MACM 101 (Discrete Mathematics I)

Final Exam, Dec. 16, 2011

Answer all the questions (worth 26 points) in Part A.
Answer questions worth 50 points in Part B.

Part A

1. (4 points) Consider the experiment of throwing two 6-sided dice, both colored red, where the faces of a die are labeled from 1 through 6.
   (a) What is the sample space of the experiment?
   (b) Consider the event E where the sum of the faces of the two dice is even. Write down the elements of E.

2. (5 points) Consider the statement "If x is a perfect square and x is even, then x is divisible by 4" .
   (a) Designate propositional variables to stand for the three conditions about x mentioned in the statement.
   (b) State the contrapositive statement of the original statement.

3. (5 points) Find the number of integral solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20$ subject to the conditions
   (a) $x_1 \geq 5$, $x_i \geq 1, i = 2, 3, 4, 5, 6$

4. (5 points) Suppose A and B are two sets containing n and 2 elements respectively. In this question we will consider the functions:
   (a) Describe the functions from $A \to B$ that are not onto.
   (b) How many different one-to-one functions g are there from B to A?

5. (7 points) Prove by induction the following:
   If $a_1 = 1$ and $a_n = a_{n-1} + n$, $n \geq 2$, then $a_n = n(n + 1)/2$, $n \geq 1$.

The proof must include all the details.

Part B

1. (3 points) Find the coefficient of x^{10} in $(3x+2)^{10}$.

2. (3 points) How many ways are there to arrange 4 men, 5 women, 6 boys and 7 girls in a row.

3. (5 points) Prove or disprove each of the following propositions:

   (a) If n is a multiple of 4 and k is a multiple of 3, the nk is a multiple of 12.

   (b) If n is a multiple of 4 and k is a multiple of 3, the $n+k$ is a multiple of 7.

4. (4 points) Determine with justification whether $p \rightarrow (q \vee r)$ and $(p \wedge \neg q) \rightarrow r$ are logically equivalent.

5. (5 points) Consider the propositions $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$.

   (a) Write out the first of these completely in English.

   (b) Write out the second of these completely in English.

   (c) Give an example to show that the two propositions are not logically equivalent.

6. (5 points) How many ways are there to distribute 25 identical balls among 5 players where each player must get at least 1 and no player may get 10 or more.

7. (4 points) Define the infinite sequence of values a_n for $n \geq 1$ as follows:

   - $a_1 = 2$
   - $a_n = a_{\lfloor n/2 \rfloor} * a_{\lceil n/2 \rceil}$ for $n \geq 2$.

   Give the values of a_2, of a_3 and of a_4.

8. (5 points)

   (a) State the well-ordering property for the set of positive integers.

   (b) Determine whether each of the following sets is well-ordered.

      i. the set of integers

ii. the set of integers greater than -100

iii. the set of positive rationals

9. (6 points) Show that $\sum_{k=0}^{n} 2^k C(n,k) = 3^n$ by computing in two different ways the number of ways to sell (possibly all or none) of your n distinct objects to two different collectors. (Hint. each object can go to one of three places; on the other hand yoy can decide to sell k objects in all.)

10. (5 points) Suppose $A = \{1,2,3,4,5,6\}$ and $B = \{a,b,c\}$.

    (a) Give an example of a function from A to B that is neither one-to-one nor onto. Explain why it is so.

    (b) State the domain, codomain, and range of the function in (a).

11. (4 points) Explain how to tell whether a relation is symmetric under each of the following representations: (a) matrix, (b) digraph.

12. (6 points)

    (a) Write down the definition of what it means for a collection C to be a partition of A.

    (b) Explain how a partition C on a set A determines the equivalence relation R on A (give an explicit definition of R).

    (c) The set $C = \{\{1,3,5\},\{2,4\},\{6\}\}$ is a partition of $\{1,2,3,4,5,6\}$. Write down the equivalence relation determined by the partition and find the partition determined by this relation.

13. (6 points) Let a partial order be given by the following matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

    Answer the following.

    (a) Draw the Hasse diagram of the partial order.

    (b) Is it a total order? Explain.

    (c) Determine its minimal, maximal, greatest and the least elements.