_i)$ is O(1).
- This results in an O(nlogn) expected time algorithm for the incremental construction of the trapezoidal map of n segments.

Inserting s_i

- Worst case situation: Tracing cost is $O(n^2)$



Inserting s_i

- On an average each insertion results in a constant # of trapezoids being created
- Intuition:
  - Short segment might not intersect very many trapezoids
  - long segment may cut many trapezoids, but it shields later segments from cutting through many trapezoids.

Lemma A: E(k_i) is O(1).

- The analysis will be based on a backward analysis. Here T_i denote the map after s_i is inserted.
- Since each segment is inserted at random, each segment has an equal probability of 1/i to be the last segment to have been added.
- Let δ(Δ,s)=1 if segment s defines one of the sides of Δ, otherwise δ(Δ,s)=0.
- Therefore

$$E[k_i] = \frac{1}{i} \sum_{s \in S_i} \sum_{\Delta \in T_i} \delta(\Delta, s).$$

Showing that $E(k_i)$ is O(1).

- Instead of counting the number of trapezoids that depend on each segment, we count the number of segments each trapezoid depend on
- Therefore $$E[k_i] = \frac{1}{i} \sum_{\Delta \in T_i} \sum_{s \in S_i} \delta(\Delta, s).$$
- Since each trapezoid is defined by 4 segments, $E(k_i) \leq 1/i * |T_i| * 4 = 1/i * O(i) * 4 = O(1)$

Incremental algorithm

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  - Go back to these trapezoids and fix them.
  - The left and the right endpoint of s_i need to have bullets fixed from them.
  - One of the earlier bullet path might hit this line segment. We need to trim this bullet path.

Running time of inserting s_i has two parts

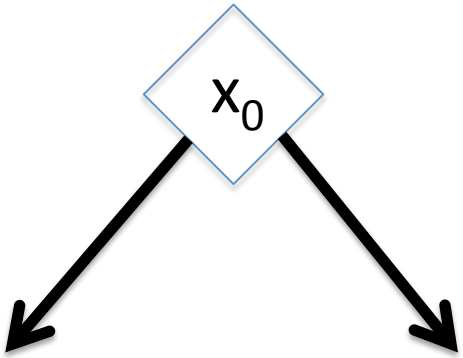
- Query time : one point location in T_{i-1}
- Tracing time through the trapezoids in T_{i-1}
  - We have seen that this takes expected O(1) time.

Step: Find the trapezoid in T_{i-1} that contains the left endpoint of s_i

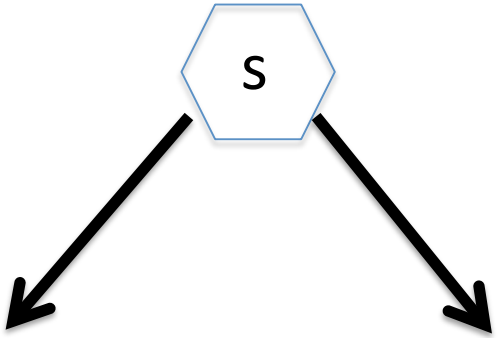
- Point Location Structure: The data structure is a rooted directed acyclic graph. Each node has either two or zero outgoing edges. Each leaf (node with zero outgoing edge) stores a trapezoid in the map. The other nodes are internal nodes that facilitate the point location search. As we will see later, this search structure is not a binary tree.

Two types of internal nodes

- x-nodes
  - each x-node contains the x-coordinate x_0 of an endpoint of one of the segment. The search takes the left branch if the x-coordinate of the query point q is less than x_0, otherwise it takes the right branch.
- y-nodes
  - each y-node contains a pointer to a line segment s of S_i. The left and the right children correspond to whether the query point is below or above the line segment s. In this case the x-coordinate of the query point lies between the x-coordinates of the endpoints of s.



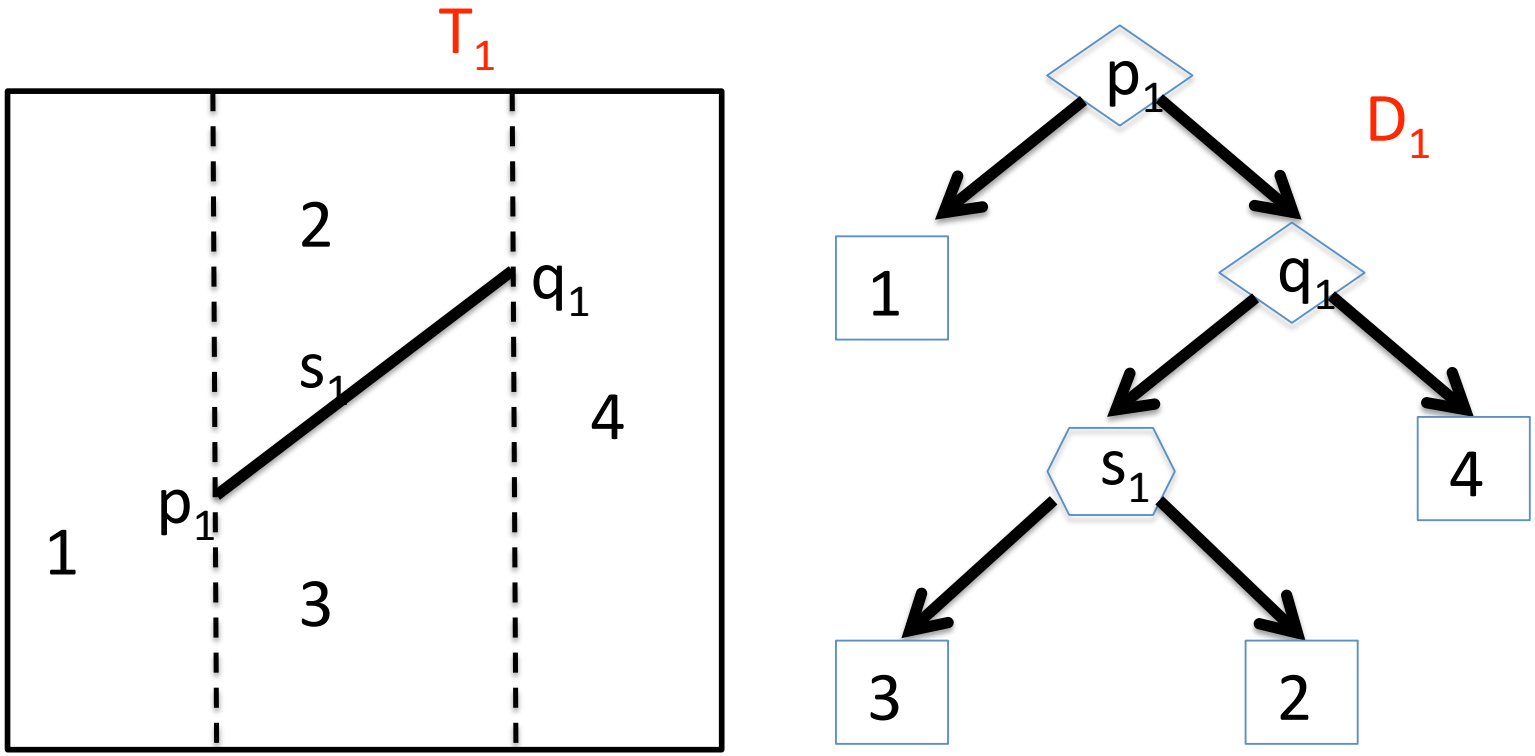
x-node



y-node

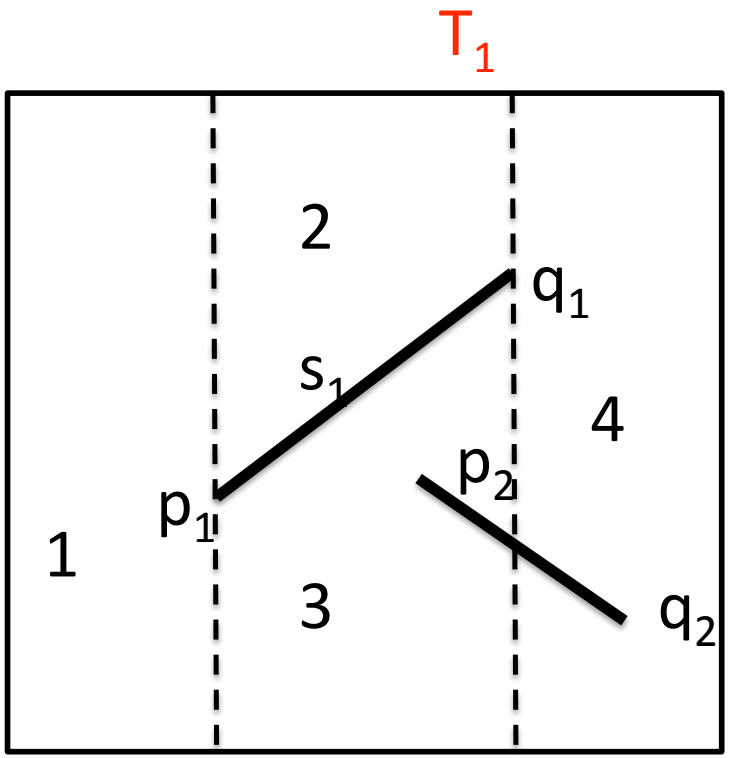
Few steps of the algorithm

- s_1 is added

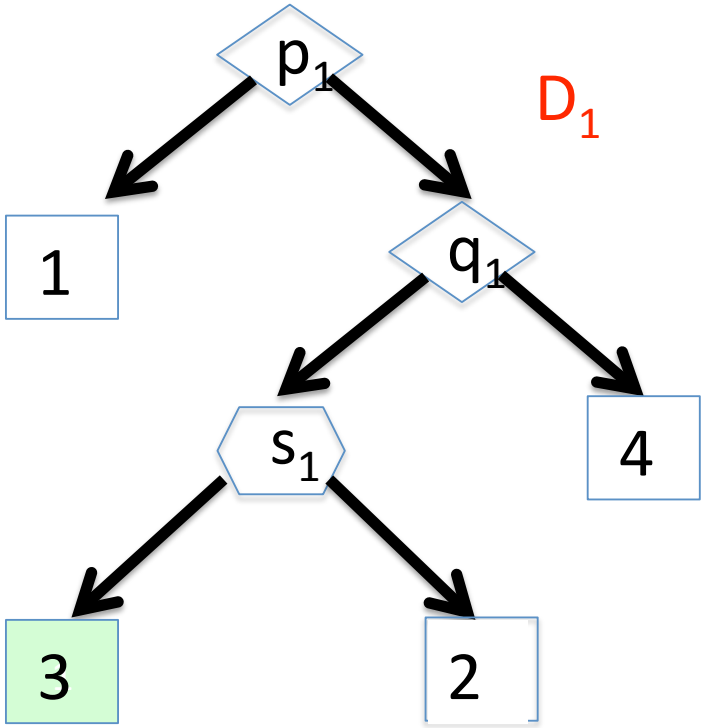


Few steps of the algorithm

- s_2 is added

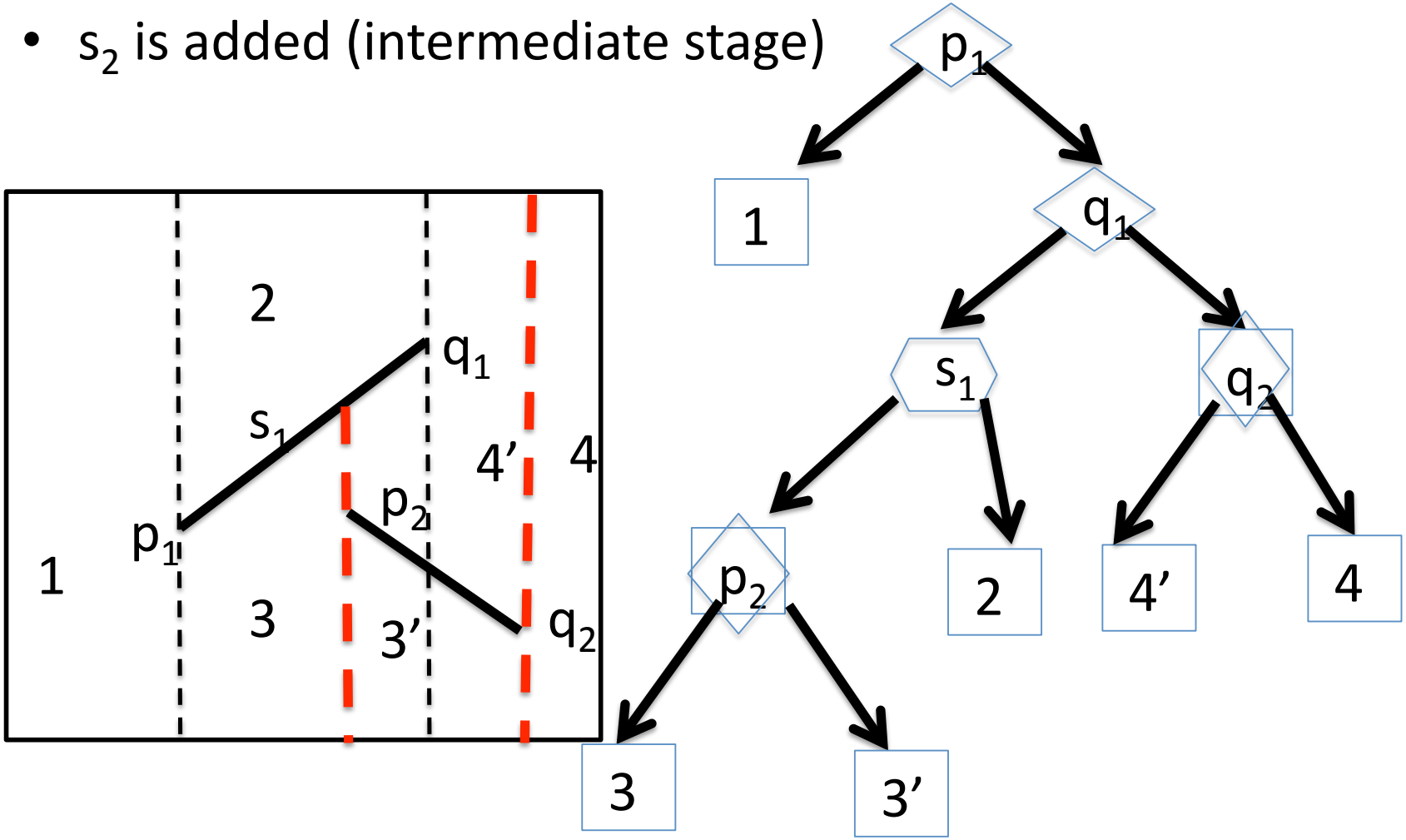


Perform search on D_1 to locate the trapezoid that contains the left point p_2.



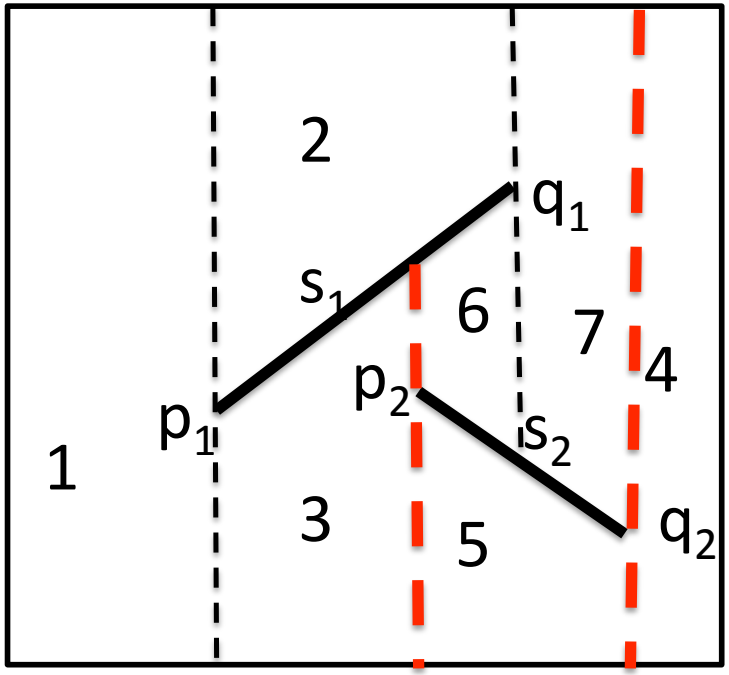
Few steps of the algorithm

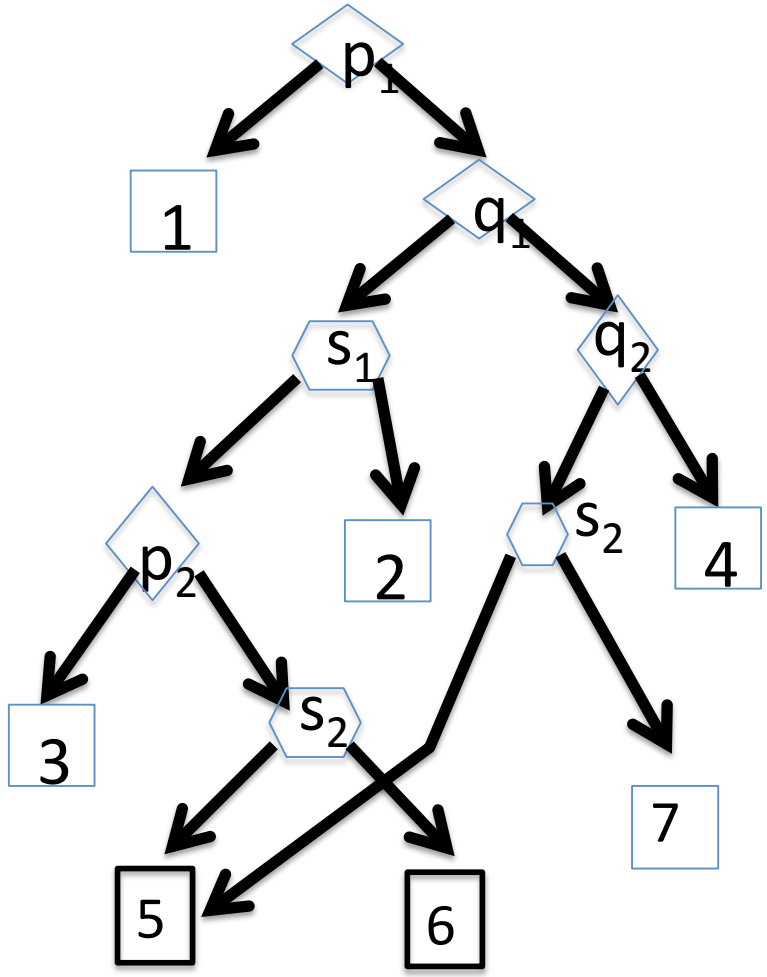
- s_2 is added (intermediate stage)



Few steps of the algorithm

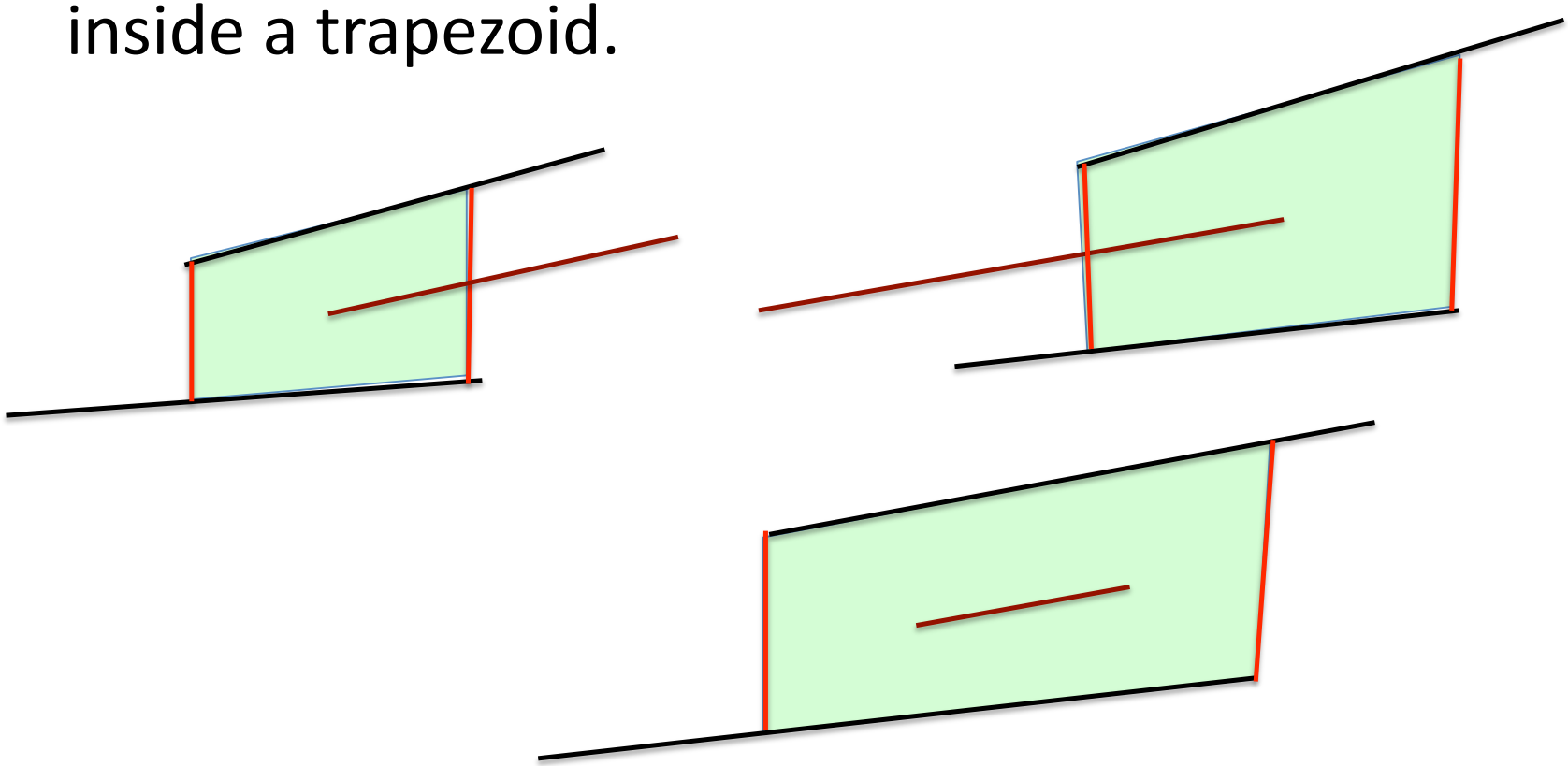
- s_2 is added (Final)





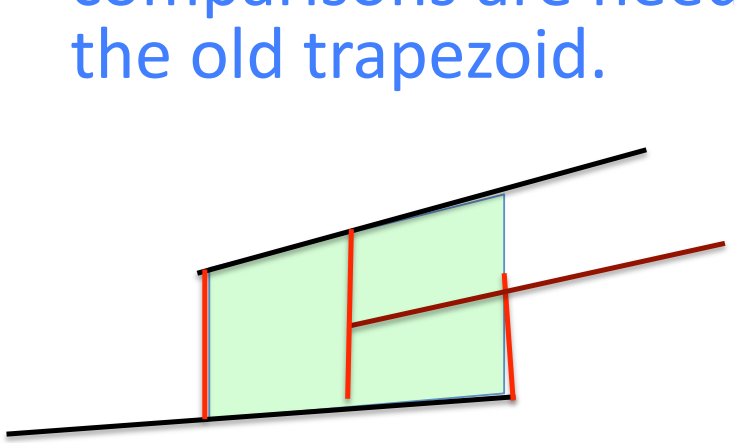
Analysis of point location structure

- Three cases of how the end points of s_i can lie inside a trapezoid.

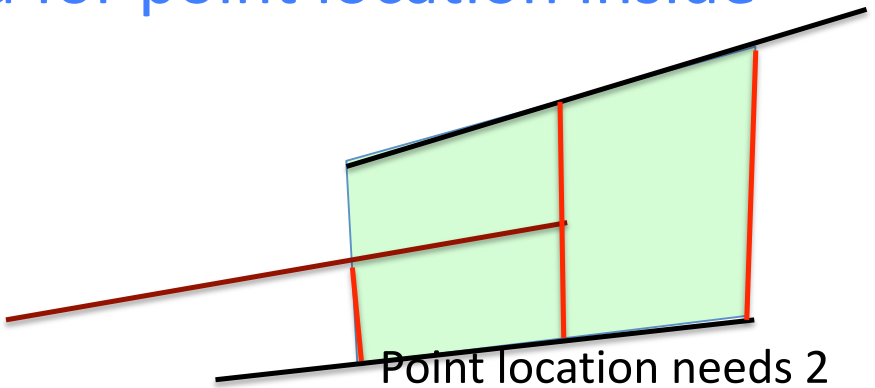


Analysis of point location structure

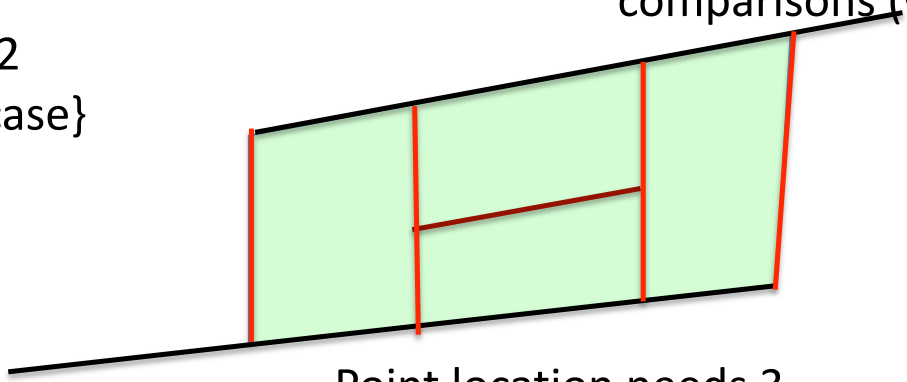
- Local modifications after s_i is added. More comparisons are needed for point location inside the old trapezoid.



Point location needs 2 comparisons (worst case}



Point location needs 2 comparisons (worst case)



Point location needs 3 comparisons (worst case)

Analysis of T_n

- Expected size of T_n is O(n)
  - The number of new nodes is proportional to the number of newly created trapezoids.
  - The expected number of new trapezoids after an insertion is O(1)
  - Expected total size is O(n).

Claim: Expected query time in T_n is O(logn)

- Consider one query point q, chosen arbitrarily.
- Let us consider how q moves incrementally through the structure with the addition of new line segment.
- Let Δ_i be the trapezoid q lies after the insertion of i segments.
- If $\Delta_{i-1} = \Delta_i$, insertion of s_i did not affect the trapezoid that q was in.
- Suppose $\Delta_{i-1} \neq \Delta_i$. In this case q must be relocated.
- In the worst case we need to make 3 comparisons to relocate (i.e. q falls as much as 3 levels)
- The probability that q changes at the i^{th} step is 4/i.
- The expected length of a path is at most

$$3.4 .(1+1/2+1/3+ \ldots + 1/n) = 12H_n = O(\log n)$$

Some useful Lemmas

- Lemma B:

  For any q, if we know that q ε Δ_j where Δ_j ε T_j, the expected cost of locating q in T_k, k ≥ j is at most 12.(H_k – H_j) which is O(log k/j).

  – This follows easily from the query time analysis

Some useful Lemmas

- Lemma C: Let R be a random subset of S, |R|=r. Let Z be the intersections between T(R) and S\R. The expected value of Z is O(n-r).

  Proof: For any s ε S\R, the number of intersections between s and the trapezoidal map of R, T(R) is *deg(s, T(RU{s}))*.

$$E(Z) = \frac{1}{C(n,r)} \sum_{\substack{R \subset S \\ |R|=r}} \sum_{s \in S \setminus R} deg(s,\ T(RU\{s\}))$$

$$= \frac{1}{C(n,r)} \sum_{\substack{R' \subset S \\ |R'|=r+1}} \sum_{s \in R'} \deg(s, T(R')) \leq \frac{1}{C(n,r)} \sum_{\substack{R' \subset S \\ |R'|=r+1}} 4\,|R'|$$

$$= 4(r+1)\frac{C(n,r+1)}{C(n,r)} = 4(n-r)$$

Seidel's Trapezoidal Partitioning Algorithm

- Define $\log^{(i)}n = \log \log \ldots \log n$ (i times)
- $\log^* n = \max(h \mid \log^{(h)} n \geq 1\}$
- $N(h) = \text{ceiling}(n/\log^{(h)}n)$; $N(0)=1$
- Generate a random order $s_1, s_2, \ldots, s_n$.

  Let $S_i = \{ s_1, s_2, \ldots, s_i\}$
- $T(S_i)$: Trapezoidal map of S_i
- $D(S_i)$: Point location data structure for $T(S_i)$

Algorithm

1. Generate $T(S_1)$ and $D(S_1)$
2. For h = 1 to log*n do

   (a) for i = N(h-1) + 1 to N(h) do

   Insert segment s_i, producing $T(S_i)$ and $D(S_i)$ from $T(S_{i-1})$ and $D(S_{i-1})$.

   (b) Trace the edges of polygon P through T(N(h)) to locate the endpoints of all s_j, j > N(h).
3. i = N(log*n) + 1 to n do

   Insert s_i, producing $T(S_i)$ and $D(S_i)$ from $T(S_{i-1})$ and D (S_{i-1}).

Algorithm Analysis

1. Generate $T(S_1)$ and $D(S_1)$

   Takes O(1) time

2. For h = 1 to log*n do

   (a) for i = N(h-1) + 1 to N(h) do

   Insert segment s_i, producing $T(S_i)$ and $D(S_i)$ from $T(S_{i-1})$ and $D(S_{i-1})$.

   ≤ N(h)*[O(1) (Lemma A) + O(log(N(h)/N(h-1)) (Lemma B)]

   i.e $N(h)*[O(1) + O(\log(n/N(h-1)))$

   i.e $N(h)*[O(1) +O(\log(n/\text{ceiling}(n/\log^{(h-1)}n)))]$

   i.e $N(h)*O(\text{loglog}^{(h-1)}n)$

   i.e $N(h)*O(\log^{(h)}n) = O(n)$.

   (b) Trace the edges of polygon P through T(N(h)) to locate the endpoints of all s_j, $j > N(h)$.

   O(n) by Lemma C

Algorithm

1. Generate $T(S_1)$ and $D(S_1)$
2. For h = 1 to log*n do
   (a) for i = N(h-1) + 1 to N(h) do
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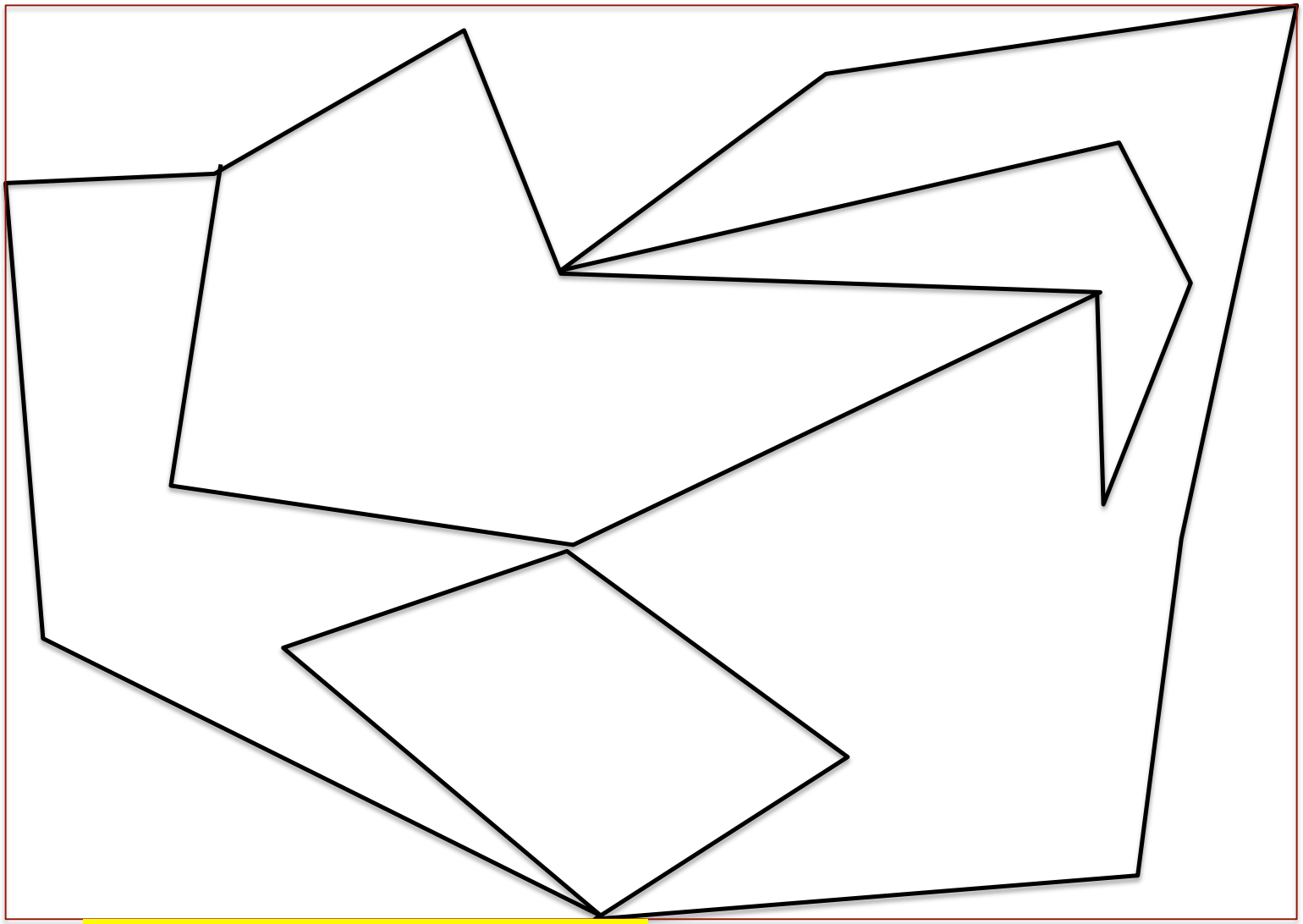
   (b) Trace the edges of polygon P through T(N(h)) to locate the endpoints of all s_j, j > N(h).
3. i = N(log*n) + 1 to n do
   Insert s_i, producing $T(S_i)$ and $D(S_i)$ from $T(S_{i-1})$ and $D(S_{i-1})$.

= O(n)* O(log(n/N(log*n))) (Lemma A and B)

i.e O(n)*O(1) = O(n)

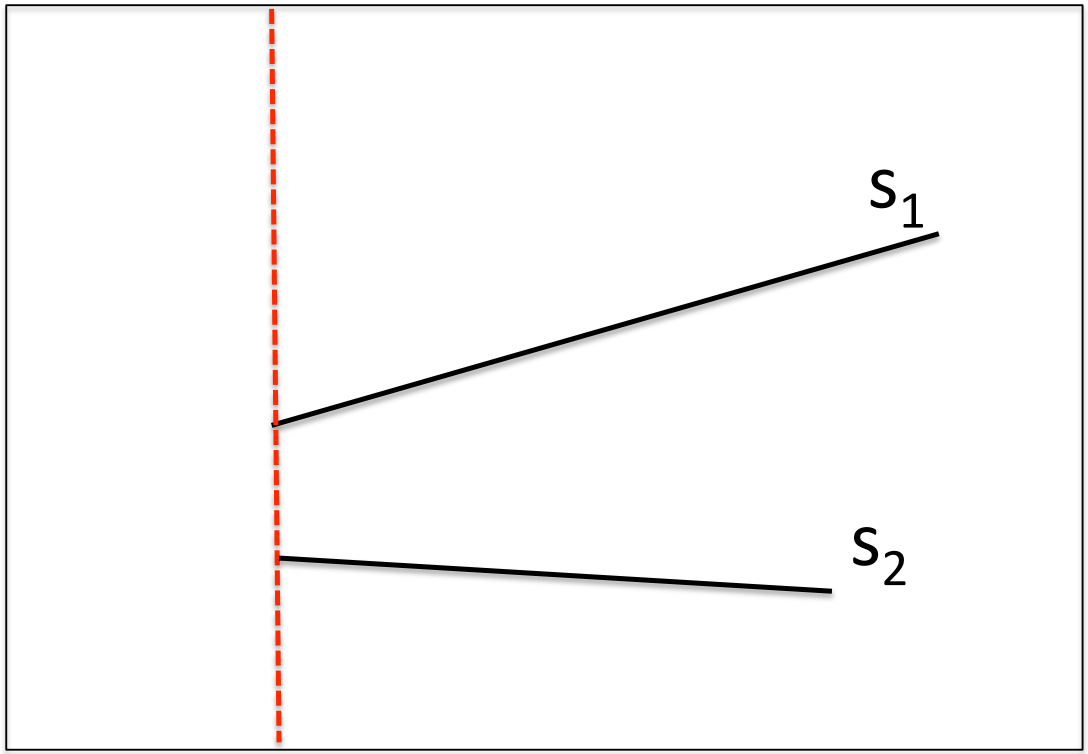
Finally

- Theorem: Let S be a set of n segments that form a simple polygon. Then
  - One can build T(S) and D(S) in O(nlog*n) time
  - Expected size of D(S) is O(n)
  - Expected query time for any q in D(S) is O(log n)
- The same results holds if P is a connected polygonal subdivision. In step 2(b) of the algorithm, a graph traversal algorithm is used to trace P through T(N(h)).

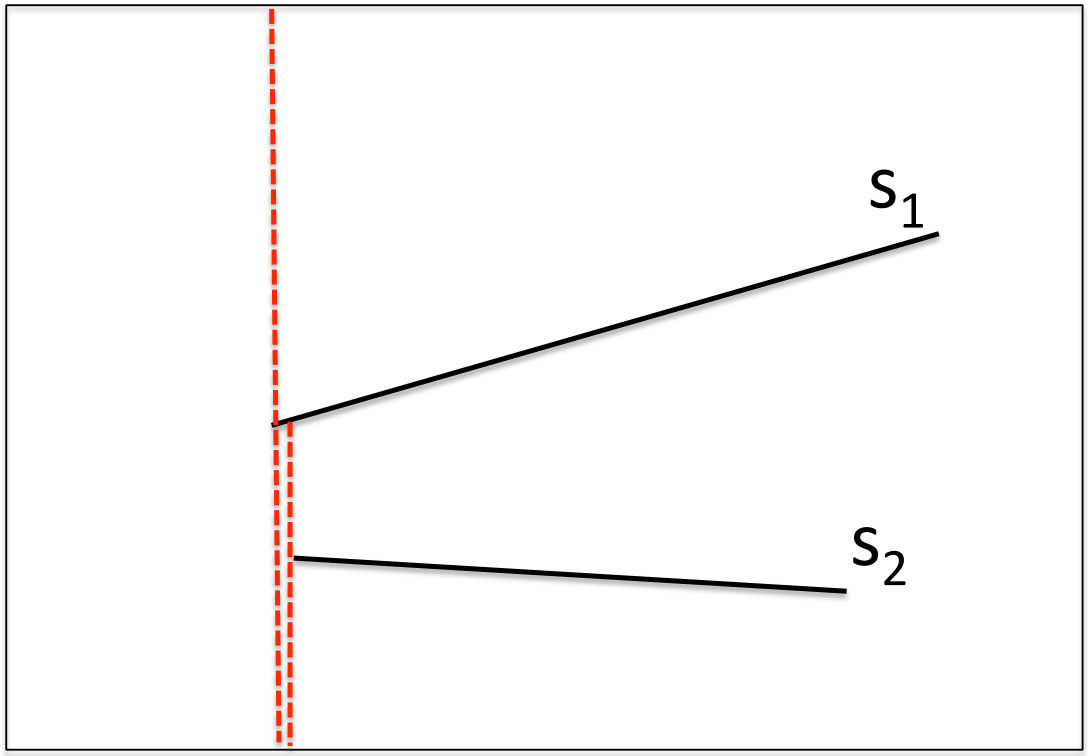


Planar Subdivision

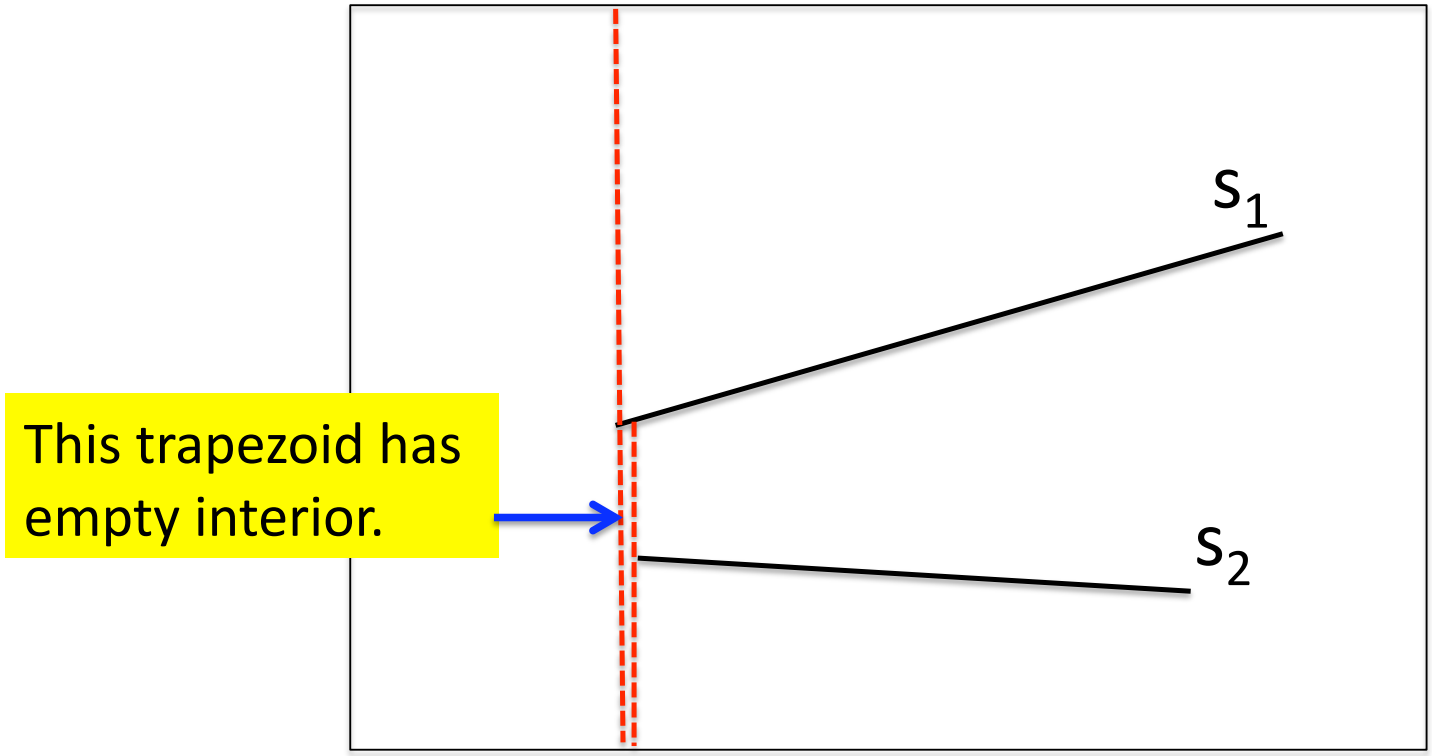
Degeneracy



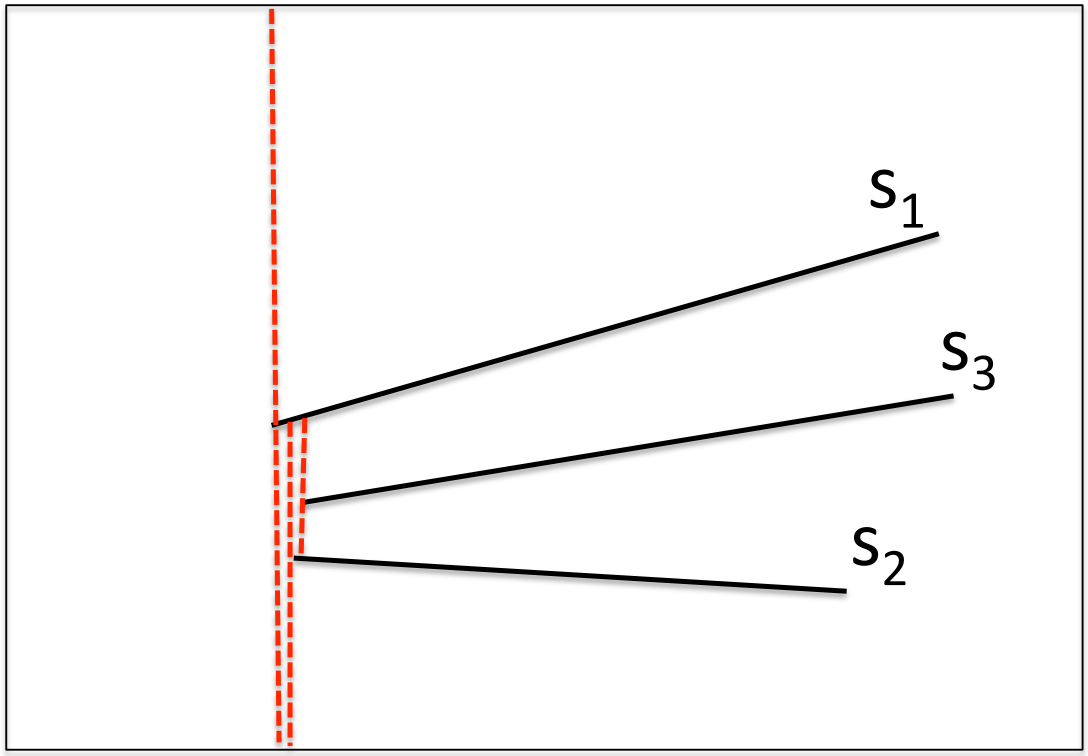
Degeneracy



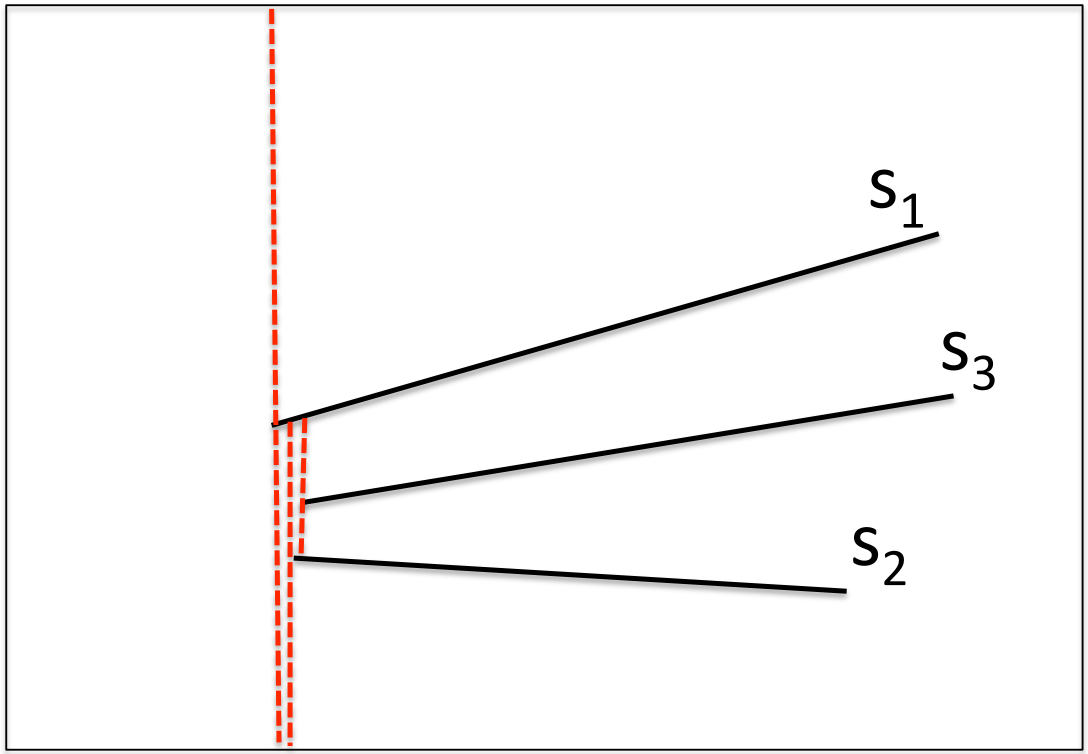
Degeneracy



Degeneracy



Degeneracy



Total number of trapezoids (including the degenerate ones) is still at most 3n+1