MACM 101 (D200)
Homework WHW 3
Due October 6, 2020; 12:30 pm

October 6, 2020

Exercises from the text (To be handed in).

(A) 2.1.2, 2.1.6

(B) 2.2.1, 2.2.4, 2.2.5

(C) 2.3.1, 2.3.3

(D) 2.4.3, 2.4.4

(E) 2.5.1, 2.5.5

(F) 2.6.3, 2.6.6

(G) 2.7.1, 2.7.3

Other Problems (Not To be handed in).

1. Give an example to show that

$$(\forall y)(\exists x)\ p(x, y) \leftrightarrow (\exists y)(\forall x)\ p(y, x)$$

   Solution: Suppose the open statement $p(x, y)$ is "$x.y = 0$" where the universe of x and y are the reals.

2. Suppose n is an arbitrary integer.

(a) Show that $n(n+1)$ is divisible by 2.
Solution: Proof by cases: If $n = 2t$, then

$$n(n+1) = 2t(2t+1) = 2(t^2+t)$$

is even.
If n=2t+1, then

$$n(n+1) = (2t+1)(2t+2) = 2(2t^2+3t+1)$$

is also even.
Therefore, whether n is even or odd, the product $n(n+1)$ is always even.

(b) Show that $n(n+1)(n+2)$ is divisible by 3!.
Solution: We can prove this in more than one way.

Proof by cases: Any integer can be expressed as $6t+u$ where is one of $0, 1, 2, 3, 4, 5$. Now

$$n(n+1)(n+2) = (6t+u)(6t+u+1)(6t+u+2)$$

which is divisible by 6 if $u(u+1)(u+2)$ is divisible by 6 (check). We prove this claim using an exhaustive proof. We show that for each value of u in $\{0, 1, 2, 3, 4, 5\}$, 6 divides $u(u+1)(u+2)$.

u	$u(u+1)(u+2)$
0	$0.1.2 = 0 = 6.0$
1	$1.2.3 = 6 = 6.1$
2	$2.3.4 = 24 = 6.4$
3	$3.4.5 = 60 = 6.10$
4	$4.5.6 = 120 = 6.20$
5	$5.6.7 = 210 = 6.35$

This completes the proof. This is a correct proof, but is not elegant.

Another proof by cases approach We have already seen that $n(n+1)$ is divisible by 2. We can also show that $n(n+1)(n+2)$ is divisible by 3. This can be done by showing that any integer of the type $n = 3t+u, u = 0, 1, 2$ is divisible by 3 (use arguments similar to the one described above). Since 2 and 3 do not have a common factor, therefore $n(n+1)(n+2)$ is divisible by $2 \cdot 3$.

3. (a) Prove that $\sqrt{7}$ is an irrational number.
Solution: We can prove this by contradiction. Suppose $\neg p$ is true, i.e.

$\sqrt{7}$ is rational. Therefore, we can use the fact that $\sqrt{7}$ can be expressed as $\sqrt{7} = \frac{a}{b}$ where integers a and b have no common factors. We can write $a^2 = 7b^2$. This implies that a^2 is divisible by 7. **Since 7 is a prime number, 7 divides a^2 implies 7 divides a.** Thus $a = 7 \cdot t$ for some integer t. Now $a^2 = 7b^2$ can be written as $49t^2 = 7b^2$. This means that 7 divides b^2 as well. Since 7 is a prime number, 7 divides b. We now arrive at a contradiction: We started with the fact that a and b have no common factor. We then showed that 7 is a common factor of a and b. This leads to the conclusion that $\neg p$ is false. This implies that $\sqrt{7}$ is an irrational number.

(b) Show where your arguments in (a) get violated if you want to show in a similar manner that $\sqrt{9}$ is an irrational number.
Solution: The arguments used above cannot be applied for the case of $\sqrt{9}$ since the highlighted statement above is not true for 9, since 9 is not a prime number. (9 divides 6^2 doesn't mean that 9 divides 6.)

4. Find a counterexample to the statement that every positive integers can be written as the sum of the squares of three integers. What is the smallest integer for which it is a counterexample.
Solution: We see that

- $1 = 1^2 + 0^2 + 0^2$
- $2 = 1^2 + 1^2 + 0^2$
- $3 = 1^2 + 1^2 + 1^2$
- $4 = 2^2 + 0^2 + 0^2$
- $5 = 2^2 + 1^2 + 0^2$
- $6 = 2^2 + 1^2 + 1^2$

We are unable to express 7 as the sum of the squares of three integers. Therefore, $n = 7$ is the smallest integer for which it is a counterexample.