Assignment 3 for CMPT 307 (2013)
October 15, 2013

These problems on shortest paths and minimum spanning trees will prepare you for Quiz 2, scheduled to be held on October 29, 2013.

1. Prove that the shortest paths from s to all the vertices of a connected undirected $G = (V, E)$ is a tree.

2. Consider the following algorithm for finding the shortest path from a node s to a node t in a directed graph with some negative edges. Add a large constant to each edge weight so that all the weights become positive, then run Dijkstra's algorithm starting at node s, and return the shortest path found to node t.

   Is this a valid method? (Hint: What happens when the graph has a negative cycle or doesn't have a negative cycle?)

3. Consider a directed graph in which the only negative edges are those that leave s; all other edges are positive. Can Dijkstra's algorithm, started at s, fail on such a graph?

4. Give an algorithm that takes as input a directed graph with positive edge lengths, and returns the length of the shortest cycle in the graph (if the graph is acyclic, it should say so). Your algorithm should take at most $O(|V|^3)$ time. (Hint: Try to report the shortest cycle containing node u, if it exists for each u.)

5. Give an $O(|V|^2)$ algorithm for the following task.

   - Input: An undirected graph $G = (V, E)$; edge length $l_e > 0$; and edge $e \in E$.
   - Output: The length of the shortest cycle containing edge $e = (u, v)$.

   (Hint: Find the shortest path between u and v that doesn't contain e.)

6. You are given a set of cities, along with the pattern of highways between them in the form of graph $G = (V, E)$. Each stretch of highway $e \in E$ connects two of the cities of l_e km distance apart. You want to go from

city s to city t. There is one problem: your car can only hold enough gas to cover L kilometers. There are no gas stations between cities. Therefore, you can only take a route if every one of its edges has length $l_e \leq L$. Given the limitation on your car's fuel tank capacity, show how to determine in $O(|V|+|E|)$ time whether there is a feasible route from s to t.

7. Given the above road network, the problem is to add an extra edge e' to E. There is a potential list E' of highways to be built. As a designer, you are asked to determine the road $e' \in E'$ whose addition to the existing network G would result in the maximum decrease then the driving distance between two fixed cities s and t. Give an efficient algorithm for solving this problem.

8. The following statements may or may not be correct. In each case, either prove it is correct or give a counterexample. $G = (V, E)$ is assumed to be undirected and connected. The edge weights are all all distinct unless otherwise stated.

   (a) If G has more than $|V| - 1$ edges, and there is a unique heaviest edge, then this edge cannot be a part of a minimum spanning tree.

   (b) If G has a cycle with a unique heaviest edge, then e cannot be a part of some minimum spanning tree.

   (c) If e is part of some MST of G, then it must be a lightest edge across some cut of G.

   (d) Let e be any edge of minimum weight in G. Then e must be part of some MST.

   (e) If the lightest edge in a graph is unique, then it must be a part of every MST.

   (f) If G has a cycle with with unique lightest edge e, then e must be part of every MST.

   (g) The shortest path tree computed by Dijkstra's algorithm is necessarily an MST.

   (h) Prim's algorithm works correctly when there are negative edges.