MACM 101 : Homework 6 (November 23, 2019)
Homework is due December 1, 2019 (through the server)
Exercises on Relations (Chapter 7).

Practice Problems

1. Problems from the text: $7.1 : 5, 6, 7, 8, 9, 10, 16$
2. Problems from the text: $7.3 : 3, 4, 6(a, b), 16$
3. Problems from the text: $7.4 : 2, 3, 4, 6, 8, 11, 13, 17$

Homework Problems

1. What is the probability that a random relation from set $A = \{a, b, c, d\}$ to set $B = \{1, 2, 3, 4, ..., 8\}$ is a one-to-one function?

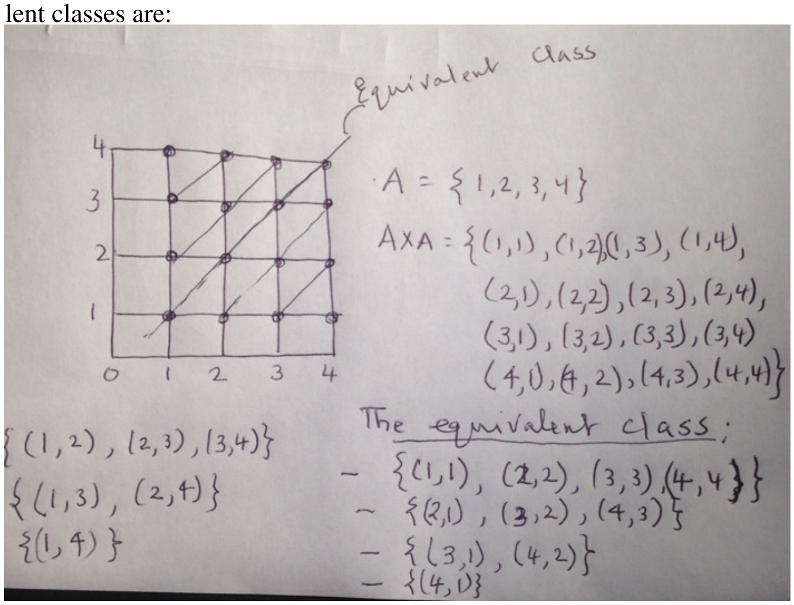
   Ans: We know that any relation from A to B is a subset of the cartesian product $A \times B$. There are 4×8 elements in $A \times B$. Therefore, there are 2^{32} different relations one can describe. Note that the number of elements in the power set of $A \times B$ is also $2^{|A|+|B|}$. The number of one to one function $f : A \to B$ is $8 \times 7 \times 6 \times 5$. Therefore, the probability of a random relation to be a one-to-one function is $\frac{8 \times 7 \times 6 \times 5}{2^{|A|+|B|}}$.

2. Consider the set $A = \{1, 2, 3, 4\}$. On the cartesian product $A \times A$ we define the relation R by

   $$(x_1, y_1) R (x_2, y_2) \leftrightarrow y_1 - x_1 = y_2 - x_2$$

   Show that R is an equivalence relation and illustrate the different equivalence classes in a figure.

   Ans: It is easy to show that R is reflexive, symmetric and transitive. The equivalent classes are:



3. Consider the following set $S = \{(a,b)|a,b \in Z, b \neq 0\}$ where Z denotes the integers. Show that the relation

$$(a,b)R(c,d) \leftrightarrow ad = bc$$

on S is an equivalence relation. Give the equivalence class $[(1,2)]$. What can an equivalence class be associated with?

Ans: Here R is a relation on S, i.e. $R \subseteq S \times S$. An element of S is related to another element of (c,d) of S if their ratios are the same, i.e. $\frac{a}{b} = \frac{c}{d}$. We can easily show that this relation R is an equivalence relation. The elements of the equivalent class that contains $(1,2)$ $\{(a,2a)$ where $a \in Z\}$.

4. Consider a set $A = \{1,2,3,4,5,6\}$. Define an equivalence relation R on A which realizes $\{1,3,5\}$ and $\{2,4,6\}$ as the partition of A.

Ans: Let $A_1 = \{1,3,5\}$ and $A_2 = A - A_1$. The equivalence relation R, which is a subset of $A \times A$, given by $\{(a,b),$ for all $a,b \in A_1, (c,d),$ for all $c,d \in A_2\}$ partitions A into two equivalent classes A_1 and A_2.

5. Two Hasse diagrams are shown below. Unfortunately, the labels at the vertices are missing. The diagram on the right the relation is the divides relation, i.e. $(x,y) \in R$ if and only if $x|y$, and the set is some subset of the positive integers. For the diagram to the left, the relation is the inclusion relation, i.e. $(X,Y) \in R$ if and only if $X \subseteq Y$, and the set is a subset of the power set of a finite set.

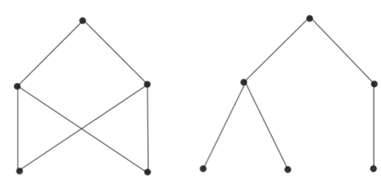
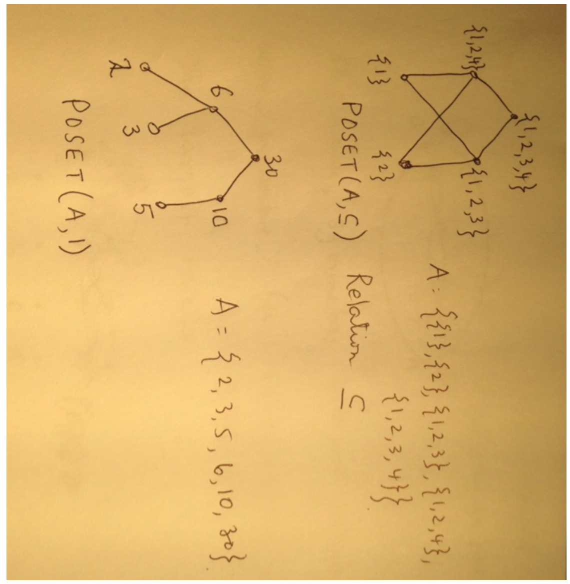


Figure 1: Hasse Diagram

Ans: 

6. Construct maximum size relations on the set $\{a,b,c,d\}$ with each of the following properties, or prove that no such relation exists.

   (a) reflexive and symmetric, but not transitive

   Ans: Consider all possible arcs between the nodes of A. There are 16 such arcs. The relation thus defined is an equivalence relation. If we remove any one of the pair of arcs $(a,d),(d,a);(a,c),(c,a);(b,d),(d,b)$ will make the the relation not transitive.

   (b) irreflexive, symmetric and transitive

   Ans: $(a,b),(b,a) \in R \rightarrow (a,a)$, will violate the property of irreflexivity. Therefore, R is an empty set.

   (c) irreflexive, antisymmetric, not transitive

   Ans: I believe the relation $R = \{(a,b),(b,c),(c,d),(a,c),(a,d),(d,b)\}$ has the required property.

   Ans:

   (d) reflexive, neither symmetric, nor antisymmetric, and transitive

7. Let $A = \{1,2,3\}$. Determine all the equivalence relations R on A. For each of these, list all ordered pairs in the relation.

   Ans: The following are the partitions of A.

   (a) $\{\{1,2,3\}\}$: size 1

   (b) $\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{2,3\},\{1\}\}$: size 2 each

   (c) $\{\{1\},\{2\},\{3\}\}$: size 3

Now apply question 4 strategy to determine an equivalence relation for each partition.

8. Let $g : B \to C$ be any surjective function. Consider the relation R defined as $\forall\ b_1 \in B\ \forall\ b_2 \in B\ [(b_1, b_2) \in R \Leftrightarrow g(b_1) = g(b_2)]$. Identify the equivalence classes.

   Ans: Two elements of B are related if they are mapped to the same image. The relation is equivalence. The equivalence classes are the elements of C, since the function g is an onto function. Every element of C is an imgae of some element of A.