Name: ———————————- **Student Id:** ——————— **Group:** ——

Midterm 1 (MACM101-D2)
February 3, 2014.
Test duration: 50 minutes

There are eight questions in this test. Answer questions worth 60 points.

1. (10 points) Consider selecting 4 objects from the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
   (a) How many ordered sequences without repetition can be chosen from A? **ans= P(8,4)**
   (b) How many ordered sequences with repetition can be chosen from A? **ans=** 8^4**; there are 8 choices for each position.**
   (c) How many unordered sequences without repetition can be chosen from A? **ans= C(8,4)**
   (d) How many unordered sequences with repetition can be chosen from A? $\binom{8+4-1}{8-1}$**; this r-combinations with repetitions.**
   (e) How many strictly increasing sequences can be chosen from A? $\{ <2, 4, 4, 7>$ is not a strictly increasing sequence.$\}$ **It is the same as the number of** $4-$**combinations without repetitions, since every such** $4-$**element combination, there is only one strictly increasing sequence. Hence the answer is C(8,4).**

2. (10 points) Consider a eight letter word *aeemrryt*.
   (a) How many different arrangements of these seven letters are there? **no constraint:** $\frac{8!}{2!2!}$
   (b) How many such arrangements are there that contain *eye*? **Arrangements with eye: use** $\{eye, a, r, m, r, t\} : \frac{6!}{2!}$
   (c) How many such arrangements are there that contain *eye* and *ram*? **Arrangements with eye and ram: use** $\{ram, eye, r, t\} : 4!$
   (d) How many such arrangements are there that do not contain either *eye* or *ram*? **Arrangements with neither eye nor ram** $= \frac{8!}{2!2!} - \frac{6!}{2!} - \frac{6!}{2!} + 4!$

3. (10 points) Suppose you are interested in buying pizzas, and each pizza gets up to 10 toppings from 10 possible types (no double toppings).

(a) How many ways can you choose toppings for a pizza?
Each pizza can have 0 to 10 toppings. Therefore, there are 2^{10} different pizzas.

(b) How many ways can you choose two pizzas with the same toppings?
It is the same as the number of different pizzas. The answer is 2^{10}.

(c) How many ways can you choose toppings for two pizzas?
Since we can have two pizzas with the same toppings, the problem is combination with repetitions. There are 2^{10} different pizzas, and we need to select two of them where repetitions are allowed. Therefore, the answer is $\binom{2-1+2^{10}}{2^{10}-1} = \binom{2^{10}+1}{2}$

(d) How many ways can you choose toppings for n pizzas?
We now select n pizzas from 2^{10} different toppings ones. The answer is $\binom{n-1+2^{10}}{2^{10}-1}$.

4. (10 points) We have seen that the following problem captures many counting problems.
Determine the number of non-negative integer solutions to

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 0, i = 1, 2, ..., k.$$

Formulate each of the following problems as a variation of the above problem.

(a) Determine the number of ways to select k objects with replacements from a set of n objects.
Ans:

$$x_1 + x_2 + ... + x_n = k$$
$$x_i \geq 0, i = 1, 2, ..., n.$$

(b) Determine the number of ways to place n nondistinguisable balls in k boxes.
Ans:

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 0, i = 1, 2, ..., k.$$

(c) Determine the number of ways to distribute n pennies to k kids such that each kid gets at least 1 penny.
Ans:

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 1, i = 1, 2, ..., k.$$

(d) Determine the number of times the following pseudocode prints the PRINT statement:

```
for i = 1 to 20
   for j = i to 20
      for k = j to 20
         PRINT(i,j,k)
```

Ans:

$$x_1 + x_2 + ... + x_{20} = 3$$
$$x_i \geq 0, i = 1, 2, ..., 20.$$

Once the three integers are selected, we assign the largest one to k, the smallest one to i and the third one to j.

5. (10 points) For each of the following deductive arguments, translate it into propositional logic notation using logical operators ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$), and then identify the premises (hypotheses) and conclusions, followed by a validity test.

(a) Either $John$ or $Mary$ is telling the truth. Either Tim or $Cathy$ is lying. Thus either $John$ is telling the truth or Tim is lying.
Ans: It is valid
The arguments can be written as:

p: $John$ is telling the truth.
q: $Mary$ is telling the truth.
r: Tim is telling the truth.

1. $p \vee q$
2. $\neg r \vee \neg q$

Therefore, $p \vee \neg r$.

From (1), either p is true, or q is true. If q is true, $\neg r$ is true (from (2)). In this case the conclusion is true. If q is false, p is true. In this case also the conclusion is true.

(b) The main course will be chicken or fish, but not both. The vegetable will be carrots or broccoli, but not. We will not have both chicken as a main course and broccoli as a vegetable. Therefore, we will not have

both fish as a main course and carrots as a vegetable.
Ans: It is not valid.
The arguments can be written as:

p: The main course will be chicken.
q: The main course will be fish.
r: Carrots will be served.
s: Broccoli will be served.

$$
\begin{array}{ll}
1. & p \oplus q \\
2. & r \oplus s \\
3. & \neg(p \wedge s) \\
\hline
\multicolumn{2}{l}{\text{Therefore, } \neg(q \wedge r)}
\end{array}
$$

Let $p = false$, $s = false$. Therefore from (1) and (2), we can conclude that q and r are true. Now conditions (1), (2) and (3) are all satisfied, But the conclusion is false. We now have a contradiction. Therefore, the above arguments do not imply the conclusion.

6. (5 points) Let p, q be primitive statements for which the implication $p \to q$ is false. Determine the truth values for each of the following.
(a) $p \wedge q$, **(b)** $\neg p \vee q$, **(c)** $q \to p$ **(d)** $\neg q \to \neg p$.
Answer:
Since $p \to q$ is false, we know that p is true and q is false. Therefore,

- $p \wedge q$ is false.
- $\neg p \vee q$ is false.
- $q \to p$ is true.
- $\neg q \to \neg p$ is false.

7. (5 points) Determine a truth value assignment, if any, for the primitive statements p, q, r, s, t that make each of the following statements false. I have used $\oplus$ to indicate **exclusive-or** connective.

   (a) $[(p \wedge q) \wedge r] \to (s \vee t)$
   Answer:
   Clearly, we need to set the left side of the implication statement to false. This is achieved when p, q and r are set to true, and s and t are set to false.

   (b) $[p \wedge (q \wedge r)] \to (s \oplus t)$
   Answer:

We can make the implication false by setting p, q and r true and setting $s \oplus t$ false. We know that $s \oplus t$ is false when both s and t are true, or both s and t are false.

8. (10 points) We have seen in the class that any compound statement can be transformed to an equivalent statement involving only $\neg$, $\vee$ and $\wedge$ connectives. Show that $p \oplus q$ is equivalent to $(p \vee q) \wedge \neg(p \wedge q)$. Proving equivalence through truth table is not acceptable.
   Answer:
   We know that $p \oplus q$ is true when p is true and q is false, or p is false and q is true. Using the techniques developed in the class, we know that $p \oplus q$ is equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$. We have shown in the class that $(p \wedge \neg q) \wedge (\neg p \wedge q)$ is then equivalent to $(p \vee q) \wedge \neg(p \wedge q)$.