MACM 101 (Fall 2019)
Final (December 16, 2019)
Total marks = 80
Time: 12 to 3pm

A: Multiple choice (10 points)

1. Which of the following questions has the solution $\frac{15!}{10!}$:
   (a) How many ways can 5 out of 15 people be arranged in a lineup?
   (b) How many ways can a committee of 5 people be formed with 15 people?
   (c) How many ways are there to distribute 15 different books to 5 children?
   (d) How many ways are there to distribute 10 identical objects into 6 distinct boxes?

2. Let $f : A \to B$ be a function where $a \in A$, and $b \in B$. Which of the following is the definition of an onto function?
   (a) $\exists a \forall b \ f(a) = b$
   (b) $\forall a \forall b \ f(a) = b$
   (c) $\exists \neg b \forall a \ f(a) = b$
   (d) $\forall b \exists a \ f(a) = b$

3. The binary relation $R = \{(0,0),(1,1)\}$ on $A = \{0,1,2,3\}$ is
   (a) Reflexive, Not Symmetric, Transitive
   (b) Not Reflexive, Symmetric, Transitive
   (c) Not Reflexive, Not Symmetric, Not Transitive
   (d) Reflexive, Symmetric, Transitive

4. Which one is divisible by 4 where m is an arbitrary positive integer?
   (a) $5m^2 + 2$
   (b) $3m + 1$
   (c) $m^3 + 3m$
   (d) None of the above

5. What is the induction hypothesis assumption for the inequality $m! > 2^m$ where $m \geq 4$?
   (a) For $m = k, \ (k+1)! > 2^k$ holds
   (b) For $m = k, \ k! > 2^k$ holds
   (c) For $m = k, \ k! > 2^{k+1}$ holds
   (d) None of the above

B: Rest of the questions

1. (5 points) Six different numbers were chosen at random from numbers 1 through 49. The winning combinations do not depend on the order in which these numbers are drawn

   (a) How many different lottery outcomes are possible?

   (b) A jackpot prize occurs if all numbers are chosen correctly. What is the probability of winning the jackpot?

   (c) If you choose five out of six correctly, you share the second prize. What is the probability of winning the second prize?

2. (5 points) Formulate the following counting problems to an equivalent "balls in bins" or "finding the number of nonnegative integral solutions".

   (a) How many terms are there in the expansion of $(w+x+y+z)^{100}$?

   (b) How many times the print statement of the following piece of code is executed?

   ```
   for i = 1 to 100 do
      for j = 1 to i do
         for k = 1 to j do
            for l = 1 to k do
               print()
   ```

3. (5 points) For any positive integer n show by the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} = 1$$

4. (5 points) Consider the statement "If x is a perfect square and x is even, then x is divisible by 4".

   (a) Write the statement formally using propositional variables.

   (b) State the contrapositive equivalent of your answer in part (a).

   (c) Prove the statement using the direct proof.

5. (5 points) Show that if A, B and C are arbitrary sets then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

   (a) by showing that each side is contained in the other; and

   (b) by using a membership table.

6. (5 points) Suppose that you are given two integers, and you know the prime factorization of one of them. Describe a way of computing the greatest common divisor of these numbers. Your answer by first computing the prime factorization of the other number is not acceptable.

7. (5 points)
   (a) Write a recursive definition of Fibonacci sequence F_n.
   (b) Describe the Euclidean Algorithm to compute the greatest common divisor of two consecutive Fibonacci numbers F_8 and F_9. How many steps does the algorithm take to get the answer?
   (c) How many steps are needed in computing $GCD(F_{n+1}, F_n)$?

8. (5 points) Define the well-ordered property of a set. Determine which of the following sets S is well ordered.
   (a) the set of integers
   (b) the set of integers greater than -100
   (c) the set of positive reals
   (d) the set of positive integers

9. (5 points) Prove by induction the following:

   $$\text{If } a_1 = 1 \text{ and } a_n = a_{n-1} + n, n \geq 2, \text{ then } a_n = n(n+1)/2, n \geq 1.$$

   The proof must include, among others, formal statement of the problem, basis and the induction hypothesis.

10. (5 points) Determine whether each of the following statements is true or false. For each true statement give justification. For each false statement give a counter example.
    (a) If $f : A \to B$ and $(a,b),(a,c) \in f$, then $b = c$.
    (b) If $f : A \to B$ is a one-to-one and onto function, and A and B are finite, then $|A| = |B|$.
    (c) If $f : A \to B$ and $A_1, A_2 \subseteq A$, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.
    (d) If $f : A \to B$ and $B_1, B_2 \subseteq B$, then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

11. (5 points) How many ordered 2-tuples of integers are needed to guarantee that there are two ordered 2-tuples $(a_1,b_1),(a_2,b_2)$ such that $a_1 + a_2, b_1 + b_2$ are all even.

12. (5 points) There are 51 houses on a street. Each house has an address between 1 and 99 inclusive.
    (a) Show that at least two houses have addresses that are consecutive.
    (b) Show that at least two houses have addresses such that the sum of their addresses are divisible by 100.

13. (a) (5 points) Write down the definition of what it means for a collection C of subsets of A to be a partition of A?

(b) The set $C = \{\{1,3,5\},\{2,4\},\{6\}\}$ is a partition of $\{1,2,3,4,5,6\}$. Give a graphical representation of the equivalence relation determined by the partition.

14. (5 points)

(a) Let R be a relation on $A \times B$ such that $((a,b),(x,y)) \in R$ if and only if $a \leq x$ and $b \leq y$. Show that R is a partial order relation.

(b) Draw the Hasse diagram for the poset $(A \times B, R)$ where $A = \{1,2,3\}$ and $B = \{2,3\}$ and R is defined as in part (a).