Sept23- Solutions

A: Quantified propositions → English statements.

1. **Question 6, section 1.4 of Rosen**
   Let $N(x)$ be the statement "x has visited North Dakota" where the domain consists of the students in your school. Express each of these quantifications in English.
   a) $\exists x\ N(x)$
      Solution: "There is a student in my school who has been to North Dakota."
   b) $\forall x\ N(x)$
      Solution: "All students in my school have been to North Dakota."
   c) $\neg\exists x\ N(x)$
      Solution: "No one in my school has been to North Dakota."
   d) $\exists x\ \neg N(x)$
      Solution: "There is a student in my school who has never been to North Dakota."

B: English statements → quantified propositions.

1. Question 10, section 1.4 of Rosen.
   Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.
   (a) A student in your class has a cat, a dog, and a ferret.
       Solution: $\exists x\ (C(x) \wedge D(x) \wedge F(x))$
   (b) All students in your class have a cat, a dog, and a ferret.
       Solution: $\forall x\ (C(x) \wedge D(x) \wedge F(x))$
   (c) Some student in your class has a cat and a ferret, but not a dog.
       Solution: $\exists x\ (C(x) \wedge F(x) \wedge \neg D(x))$
2. Question 42, section 1.4 of Rosen.
   Express each of these system specifications using predicates, quantifiers, and logical connectives.
   a) Every user has access to an electronic mailbox.
      Solution: We can limit the domain to all users of the system, and introduce the E predicate:

      $$E(x, e) := \text{user } x \text{ has access to electronic mailbox } e$$

      Therefore, we're quantifying over two domains in the E predicate, the domain of system users and the domain of electronic mailboxes. This requires that we quantify over both domains:

      $$\forall x\ \exists e\ M(x, e)$$

   b) The system mailbox can be accessed by everyone in the group if the file system is locked.
      Solution: You may be tempted to create a predicate quantifying over all file systems here, but we're referring to a single file system, in the definite sense. It's the file system state that is uncertain, and which therefore needs to be quantified over. The states it can have are: locked, unlocked, . . . (possible further states of which we have no knowledge). So, we introduce the predicate

      $$F(s) := \text{ the file system is in state } s$$

      For users in the group, we have the predicate

      $$G(x) := \text{ user } x \text{ is in the group}$$

      and for system mailbox access, we have

      $$SM(x) := \text{ user } x \text{ can access the system mailbox}$$

      Now to write the actual statement. We want to ensure that the user x is in the group, and we want the condition that the file system is locked to hold before we assert that users in the group can access the system mailbox. This leads us to

      $$\forall x\ G(x) \wedge F(\text{locked}) \rightarrow SM(x)$$

      Notice that "locked" is a constant, representing the "locked" state of the filesystem (indeed, there is no propositional variable named "locked" we've quantified over! Any variable we introduce must be quantified).

C: Contradicting quantified propositions.

1. Question 36, section 1.4 of Rosen.
   Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.
   a) $\forall x\ (x^2 \neq x)$
   Solution: $x = 1$ and $x = 0$ are all the counterexamples, which is to say that $x^2 = x$ in either case.
   b) $\forall x\ (x^2 \neq 2)$
   Solution: $x = \sqrt{2}$ and $x = -\sqrt{2}$ are the only two counterexamples.
   c) $\forall x\ (|x| > 0)$
   Solution: $x = 0$ is the one counterexample.

D: Validity of arguments

Decide whether the following argument is valid. If it is valid argument, give a formal proof (i.e. justify which rules need to be applied to premises). If the argument is invalid, show that it is invalid by finding an appropriate assignment of truth values to the propositions.

1. $$p, p \to q, s \vee r, r \to \neg q. \text{ Prove: } \quad (s \vee t).$$
   p is true and $p \to q$ is true: imply q is true. (Modus Ponens)
   $r \to \neg q$ is true and $\neg(\neg q)$ is true: imply $\neg r$ is true. (Modus Tollens)
   $\neg r$ is true: implies r is false. (By definition)
   r is false and $s \vee r$ is true: imply s is true. (Disjunctive Syllogism)
   s is true: implies $s \vee t$ is true. (Disjunctive amplification)

2. $$\neg(r \vee s), \quad \neg p \to s, \quad p \to q. \quad \text{Prove:} \quad q$$
   $\neg(r \vee s) \Leftrightarrow \neg r \wedge \neg s$ (De Morgan's Law)
   $\neg r \wedge \neg s$ is true: implies $\neg r$ and $\neg s$ are true. (Conjunctive simplification)
   $\neg s$ is true and $\neg p \to s$ is true imply $\neg\neg p$ (Modus Tollens)
   p is true and $p \to q$ is true: q is true. (Modus Ponens)

3. $$p \to (q \to r), \quad p \vee s, \quad t \to q, \quad \neg s. \text{ Prove: } \quad \neg r \to \neg t.$$
   $\neg s$ is true and $p \vee s$ is true: p is true. (Disjunctive Syllogism)
   p is treu and $p \to (q \to r)$ is true: imply $q \to q$ is true. (Modus Ponens)
   $t \to q$ is true and $q \to r$ is true: imply $t \to r$. (Law of Syllogism)
   $t \to r \Leftrightarrow (\neg t \vee r) \Leftrightarrow (r \vee \neg t) \Leftrightarrow \neg r \to \neg t$. (By definition)