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The Mathematics of Information Technology and Complex Systems

Les mathématiques des technologies de l'information et des systèmes complexes

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Solutions to Practice Problems

Page 1. Question 4.

(a) If we look at the recursion tree, each leaf node represents a ~~s~~ subarray containing at most k nodes. On an average, there will be $\frac{n}{k}$ leaf nodes & the expected depth of the recursion tree is $\log_2 \frac{n}{k}$. Since each level takes $O(n)$ time to complete, total cost is $O(n \log_2 n/k)$ (expected).

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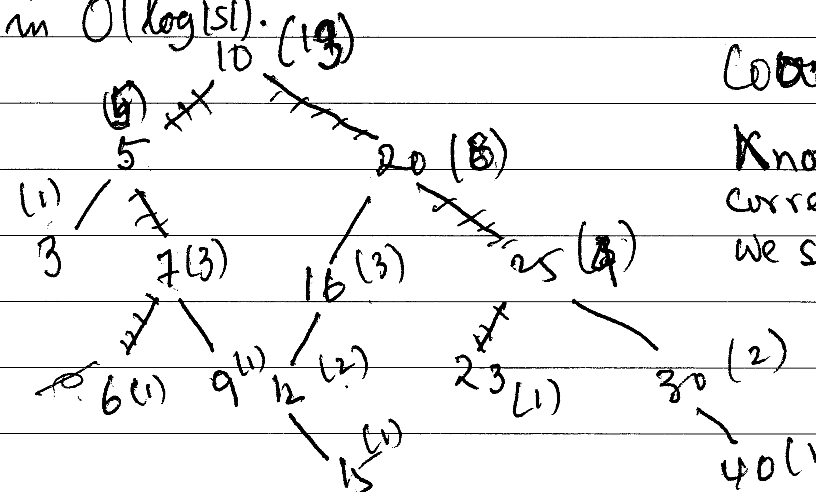
7. COUNT(x, y, S) : The B-trees (or bst) should be augmented to indicate the number of nodes stored in each subtree.
~~Given x & y, determine the paths from the root node to the node~~
Given x determine the subtrees that contain elements ≥ x. Then use y to determine the # of elements lying between x & y. This can be done in O(log|S|).



Count(5.5, 22, S)

Knowing the paths corresponding to 5.5 & 22, we see that there are 8 elements in between 5.5 & 22.

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1 (b) It is well known that it takes $\Omega(n \log n)$ time to determine if any two points in a set are equal.

2. Using Master theorem one can show that $T(n)$ is $\Theta(n^{\log_2 5})$ which is $\Theta(n^{2.\cdots})$. The running time is dominated by the divide step. $\Theta(n^{2.\cdots})$ # of leaf nodes are present in the recursion tree. Hence the programmer should focus on the basis & make it as efficient as possible.

Page 4

(a)
- Worst case lower bound is $\Omega(n)$
- Worst case upper bound is $O(n^2)$ since an ~~algo~~ algorithm already exists
- The algorithm A is not optimal. We need to show that the lower bound of P is $\Omega(n^2)$.

(d) Reverse the direction & apply DFS & show that only tree edges exist. The # of edges should not be more than $n-1$.

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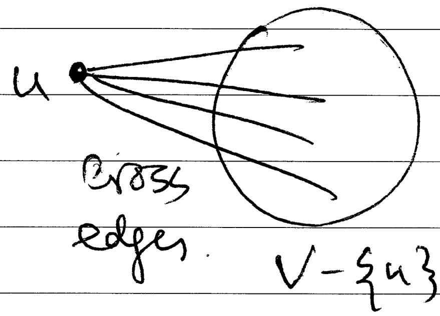
(e) ~~Use the idea that~~ Consider the ~~[illegible]~~ partition $(\{u\}, V-\{u\})$



The light edge in this case must be in every mst. (why?)

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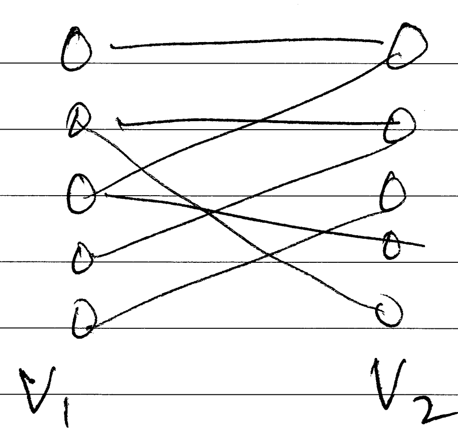
6. A bipartite graph $G=(V,E)$ has the following form



The vertex set V can be partitioned into two subsets V_1 & V_2 s.t. all the edges in E has one endpoint in V_1 & the other endpoint in V_2.

Apply DFS on G & ~~[illegible]~~ and obtain a partition.

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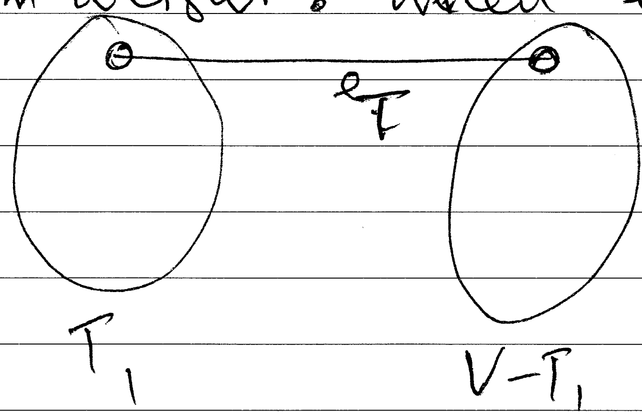
8. One can show that the ~~ma~~ mst is the spanning whose maximum (largest) edge weight is minimum over all spanning trees of G.

Let T be a mst of $G=(V,E)$. Let e_T be the edge with the maxi-mum weight. When e_T is removed



from T, we get a partition $(T_1, V-T_1)$. e_T is a cross edge. Since e_T is in a mst, e_T must be a light edge. Any other spanning tree must consist of a cross edge in $(T_1, V-T_1)$.

Other questions

Q: Is the largest weight edge in mst greater than b? Can be answered in $O(|V|+|E|)$ time by removing all the edges in E with cost $\geq b$. After the edges are removed, check if the graph is connected.

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~~6. Suppose $M \geq 200$ & suppose~~

6. Suppose $200 \leq M \leq 300$ & argue that in the optimal solution there must exist a coin of 200.

9. Construct a mst. Look at all the edges not selected as a mst edge. Use the ~~one~~ edge with the smallest weight and add it to the mst edge. Why?

11. Every vertex u has a weight w_u.

$$s = u_1 \to u_2 \to u_3 \to \cdots \to t = u_k \qquad p$$

Cost of the path p $= \sum_{i=1}^{t} w(u_i) + \sum_{i=1}^{t-1} d(u_i, u_{i+1})$

We can transform

$$s\; u_1' \to u_1'' \to u_2' \to u_2'' \to u_3' \to u_3'' \to \cdots$$

$$\left.\begin{aligned} d(u_1', u_1'') &= w(u_1) \\ d(u_1'', u_2') &= d(u_1, u_2) \quad \ldots \end{aligned}\right\} \; \& \; w(u_i') = w(u_i'') = 1$$

In this case the cost of the transformed path is still the same.

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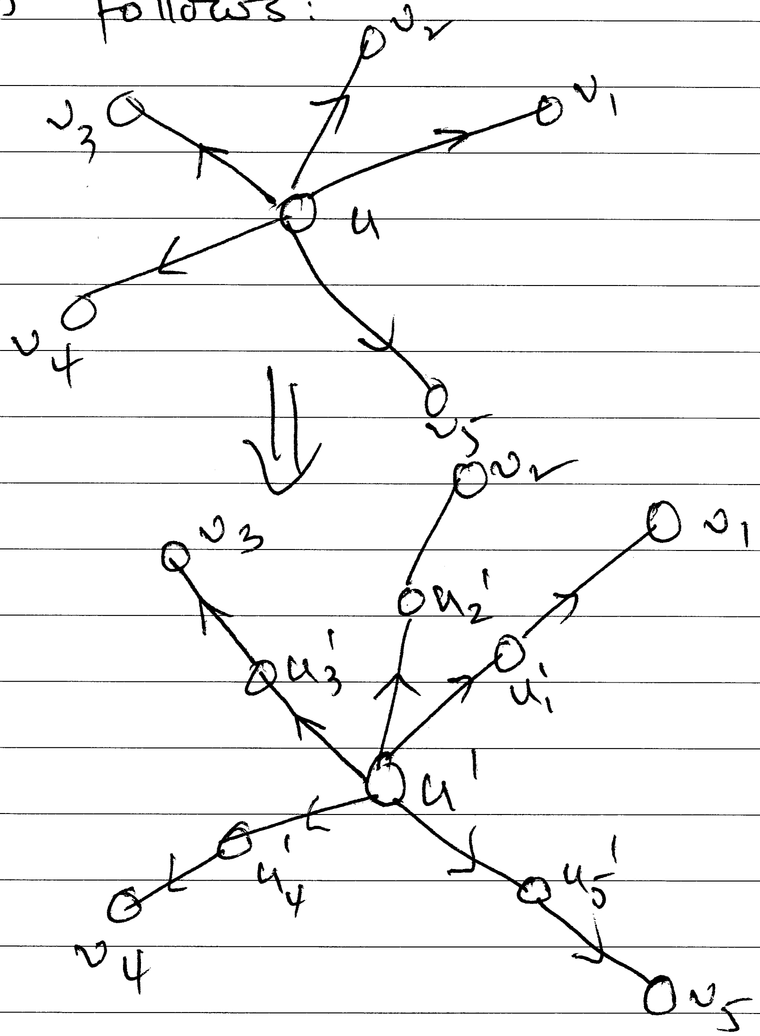
6 Thus in G, every vertex u is transformed as follows:



G':

where $d(u', u_i') = \omega(u)$ $\forall i$

& $d(u_i', v_i) = d(u, v_i)$ $\forall i$.

$\therefore$ new directed graph $G' = (V', E')$ has $|V| + |E|$ vertices & $|V| + 2|E|$ edges

Now Dijkstra's algorithm can be applied

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Problem 2. Partition the points into $\frac{n}{p}$ groups, & each group contains p elements.

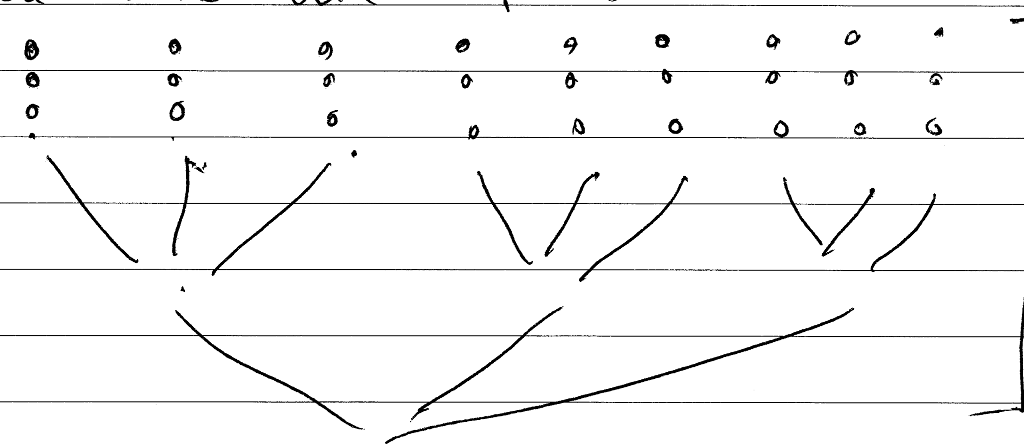
~~Each~~ Look at the following set up

$\underbrace{\quad}_{p}\ \underbrace{\quad}_{p}\ - - - - \underbrace{\quad}_{p}$ n/p group.

$\underbrace{\qquad\qquad}_{p}\ \underbrace{\quad}_{p}\ \underbrace{\quad}_{p}$

$\vdots$

The picture looks like $p = 3$



$\log_p n$

Each node is represented by a hardware
~~Each element can be~~ Extract-Min now
Costs $O(\log_p n) \Rightarrow O(n \log_p n)$ total time.

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Problem 3 Same as assignment 4 question 3(b) (determining the optimal ~~3~~ relation)

Problem 8 Each block (x_i, y_i, z_i) gives rise to three bases (x_i, y_i), (y_i, z_i) & (z_i, x_i) with heights z_i, x_i & y_i respectively. Let $(a_j, b_j)(c_j)$, ... denote a base of (a_j, b_j) with height c_j, $j = 1, 2, \dots, 3n$. Let $H(a_k, b_k)$ denote the height of the maximum tower with base (a_k, b_k).

$\therefore$ $H(a_k, b_k) = c_k$ if $\nexists$ any base (a_j, b_j) with $a_k > a_j$ and $b_k > b_j$ or $a_k > b_j$ and $b_k > a_j$.

$$= c_k + \max_{\substack{\forall \text{ base } (a_j, b_j) \\ \text{strictly smaller} \\ \text{than } (a_k, b_k)}} H(a_j, b_j).$$

Memoization method can now be employed. How can we enumerate the subproblems in increasing order?

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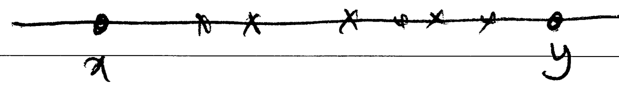
Problem 14.

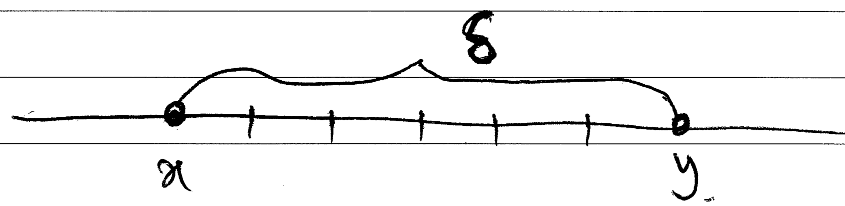
(a)



x & y are the two extreme points of S

(b) The problem is called the smallest gap problem. i.e find two points whose gap is the minimum. Clearly, these two points should be consecutive in the sorted list.

(d) Interesting problem. Let $\delta = \max_{z \in S} z - \min_{z \in S} z$



We subdivide the interval into $n-1$ subintervals of equal width. ($n=7$ in the above diagram). Determine ~~all~~ the subinterval ~~that~~ of each point of S. There will be a subinterval that contains two points (why? Pigeon-hole principle) These two points x & y satisfy the condition of the question.