MACM 101 Final Exam

Name:	
Student Number:	
Signature:	

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13	/10
TOTAL	/100

There are a total of 100 points possible on this exam.

1. (6 points) Consider the statement "If x is a perfect square and x is even, then x is divisible by 4".

   (a) Designate propositional variables to stand for the three conditions about x mentioned in the statement.

   (b) Write the statement formally in terms of these propositions.

   (c) State the contrapositive of your answer in part (b), both in terms of your propositional variables and in colloquial terms.

2. (10 points) Six different numbers were chosen at random from the numbers 1 through 49. The winning combinations do not depend on the order in which these numbers are drawn.

   (a) How many different lottery outcomes are possible?

   (b) A jackpot prize occurs if all numbers are chosen correctly. What is the probability of winning the jackpot?

   (c) If you choose five out of six correctly, you share the second prize. What is the probability of winning the second prize?

3. (10 points)

(a) In how many ways can the letters in UNUSUAL be arranged?

(b) For the arrangements in part (a), how many have all three U's together?

(c) How many of the arrangements in part (a) have no two consecutive U's.

4. (6 points) For any positive integer n show by using the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} = 1.$$

5. (4 points) There must be something wrong with the following induction proof; What is it?

   Theorem: For all positive integers n, $2^{n-1} = 1$.

   Proof. If $n = 1$, $2^{n-1} = 2^{1-1} = 2^0 = 1$. Suppose that the theorem is true for all $n \leq k$. Now we have

$$2^{(k+1)-1} = 2^k = \frac{2^{k-1}.2^{k-1}}{2^{k-2}} = \frac{1 \times 1}{1} = 1.$$

   Therefore, the theorem is true for $n = k + 1$ as well. Hence the theorem is true for all positive integers (Using the principle of strong mathematical induction) .

6. (10 points) Prove by induction the following generalization of De Morgans law to n sets.

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$$

7. (5 points) How many ordered pairs of integers are needed to guarantee that there are two ordered pairs (a_1, b_1), (a_2, b_2) such that $a_1 + a_2$ is even and $b_1 + b_2$ is even?

8. (8 points) There are 51 houses on a street. Each house has an address between 1 and 100 inclusive.

   (a) Show that at least two houses have addresses that are consecutive.

   (b) Show that at least two houses have addresses such that the sum of their addresses are divisible by 100.

9. (10 points)

(a) Let R be a relation defined on $A \times B$ such that $((a, b), (x, y)) \epsilon R$ if and only if $a \leq x$ and $b \leq y$. Show that R is a partial order relation.

(b) Draw the Hasse diagram for the poset $(A \times B, R)$ where $A = \{1, 2, 3\}$ and $B = \{2, 3\}$ and R is defined as in part (a).

10. (5 points) Let the relation R be reflexive and transitive on A. Show that $R \cap R^{-1}$ is an equivalence relation on A.

11. (8 points) Determine whether each of the following statements is true or false. For each false statement give a counterexample.

    (a) If $f : A \to B$ and $(a, b), (a, c) \epsilon f$, then $b = c$.

    (b) If $f : A \to B$ is a one-to-one correspondence and A and B are finite, then $A = B$.

    (c) If $f : A \to B$ is one-to-one, then f is invertible.

(d) $f : A \to B$ and $A_1, A_2 \subseteq A$, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

(e) $f : A \to B$ and $B_1, B_2 \subseteq B$, then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

12. (8 points) Provide a recursive definition for each of the following languages $A \subseteq \Sigma^*$ where $\Sigma = \{0, 1\}$.

    (a) $x \epsilon A$ if (and only if) the number of 0's in x is even.

    (b) $x \epsilon A$ if (and only if) $x = x^R$ where x^R is the reversal of x. (The reversal of 101100 is 001101.)

13. (10 points)

Let $\mathcal{I} = \{0, 1, 2\}$ and $\mathcal{O} = \{0, 1\}$ be the input and the output alphabet respectively. A string $x \epsilon \mathcal{I}^*$ is said to have the odd parity if it contains an odd number of 1's and odd number of 2's. Construct a finite state machine that recognizes all nonempty string of odd parity.