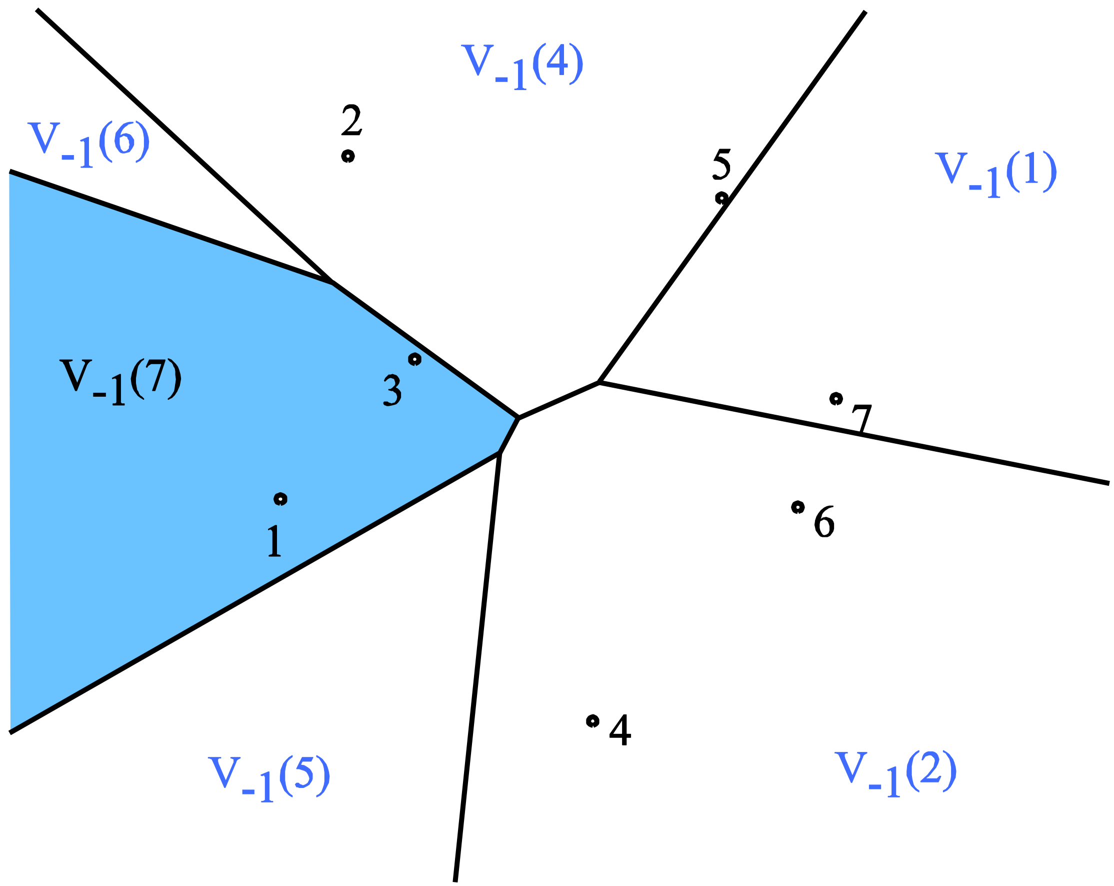
Extensions of Voronoi Diagrams

Furthest Point Voronoi Diagram

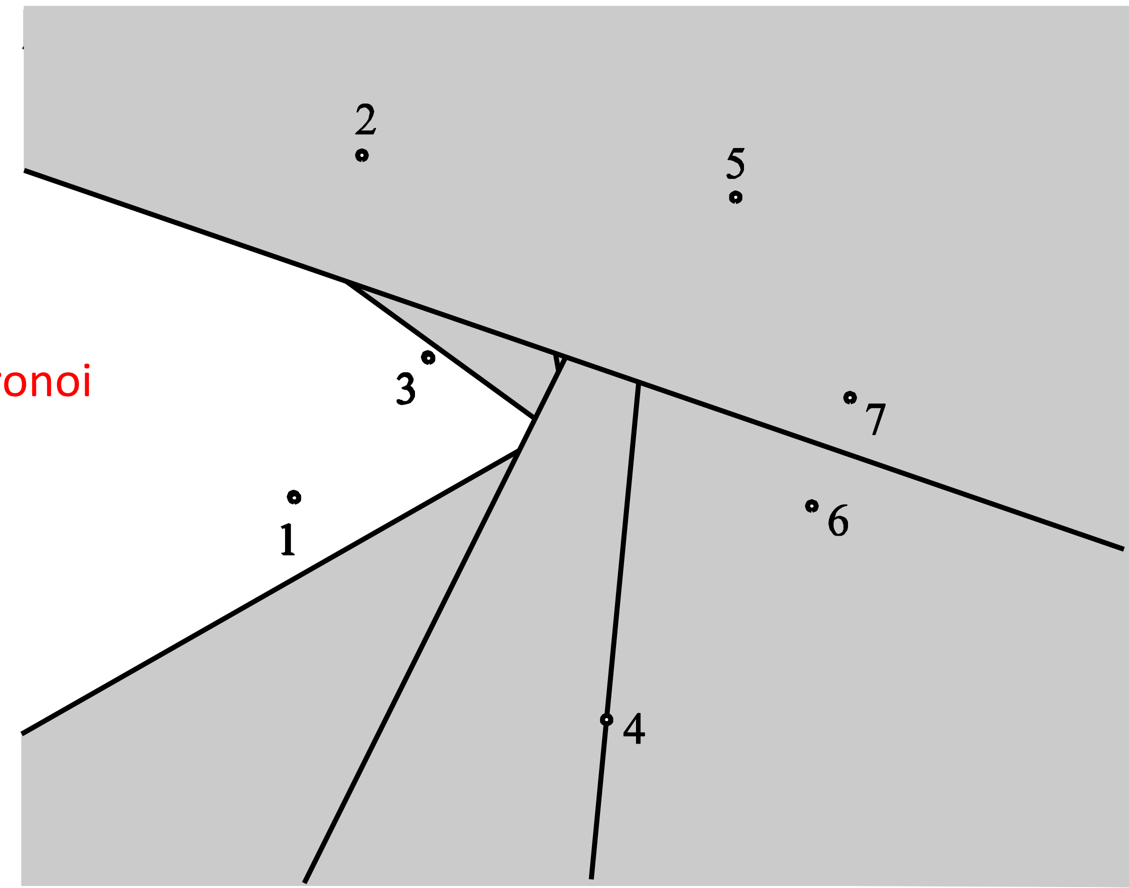
$V_{-1}(p_i)$: the set of point of the plane farther from p_i than from any other site

$Vor_{-1}(P)$: the partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



Furthest Point Voronoi Region

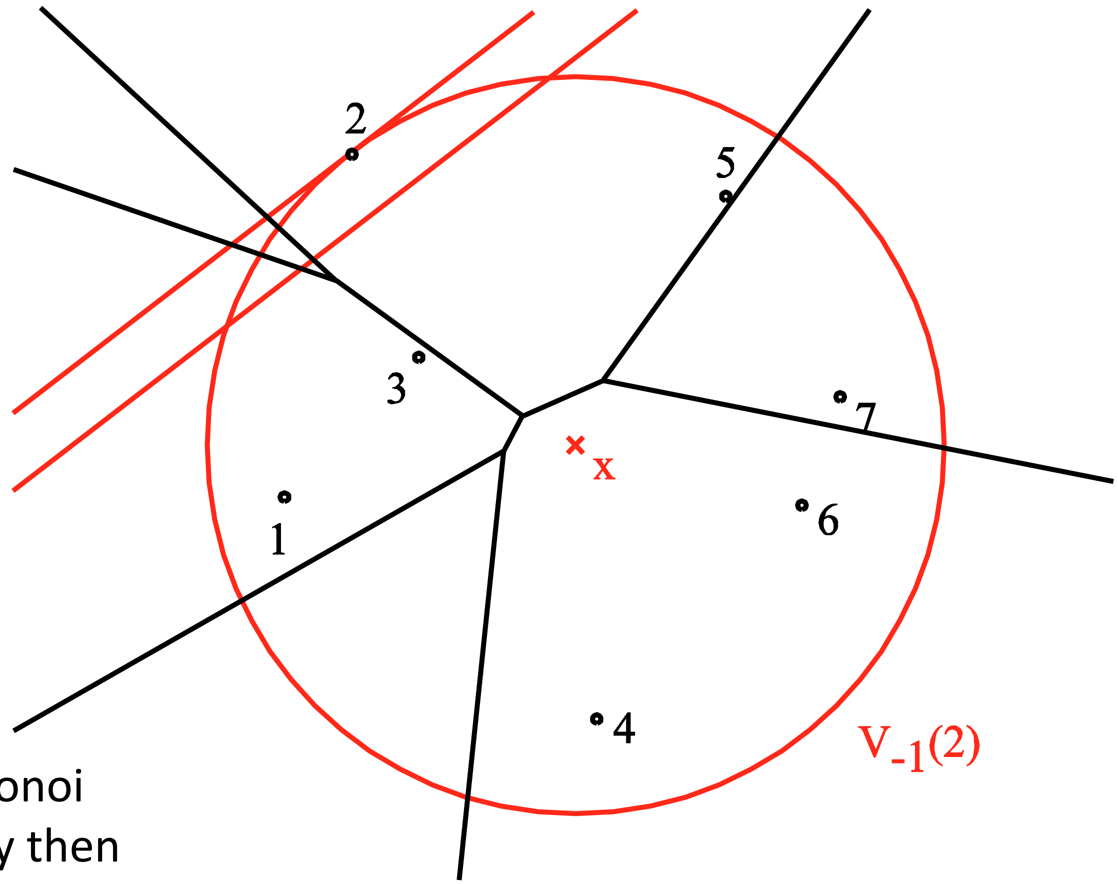
Construction of $V_{-1}(7)$



Property

The farthest point Voronoi regions are convex

Furthest Point Voronoi Region



Property

If the farthest point Voronoi region of p_i is non empty then p_i is a vertex of the convex hull of P

Furthest Point Voronoi Region

Property

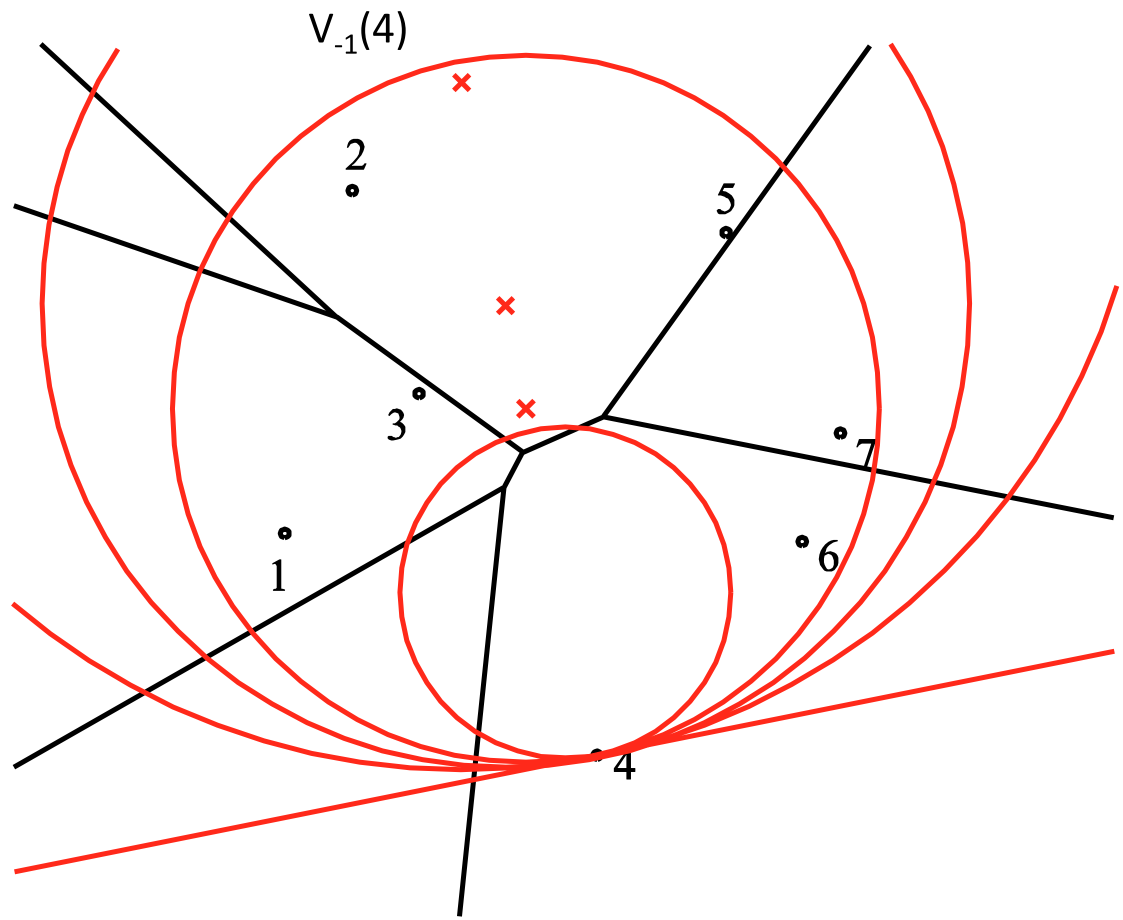
If p_i is a vertex of convex hull of P then the farthest point Voronoi region of p_i is non empty

Property

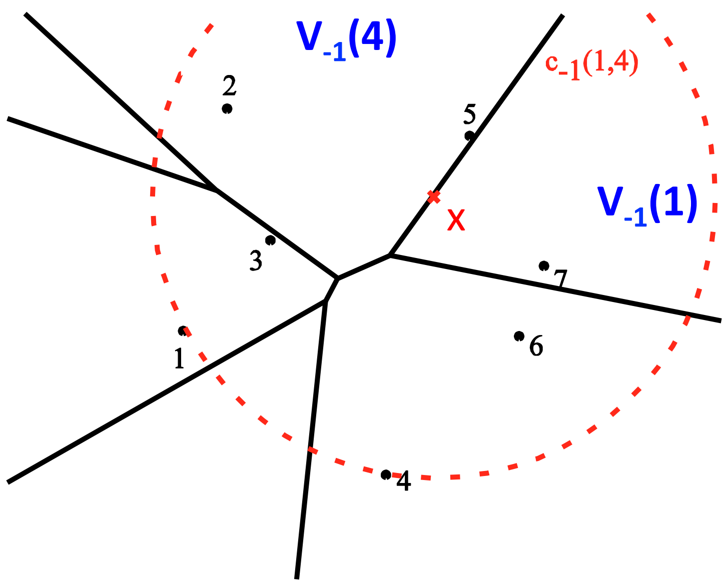
The farthest point Voronoi regions are unbounded

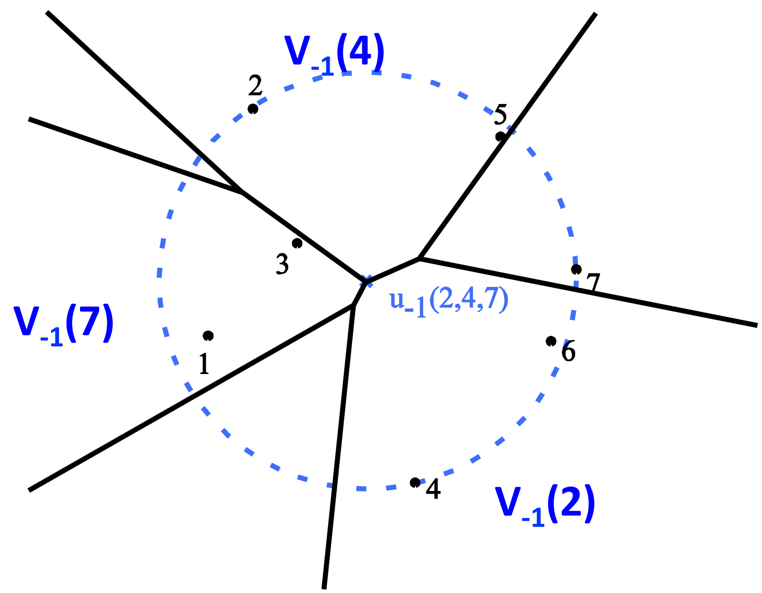
Corollary

The farthest point Voronoi edges and vertices form a tree



Farthest point Voronoi edges and vertices

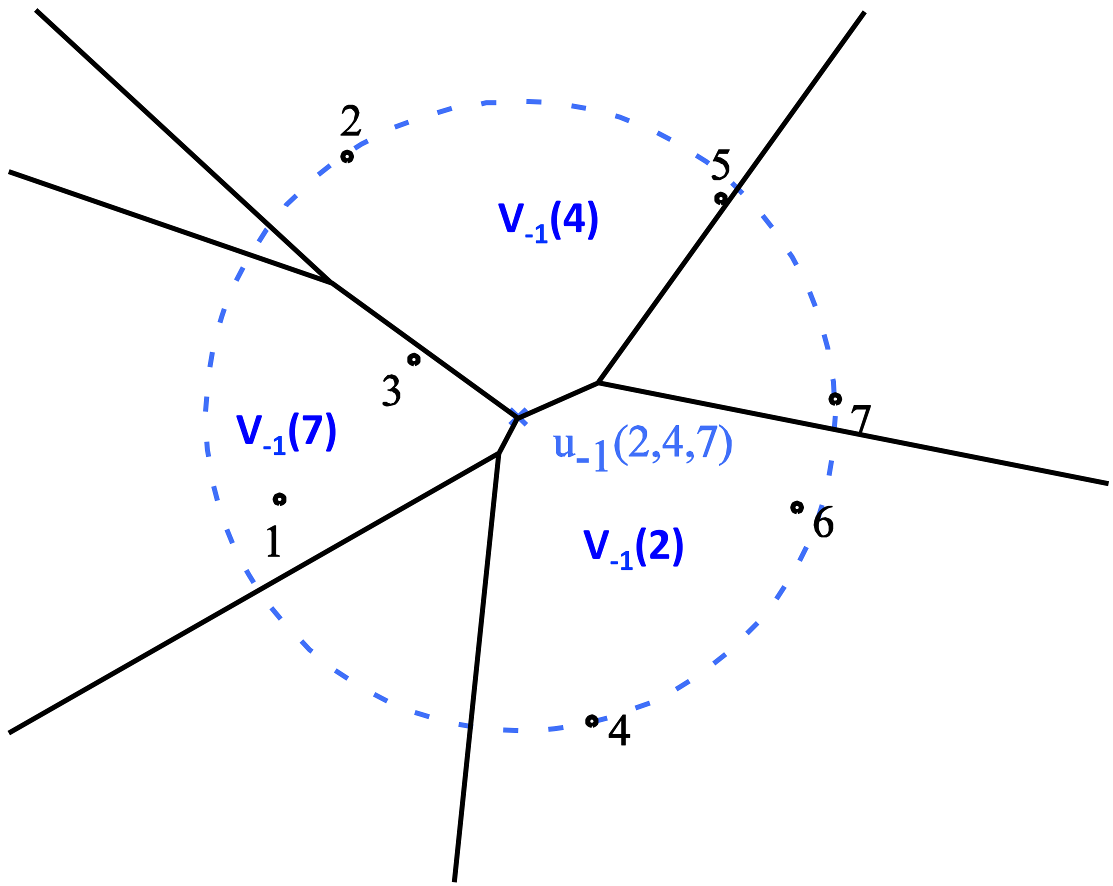




edge : set of points equidistant
from 2 sites and closer
to all the other sites

vertex : point equidistant
from at least 3 sites and
closer to all the other sites

Application: Smallest enclosing circle



Higher Order Voronoi Diagrams (VD) k^{th} order VD, $V_k(S)$

- $V_k(S)$ is a partition of the plane into regions such that points in each region have the same k closest sites.

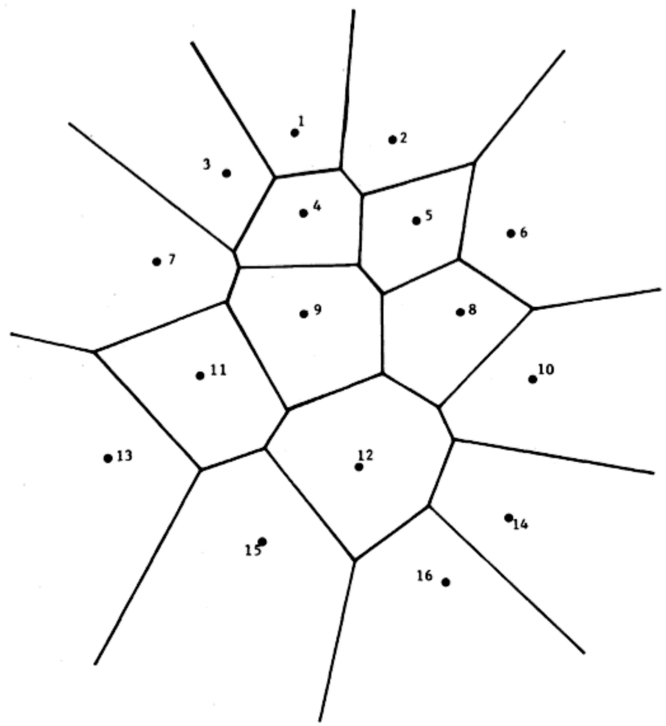


Fig. 1. The nearest neighbor Voronoi diagram for a set of 16 points.

3th order Voronoi Diagram

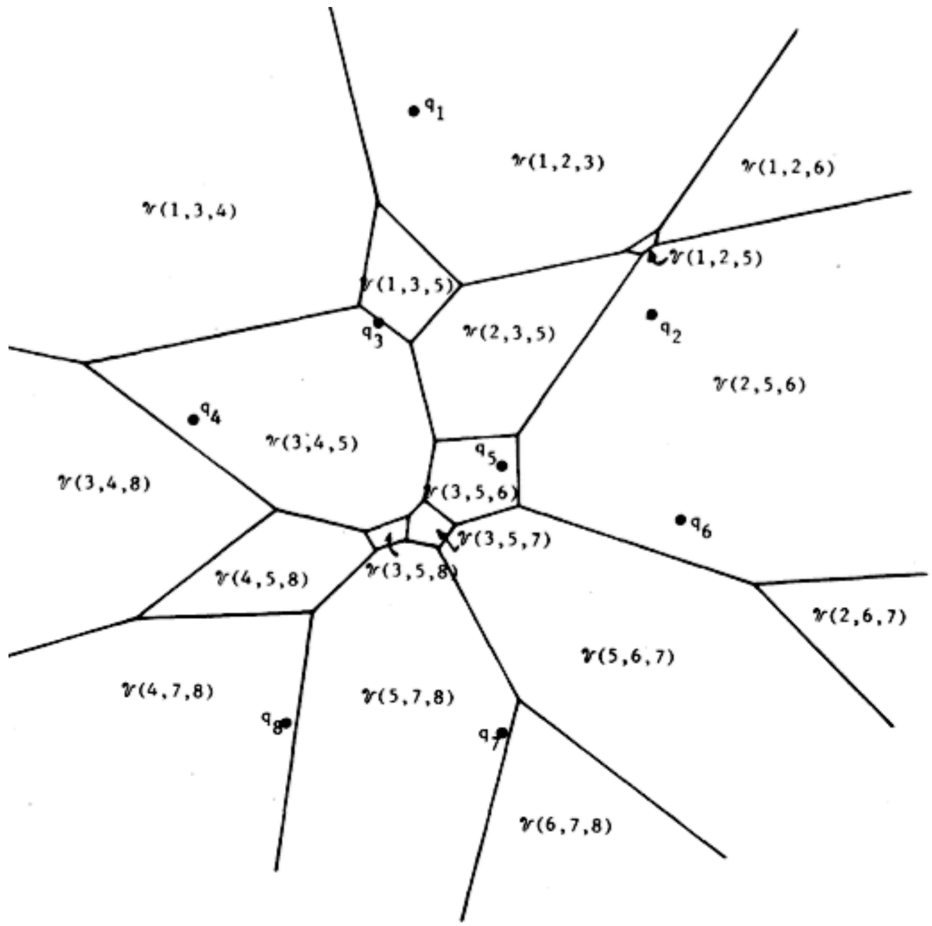


Fig. 4. The order 3 Voronoi diagram for the set of 8 points in Fig. 2.

- Let H be a subset of points of S. Let $V_S(H)$ denote the region such that for any point in the region, the |H| closest neighbors of S are the points of H.

  That is $V_S(H) = \bigcap_{q \in H, q' \in S-H} h(q,q')$
- Lee in 1982 presented an O(knlogn) algorithm where the kthe order Voronoi diagram is obtained from $(k-1)^{th}$ order Voronoi diagram.
- The size of the k^{th} order Voronoi diagram is O(k(n-k)).

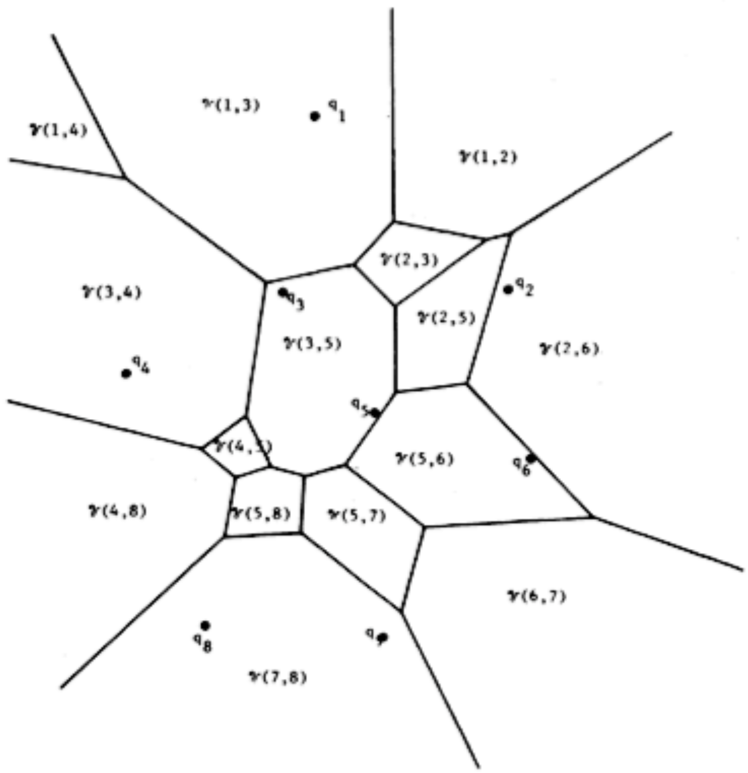


Fig. 3. The order 2 Voronoi diagram for the set of 8 points in Fig. 2.

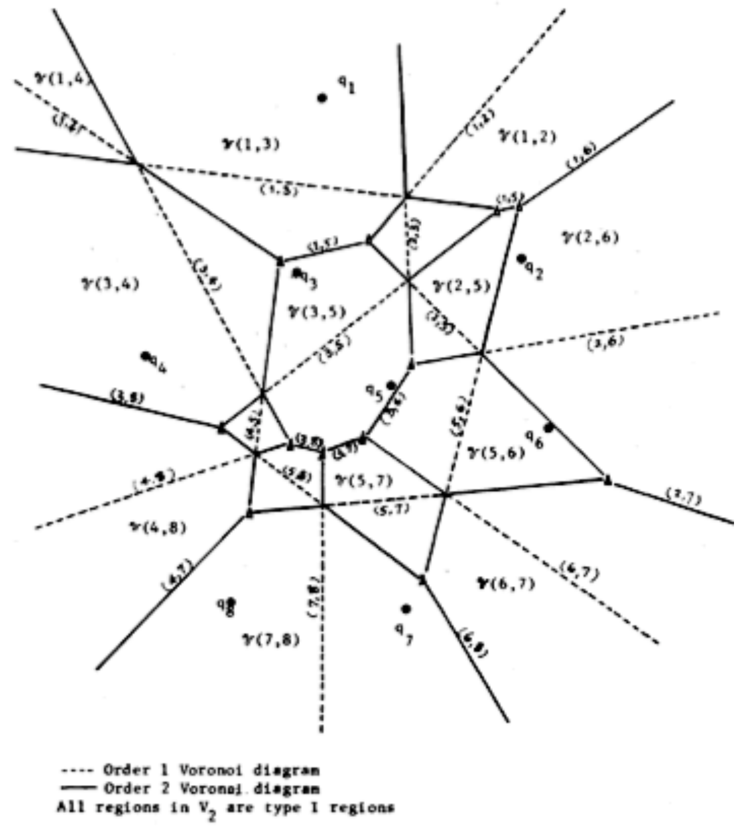


Fig. 5. The order 2 and order 1 Voronoi diagrams superimposed.

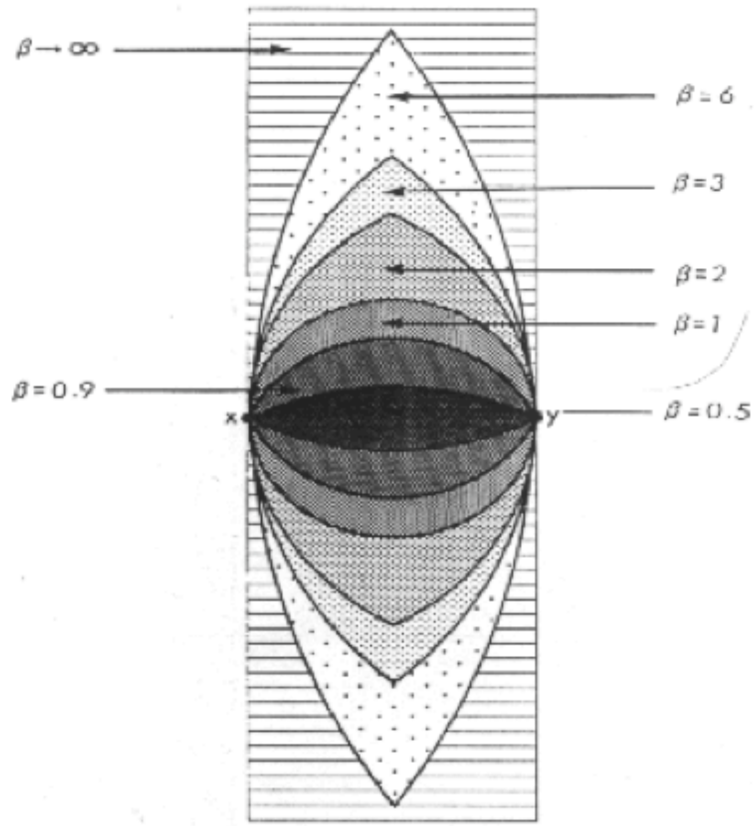
Gabriel Graphs

- Two points p and q are connected by an edge in the Gabriel graph whenever the disk having line segment pq as its diameter contains no other points from the given point set.
- More generally, in any dimension, the Gabriel graph connects any two points forming the endpoints of the diameter of an empty sphere. Gabriel graphs are named after K. R. Gabriel, who introduced them in a paper with R. R. Sokal in 1969.
- The Gabriel graph is a subgraph of the Delaunay triangulation; it can be found in linear time if the Delaunay triangulation is given (Matula and Sokal, 1980). The Gabriel graph contains as a subgraph the Euclidean minimum spanning tree and the nearest neighbor graph.
- It is an instance of a beta-skeleton.

β-skeletons

- Given a point set S, the β-Skeleton of S is the set of edges joining β -neighbors in S.
- Given a point set S, two points x and y are β-neighbors in the set S if the region of influence of x and y, N(x,y,) contains no point of S, other than x or y, in its interior. There are various kinds of definitions of N(x,y,). One of them is called Lune-Based Neighborhoods.
- First proposed by Kirkpatrick and Radke 1985, “A framework for computational morphology”

Lune-Based Neighbourhoods



Lune-based neighbourhoods N(x,y,β) for various β>0

For β ≥ 1

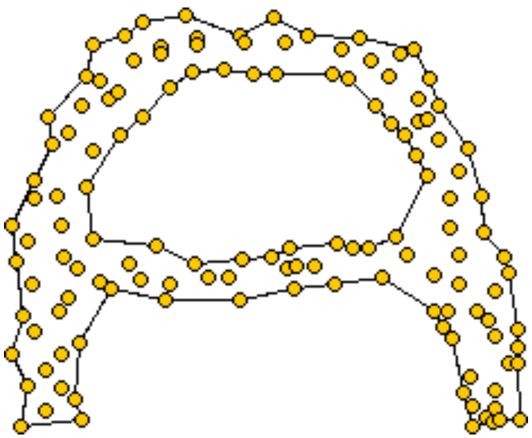
We define N(x,y, β) to be the intersection of the two circles of radius β d(x,y)/2 centered at the points (1- β/2)x+(β/2)y and (β/2)x + (1- β/2)y, respectively.

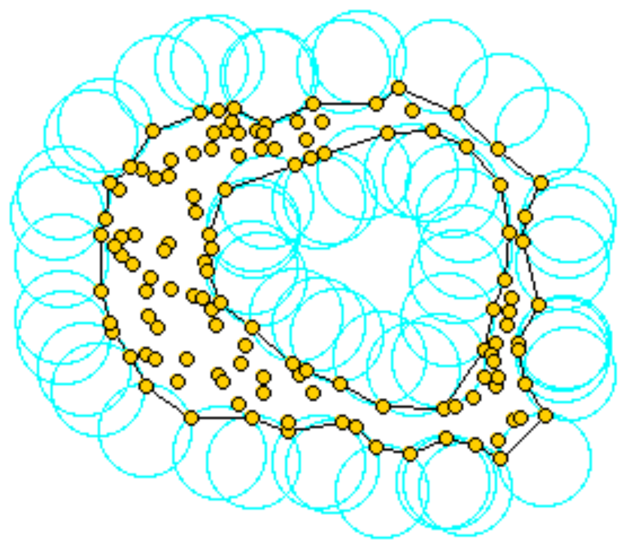
When β =1, N(x,y, β) corresponds exactly to the Gabriel neighborhood of x and y. When β =2, we get the ``relative neighborhood'' of the RNG. As β approaches infinity, the neighborhood of x and y approximates the infinite strip formed by translating the line segment (x, y) normal to itself.

For β ε [0,1]

We define N(x,y, β) to be the intersection of the two circles of radius d (x,y)/(2β) passing through both x and y. When β =1, this is consistent with the definition above. As β approaches 0, N(x,y, β) approximates the line segment joining x and y. Thus, except in degenerate situations (three or more points collinear), all point pairs are β-neighbors under this scheme for β sufficiently small.

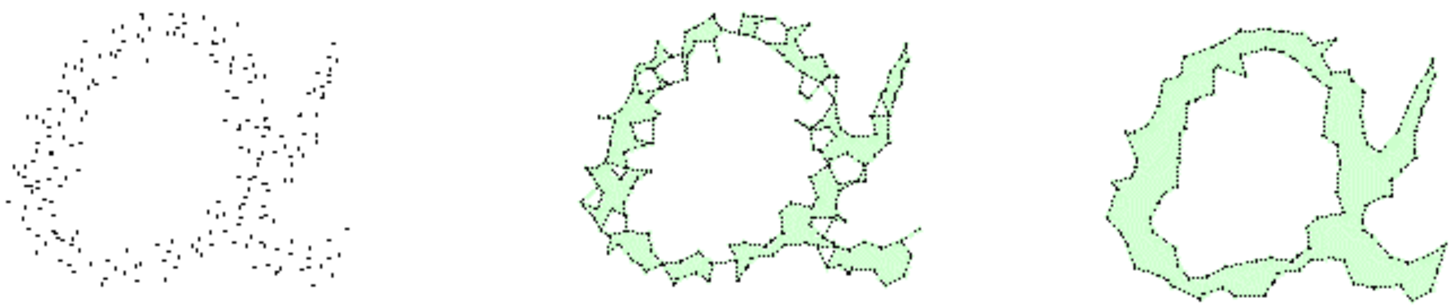
Alpha Shapes



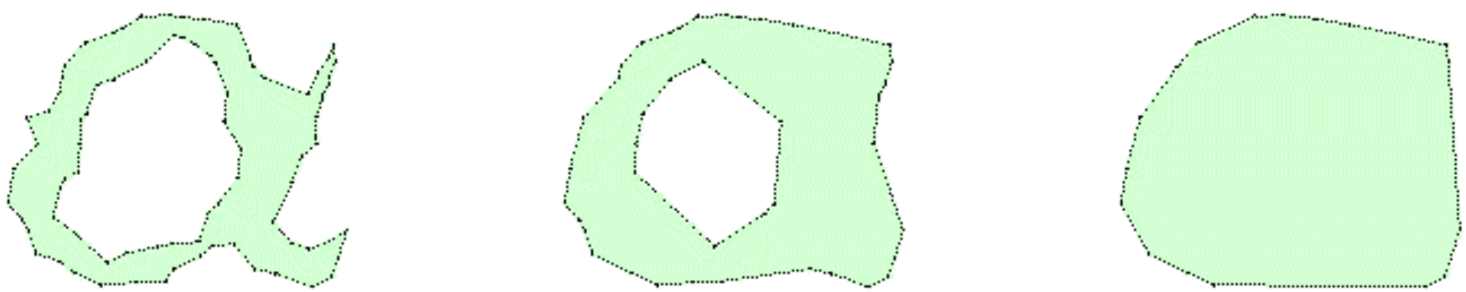


The space generated by point pairs that can be touched by an empty disc of radius alpha.

Alpha Shapes



Alpha Controls the desired level of detail.



$\alpha = \infty$

α-Shapes

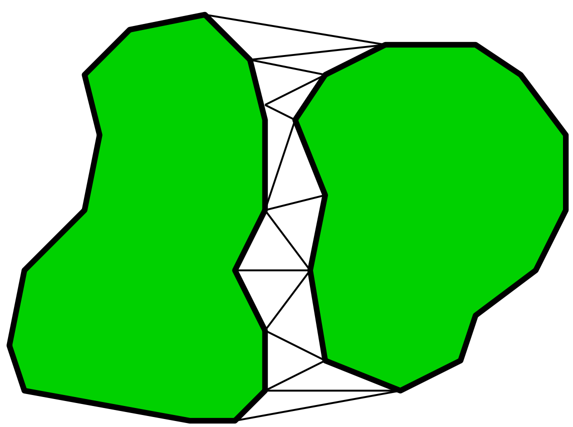
- Theorem: For each Delaunay edge, $e=(p_i,p_j)$, there exists $\alpha_{min}(e)>0$ and $\alpha_{max}(e)>0$ such that $e \in \alpha$-shape of S iff $\alpha_{min}(e) \leq \alpha \leq \alpha_{max}(e)$.
- Thus, every alpha-hull edge is in the Delaunay, and every Delaunay edge is in some alpha-shape.

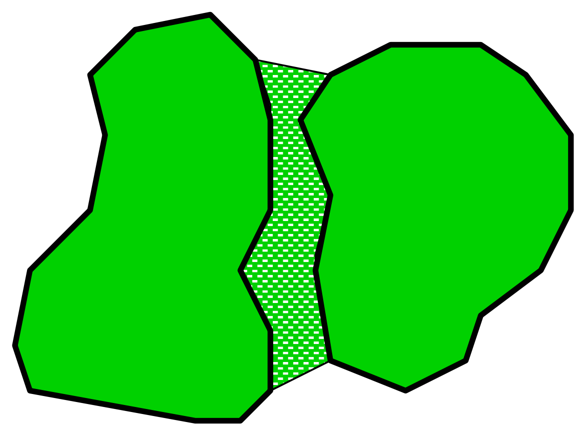
Papers

- H. Edelsbrunner, D. G. Kirkpatrick and R.Seidel. On the shape of a set of points in the plane. *IEEE Transactions on Information Theory, 1983*.
- H. Edelsbrunner and E. P. Mucke. Three dimensional alpha shapes, *ToG 1994.*
- Kaspar Fischer, Introduction to alpha shapes.

Region aggregation

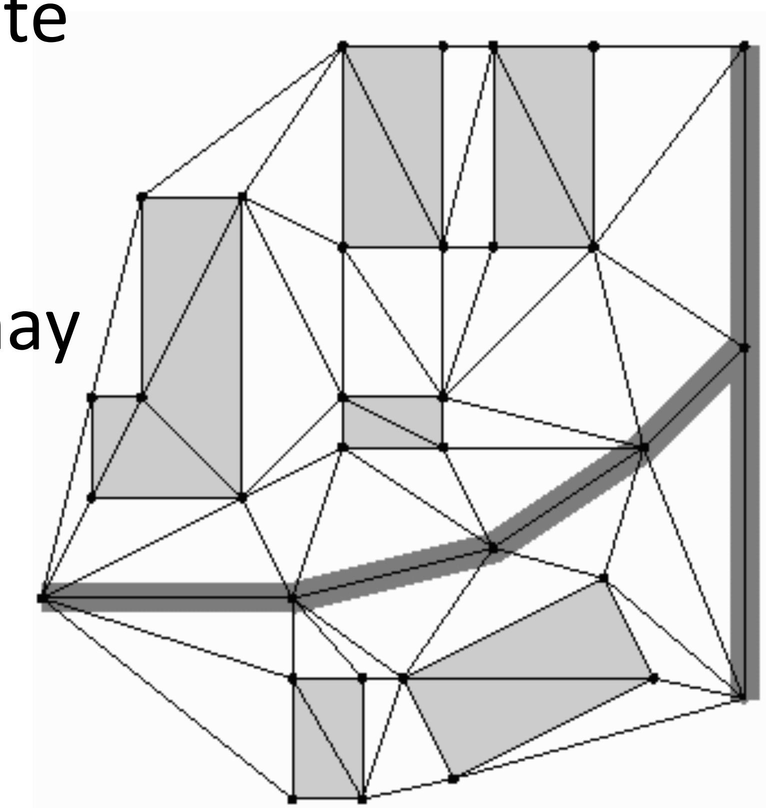
- Triangulate between objects to be aggregated
- Aggregate if:
  - the distance is small over a stretch
  - it reduces total boundary length





Region aggregation
à la Jones et al.

- More general: triangulate all and test where aggregation is good
- Use constrained Delaunay triangulation

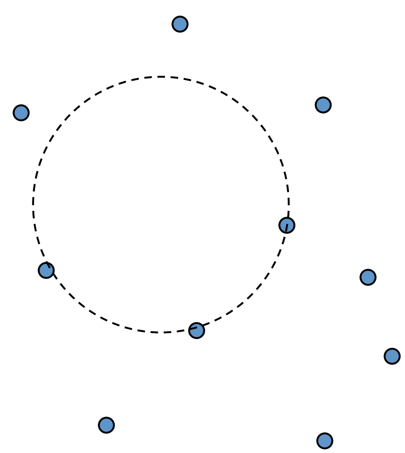


Constrained Delaunay Triangulations

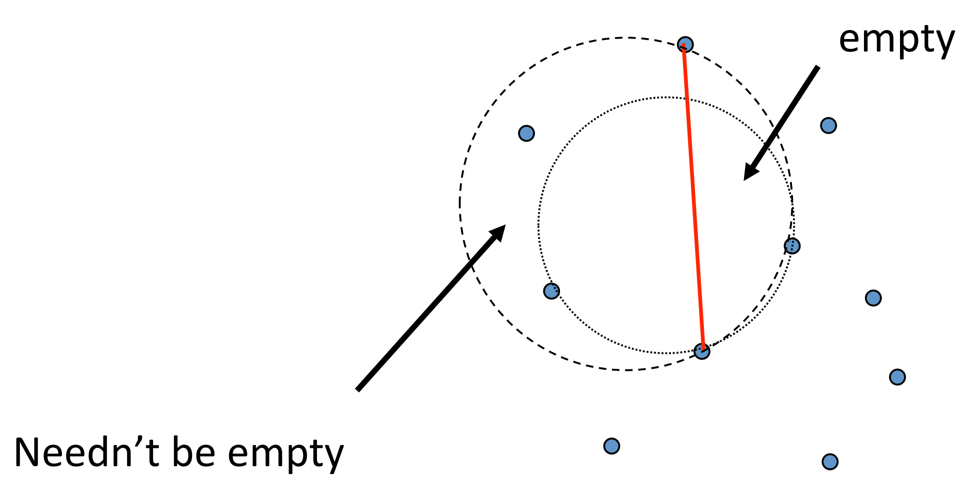
- S : Set of non-intersecting edges
- CDT(S): A triangulation of the vertices with the following properties:
  - the edges are included in the triangulation
  - it is as close as possible by Delaunay Triangulation
- The circumcircle through the vertices (say a, b and c) of each triangle does not contain
  - any endpoint of S that is visible from a, b and c.

Constrained Delaunay Triangulation

- Delaunay: empty circle

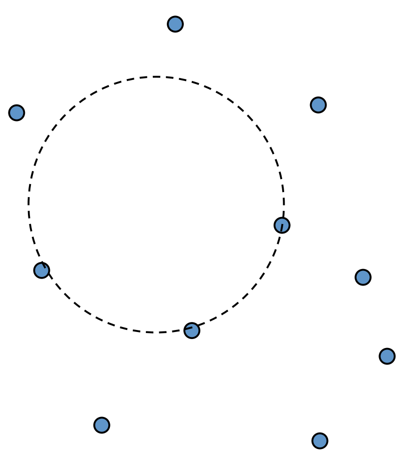


- Constrained Delaunay: empty circle but with given edges

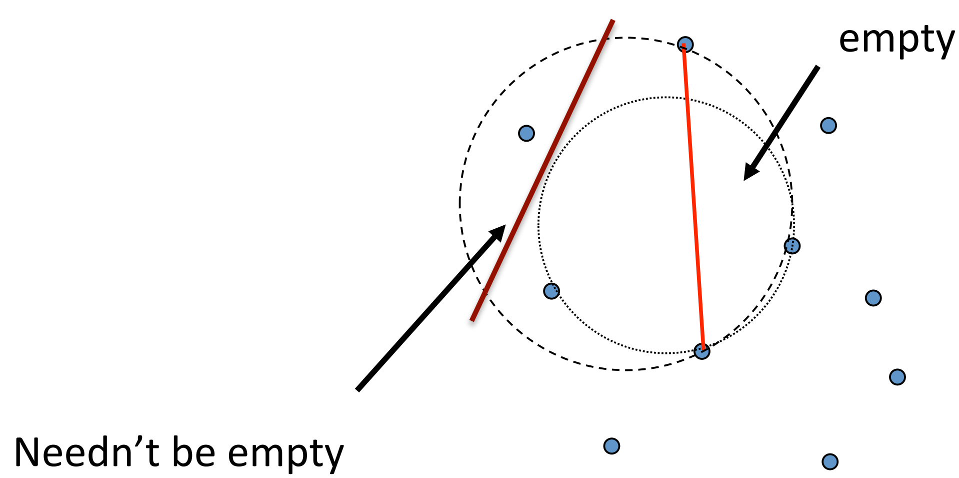


Constrained Delaunay Triangulation

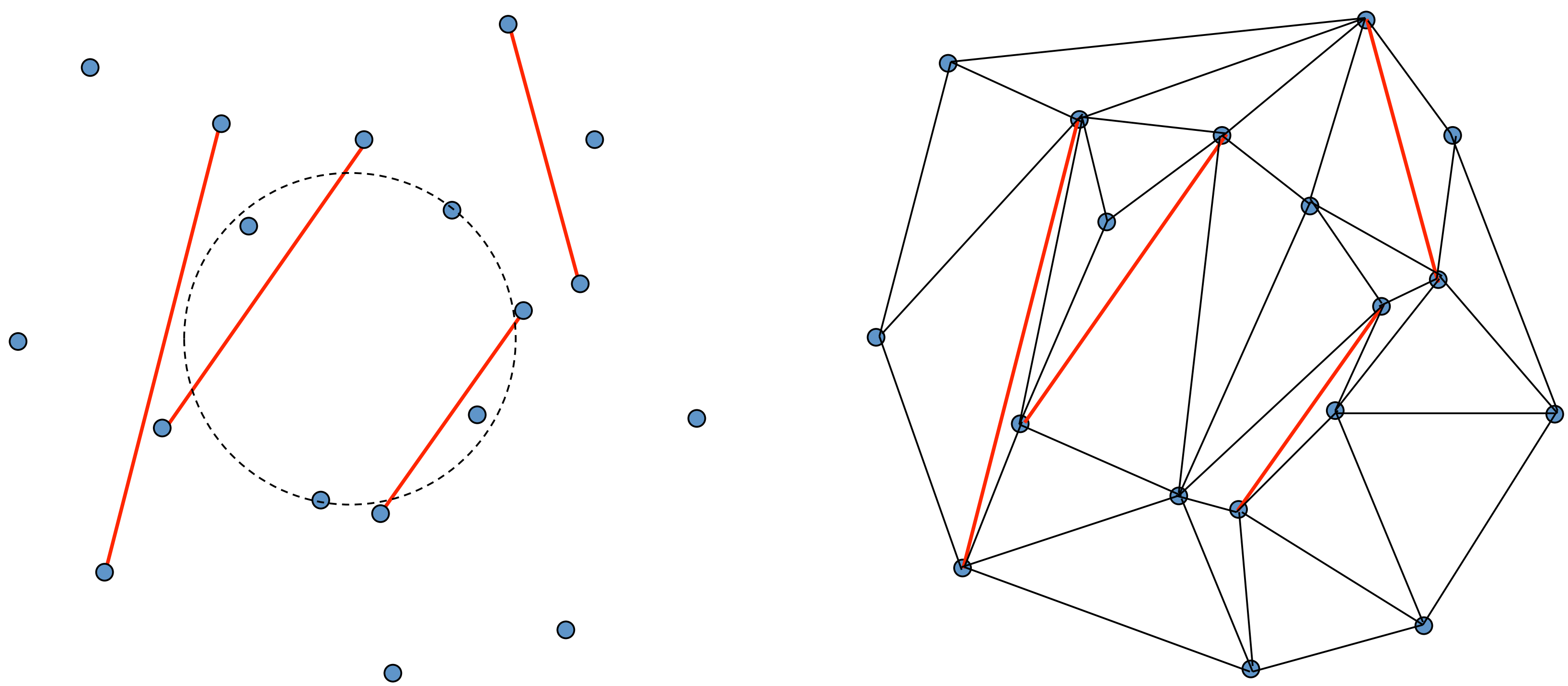
- Delaunay: empty circle



- Constrained Delaunay: empty circle but with given edges

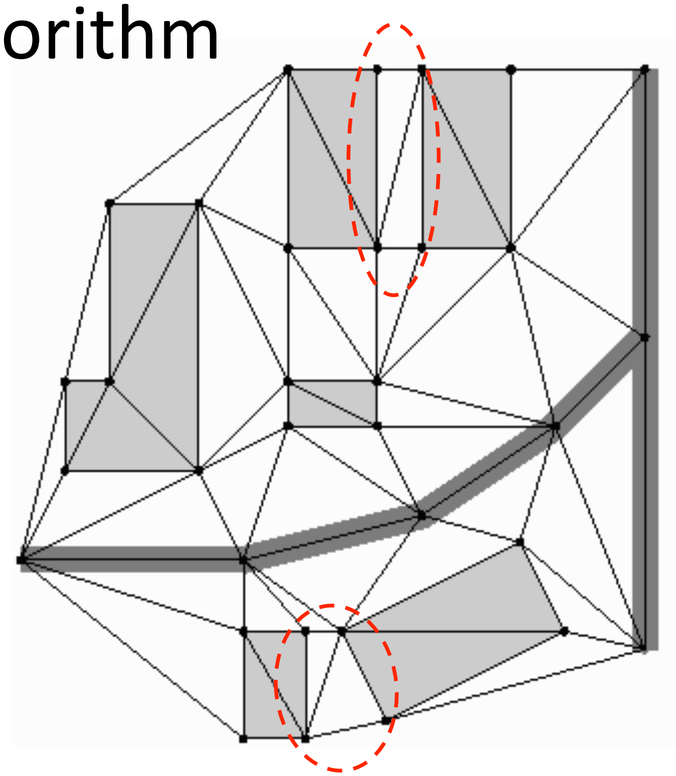


Constrained Delaunay Triangulation

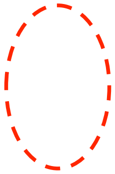


Constrained Delaunay Triangulation

- Can be constructed in O(*n* log *n*) time (Chew, ’87) with sweep algorithm
- Incremental also possible



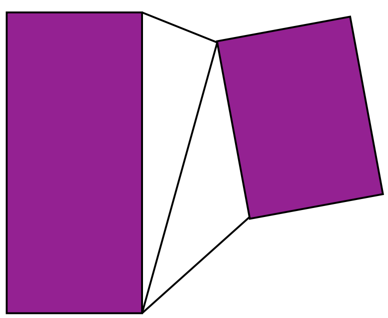
Usage:

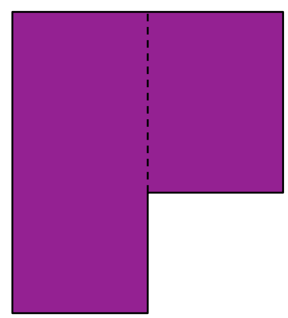


Places where aggregation is possible because objects are close enough and boundary length is reduced

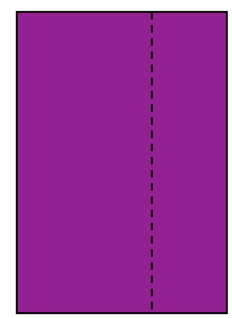
Aggregation buildings

- Flatten triangles in between
- Reorient buildings
- Test for possible conflicts





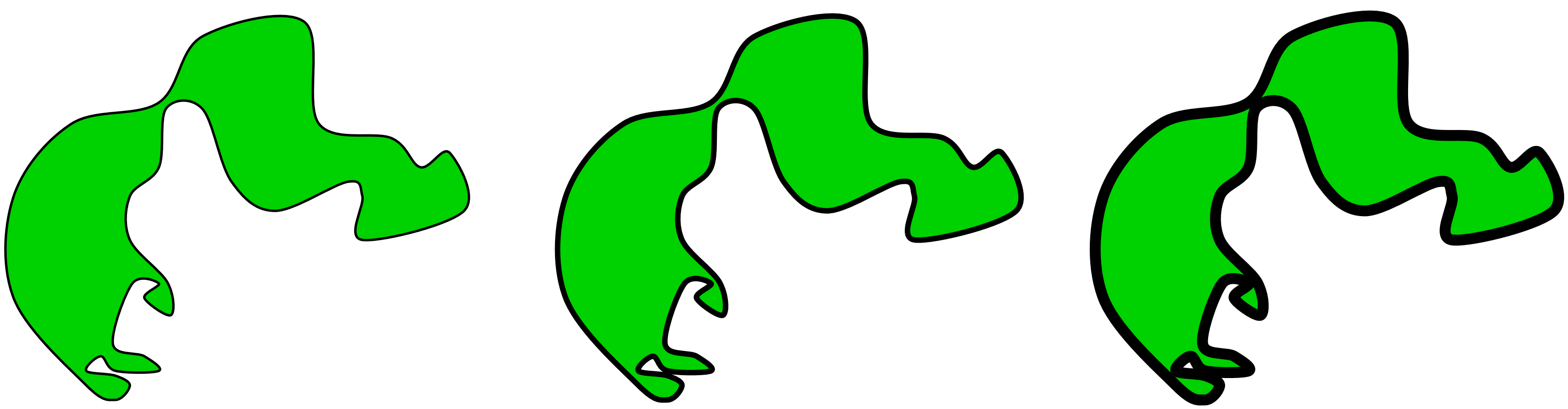
Area preservation and more or less shape preservation



Area preservation and more generalization

Shape change of natural objects

- No restrictions on shape needed
- Problem is self-coalescence, among which degree of detail



Shape change natural objects

- Using triangulations: constrained Delaunay
- Add triangles to polygon or remove them if that "improves" the shape