MACM 101 Test 4
November 30, 2020.
Answer any 5 questions.

1. (10 Points) A particular state's license plates have 7 characters. Each character can be a capital letter, or a digit except for 0.

   (a) How many license plates are there in which no two adjacent characters are the same?
   Solution: Since digit 0 is not used, we have 35 different letters. The choice for the first position of the license plate is 35. The choice for each of the subsequent positions is only 34. For the i^{th} position, you cannot select the letter used at the $(i-1)^{th}$ position, $i \geq 2$. By the product rule, the answer is $35 \times (34)^6$.

   (b) How many license plates are there in which no character appears more than once and the first character is a digit?
   Solution: The number of choices for the first position is 9. For the remaining 6 positions, we have $P(34,6)$ choices. Therefore, the number of choices is $9 \times P(34,6)$

2. (10 points) Consider a deck of 40 cards consists of cards numbered $1, 2, \ldots, 10$ in red, yellow, green and blue. Find the number of ways one can select 5-card sequence (here order is important) when

   (a) all sequences with no card restrictions.
   Solution: # of sequences = P(40,5) (order is important).

   (b) all sequences with cards all the same color
   Solution: # of sequence = 4 $\times$ P(10,5).

   (c) all sequences with all the cards having distinct numbers.

   Solution: Let us count all 5-card sequences where all the cards have distinct numbers. The number of such sequences is $40 \times 36 \times 32 \times 28 \times 24$. After selecting the first card, we are not allowed to pick any card with the same number as the first card. We use the same strategy to select the remaining cards.

3. (10 points) Consider the 13-letter word **MASSACHUSETTS**.

   (a) Determine the number of different strings of length 13 that can be formed from all the letters of the word.

Solution: In MASSACHUSETTS, there are 1 M, 2 As, 4 Ss, 1 C, 1H, 1 H, 1 E and 2 Ts. The number of strings that can be formed is $\frac{13!}{2!.4!.2!}$.

(b) Determine the number of different strings that can be formed from all the letters of the word where no two S's can appear side by side.

Solution: We first note that there are $\frac{9!}{2!.2!}$ strings that do not contain S. We now insert 4 Ss to each string of 9 letters. Since no two Ss can appear together, 4 Ss have to be inserted in 10 positions, determined by a 9-letter string. There are $C(10,4)$ ways one can determine 4 positions for Ss. Therefore, the answer is $C(10,4) \times \frac{9!}{2!.2!}$.

4. (10 points) Consider a function from a set with three elements to a set with four elements.

   (a) Answer the following:

      i. How many such functions are there?
      Solution: $4 \times 4 \times 4 = 4^3$**.**

      ii. How many are one to one?
      Solution: $4 \times 3 \times 2$**.**

      iii. How many are onto?
      Solution: The answer is 0 (zero). If a function is onto, $size(domain) \geq size(codomain)$.

   (b) Formulate each of the above three problems as a "balls-in-bins" problem.
   Solution:

      i. # of balls (distinguishable) = 3; # of bins (distinguishable) = 4; any number of balls can be placed in any bin.

      ii. # of balls (distinguishable) = 3; # of bins (distinguishable) = 4; at most one ball can be placed in each bin.

      iii. # of balls (distinguishable) = 3; # of bins (distinguishable) = 4; each bin must have at least one ball.

5. (10 points)

   (a) Give combinatorial arguments to show that $C(n+1,k) = C(n,k) + C(n,k-1), 0 < m < n$.
   Solution: Discussed in the class. Combinatorial arguments are needed.

   (b) The first few numbers in the 17^{th} row of Pascal's triangle are: $1, 17, 136, 680, 2380, \ldots..$

i. What is the value of $C(17,4)$?
Solution: $\binom{17}{4} = 2380.$**.**

ii. What is the value of $C(18,3)$?
Solution: $\binom{18}{3} = \binom{17}{3} + \binom{17}{2} = 680 + 136$**.**

6. A store sells 6 varieties of doughnuts. Chocolate is one of the varieties sold. Solve the following two problems by first formulating each one of them as the number of integral solutions to an equation, as discussed in the class.

   (a) How many ways are there to select 14 doughnuts if at least 4 Chocolate doughnuts are selected?
   Solution: The answer is
   # of integral solution to
   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 14, x_i \geq 0, i = 1, 2, \ldots, 5, x_{chocolate} \geq 4$
   which is C(10+6-1,6-1).

   (b) How many ways are there to select 14 doughnuts if at most 4 Chocolate doughnuts are selected?
   Solution: The answer is
   (# of integral solution to
   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 14, x_i \geq 0, i = 1, 2, \ldots, 6$)
   minus
   the (# of integral solution to
   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 14, x_i \geq 0, i = 1, 2, \ldots, 5, x_{chocolate} \geq 5$)**.**
   The answer is C(14+6-1,6-1) - C(9+6-1,6-1).