MACM 101 D1 Section Final Exam
April 22, 2014.

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Student Number:	
Signature:	

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This examination has two parts. All the questions in Part A need to be answered. You need to answer questions worth 60 marks in Part B.

Part A

1. (25 points) Multiple choice questions

   (a) The implication $(\neg q \vee p) \rightarrow q$ is:

   i. a tautology
   ii. true for exactly one truth assignment to the variables p and q.
   iii. false for exactly one truth assignment to the variables p and q.
   iv. true whenever p is true, but false otherwise.
   v. true whenever q is true, false otherwise.

   (b) Let $F(x, y)$: "x can fool y". What would be the appropriate equivalent of the statement "John can fool Mary, but Mary cannot fool John "?

   i. $\neg F(John, Mary)$
   ii. $\neg F(Mary, Fred)$
   iii. $\neg F(John, Mary) \wedge \neg F(Mary, John)$
   iv. $\neg F(John, Mary) \wedge F(Mary, John)$
   v. $\neg(F(John, Mary) \wedge F(Mary, John))$

   (c) The function $f : A \rightarrow B$ maps every element of A to an element of B. Suppose $|A| > |B| > 0$. From this information, we can conclude that:

   i. f is injective but not surjective
   ii. f is surjective but not injective
   iii. f is injective but not necessarily surjective
   iv. f is surjective but not necessarily injective
   v. none of the above

   (d) Which of the following statements would be least likely to appear in an inductive proof as the induction hypothesis?

   i. "Assume that $S(k)$ is true for some fixed but arbitrary $k \in \mathbb{N}$".
   ii. "Assume that $S(k)$ is true for all $0 \leq k < n$".
   iii. "Assume that $S(k)$ is true for all $0 \leq k \leq n$."
   iv. "Assume that $S(k)$ is true for all for all natural numbers k."

   (e) Suppose that a committee of 5 people from a group of 7 women and 9 men is to be formed such that at least one woman serves on a committee. How many such committees can be formed?

i. $\binom{7}{5}$
ii. $\binom{7}{1} \times \binom{15}{4}$
iii. $\binom{16}{5} - \binom{9}{5}$
iv. $\binom{16}{5} - \binom{7}{5}$
v. none of the above

(f) Two labeled fair dice are rolled. What is the probability that the product of the two spots is odd?

$(i)\ \frac{1}{2} \quad (ii)\ \frac{1}{4} \quad (iii)\ \frac{2}{3} \quad (iv)\ \frac{3}{4} \quad (v)\ \frac{1}{8}$

(g) A person is positioned at the origin of 2-dimensional coordinate system, and need to reach the point (4,5). The movement is restricted by one-unit steps in only the directions of positive x and y. Under these conditions, in how many ways can it make the trip to the destination point $(4, 5)$?

$(i)\ 9 \quad (ii)\ 20 \quad (iii)\ 9! \quad (iv)\ \frac{9!}{4!5!} \quad (v)\ \frac{20!}{4!5!}$

(h) How many ordered 4-tuples (w, x, y, z) are solutions to the equation $w + x + y + z = 50$ if each of w, x, y and z is to be positive multiple of 5?

$(i)\ \binom{33}{3} \quad (ii)\ \binom{50}{3} \quad (iii)\ \binom{10}{3} \quad (iv)\ \binom{13}{3} \quad (v)$ none of the above

(i) Consider the function $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{0, 1, 2, 3, 4\}$. How many such functions have the property that $|f^{-1}(\{3\})| = 3$?

$(i)\ \binom{7}{3} \quad (ii)\ \binom{4}{3} \times \binom{7}{3} \quad (iii)\ 4^4 \times \binom{7}{3} \quad (iv)$ none of the above

(j) A set is well ordered if every nonempty subset of this set has a least element. Determine which of the following sets S is well ordered.

i. the set of integers
ii. the set of integers greater than -100
iii. the set of positive rationals
iv. the set of positive rationals with denominator less than 100.

2. (15 points) TRUE/FALSE QUESTIONS.

   (a) $|A \cup B| + |A \cap B| = |A| + |B|$

   (b) There are exactly 130 integers between 1 and 1000 (inclusive) which are divisible 7 but not by 11.

   (c) The propositional expression $[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$ is a tautology.

   (d) The prefix relation on strings is a partial order. (We know that y is a prefix of x if there exists strings z such that $x = yz$. Note that when $x =$ **binary** , the prefixes of x are {b, bi, bin, ..., binary}.

   (e) If R is reflexive and transitive, $R \cap R^{-1}$ is an equivalence relation.

   (f) For every set A, there exists onto functions $f : A \to 2^A$.

   (g) Every function is a relation.

   (h) $\forall x \in \mathbb{R}, \lfloor \frac{x}{2} \rfloor + \lfloor \frac{x+1}{2} \rfloor = \lfloor x \rfloor$.

   (i) The statement $\exists x \forall y P(x, y) \Rightarrow \forall y \exists x P(x, y)$

   (j) Among any 800 distinct integers chosen from the set $\{n : (n \in \mathbb{N}) \wedge (1 \leq n \leq 1600)\}$ at least two must be consecutive.

Part B

1. (15 points) This problem is on "placing balls in bins".

   (a) We are interested in placing m balls in n bins. Placement is either unrestricted (some bins may be left empty), one-to-one (each bin has at most one ball), or onto (each bin must have at least one ball). Determine the number of ways of placing balls in bins for each of the following cases:

Constraints	unrestricted	one-to-one	onto
balls labeled bins labeled			
balls unlabeled bins labeled			
balls labeled bins unlabeled			

   (b) Transform each of the following counting problems to an equivalent "placing balls in bins" problem. You need to specify the number of balls and bins, labeled or unlabeled, in each case along with the placement category.

   i. The number of integer solutions of the equation

$$x_1 + x_2 + \ldots + x_n = r,\ x_i \geq 1,\ 1 \leq i \leq n.$$

   ii. Determine the number of times the following pseudocode prints the PRINT statement:

```
for i = 1 to 25
   for j = i to 25
      for k = j to 25
         PRINT(i,j,k)
```

   iii. Determine the number of functions of the type $f : A \to B$, $|A| = m$ and $|B| = n$.

iv. Determine the number of outcomes of throwing 3 coins (indistinguishable).

v. Count the number of ways of writing t as a sum of n positive integers where different orderings are counted as different. ($t = 5$ can be written as a sum of $n = 3$ integers as: $3+1+1, 1+3+1, 2+1+2, ...$)

2. (5 points) For the following formulas, let the universe be $\mathbb{R}$. Translate each of the following sentences into a formula using quantifiers.

   (a) There is no largest number.

   (b) There is no smallest positive number.

   (c) Between any two distinct numbers, there is a third number not equal to either of them.

3. (4 points) Give reasons for each step in the proof of the following. A proof based on the truth table is not acceptable.

$$[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$$

4. (5 points) We wish to define what it means for a string of parentheses of two types ( ) and [ ] to be balanced. Intuitively, each left parenthesis should match with a right parenthesis of the same type to the right, and the matched pairs must be well-nested Here are some examples of balanced strings:

   λ (the empty string)
   ( )   [ ]   ([ ])   [ ]( )   [( )( )]
   [([ ])([ ])]   [[(( ))( )[ ]]( )]( )

   Here are some examples of strings that are not balanced:

   ( ]   [[ ]   [(])   )([[[

   Give a recursive definition of balanced that captures the definition.

5. (4 points) Recursively define the set $S = \{n^2 | n \in \mathbb{Z}^+\}$, that is $S = \{1, 4, 9 \ldots\}$. What should we do to generate the elements in increasing sequence? This means that after generating 4^2, we need to generate 5^2.

6. (8 points) Answer any two of the following three.

   (a) Suppose $a, b, c, d \in \mathbb{Z}$. Prove, using a direct proof approach, that if $a|b$ and $c|d$, then $ac|bd$.

   (b) Use the contrapositive proof method to show "if the average of n numbers (all of them are not the same) is 100, then one of the numbers is greater than 100".

   (c) Prove by contradiction that there are no integers, a, b, c, d such that

$$x^4 + 2x^2 + 2x + 2 = (x^2 + ax + b)(x^2 + cx + d)$$

7. (4 points) Suppose that $h : X \to Y$ is any one-to-one function, and $g : Y \to Z$ is any onto function. Prove or disprove

   (a) $g \circ h$ must be onto.

   (b) $g \circ h$ must be one-to-one.

8. (4 points) Find the probability that a family of 4 children there will be (a) at least 1 boy and (b) at least 1 boy and 1 girl. Assume that the probability of a male or a girl birth is the same. What is the sample space?

9. (8 points) Using the Mathematical Induction Principle solve any two of the following. Identify the parts very clearly. Marks are alloted for the correct structure of the proof.

   (a) Prove the validity of the following Rule of Inference for all integers $n \geq 1$:

$$\begin{array}{ccc} p_1 & \rightarrow & p_2 \\ p_2 & \rightarrow & p_3 \\ \\ \cdots & \cdots & \cdots \\ \\ p_n & \rightarrow & p_{n+1} \\ & \neg p_{n+1} & \\ \hline & \neg p_1 & \end{array}$$

   (b) A sequence $a_1, a_2, \ldots$ is defined recursively by $a_1 = 3$ and $a_n = 7a_{n-1}$ for $n \geq 2$. Show that $a_n = 3.7^{n-1}$ for all $n \geq 1$.

   (c) Show that 23 is the largest integer which cannot be written as a sum of $5's$ and/or $7's$.

10. (5 points)

(a) State the generalized pigeonhole principle.

(b) Consider the points $(x_i, y_i, z_i), x_i, y_i, z_i \in \mathbb{N}, i = 1, 2, \ldots, n$ in 3-dimension. What is the smallest value for n such that for any given n points, there exist at least two pair of points whose midpoints also have integer coordinates. The midpoint of (a, b, c) and (α, β, γ) is $(\frac{a+\alpha}{2}, \frac{b+\beta}{2}, \frac{c+\gamma}{2})$.

11. (8 points) Let R be a relation on the power-set of a finite set A. Fill in the blanks in the following table on the properties of R when $R \in \{\subset, \subseteq, =, \not\subset\}$

Relation on $\mathcal{P}(A)$	$\subset$	$\subseteq$	$=$	$\not\subset$
reflexive				
symmetric				
antisymmetric				
transitive				

12. (8 points) Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. Let R be a relation on A such that $(a, b) \in R$ if $a|b$ (a divides b **or** $a = b$).

(a) Show that (A, R) is a poset.

(b) Draw the Hasse diagram of the poset (A, R). Identify the minimum, maximum, minimal and the maximal elements.

(c) Is (A, R) a total order? Explain.