Homework 3
MACM 101-D1
October 4, 2019
Date due: October 11, 2019 in the Class.

1 Practice Problems (Not to be handed in)

1. Problems (pages 116-117) 4, 6, 8, 10, 13, 20.

2. Prove the following statements using either direct or contrapositive proof, which ever is easer.

   (a) If $a, b \in Z$ (set of all integers) and a and b have the same parity, then $3a + 7$ and $7b - 4$ do not.

   (b) If n is odd, then $8|(n^2 - 1)$.

   (c) If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

3. Prove the following statements using proof by contradiction method.

   (a) There exist no integers a and b for which 21a + 30b = 1.

   (b) Every non-zero rational number can be expressed as a product of two irrational numbers.

4. Write the following compound statements in symbols. Use the following letters to represent the statements:

c	:	It is cold.
d	:	It is dry.
r	:	It is rainy.
w	:	It is warm.

   (a) It is neither cold nor dry.

   (b) It is rainy if it is not dry.

   (c) To be warm it is necessary that it be dry.

   (d) It is cold or dry, but not both.

2 Homework Problems (To be handed in)

1. Suppose the variables x, y represent students and courses, respectively. $T(x, y)$ is an open statement "x is taking y". For each of the following symbolic statements state its equivalent English statements.

   (a) $\exists y \forall x T(x, y)$
   There exists a course which is taken by all the students.

   (b) $\neg \exists x \exists y T(x, y)$
   There does not exist a student who is taking at least one course.

   (c) $\forall y \exists x T(x, y)$
   For each of the courses there exists a student taking the course.

   (d) $\neg \forall x \exists y T(x, y)$
   Not all students are taking a course.

   (e) $\forall x \exists y \neg T(x, y)$
   Every student is not taking at least one course.)

2. Give an example to show that

$$(\forall y)(\exists x)\ p(x, y) \leftrightarrow (\exists y)(\forall x)\ p(y, x)$$

   Solution: Suppose the open statement $p(x, y)$ **is** "$x.y = 0$" **where the universe of** x **and** y **are the reals.**

3. Prove or disprove the following statements about integers whose domain is non-zero integers.

   (a) If $a|b$ and $c|d$, then $(a + b)|(c + d)$.
   Not true. Counterexample: $a = 2, b = 4, c = 3, d = 9$.

   (b) If $a|b$ and $b|c$, then $a|c$.
   True. We can write $b = a.t$ **and** $c = b.t'$, t, t' **are integers. Therefore,** $a|c$ **since** $c = a.t.t'$.

   (c) If $a|b$ and $b|c$, then $(a + b)|c$.
   Not true. Counterexample: $a = 4, b = 8, c = 32$.

   (d) If $a|b$ and $b|c$, then $ab|c^2$.
   True. We can write $b = a.t$ **and** $c = b.t'$. **Therefore,** $ab = a^2t$. **Now** $c^2 = b^2.t'^2$, **i.e.** $c^2 = a^2t^2t'^2$, **i.e.** $c^2 = ab.t.t'^2$. **Thus,** $ab|c^2$.

4. Suppose n is an arbitrary integer.

(a) Show that $n(n+1)$ is divisible by 2.
Solution: Proof by cases: If $n = 2t$, then

$$n(n+1) = 2t(2t+1) = 2(t^2+t)$$

is even.
If n=2t+1, then

$$n(n+1) = (2t+1)(2t+2) = 2(2t^2+3t+1)$$

is also even.

Therefore, whether n is even or odd, the product $n(n+1)$ is always even.

(b) Show that $n(n+1)(n+2)$ is divisible by 3!.
Solution: We can prove this in more than one way.

Proof by cases: Any integer can be expressed as $6t+u$ where is one of $0, 1, 2, 3, 4, 5$. Now

$$n(n+1)(n+2) = (6t+u)(6t+u+1)(6t+u+2)$$

which is divisible by 6 if $u(u+1)(u+2)$ is divisible by 6 (check). We prove this claim using an exhaustive proof. We show that for each value of u in $\{0, 1, 2, 3, 4, 5\}$, 6 divides $u(u+1)(u+2)$.

u	$u(u+1)(u+2)$
0	$0.1.2 = 0 = 6.0$
1	$1.2.3 = 6 = 6.1$
2	$2.3.4 = 24 = 6.4$
3	$3.4.5 = 60 = 6.10$
4	$4.5.6 = 120 = 6.20$
5	$5.6.7 = 210 = 6.35$

This completes the proof. This is a correct proof, but is not elegant.

Another proof by cases approach We have already seen that $n(n+1)$ is divisible by 2. We can also show that $n(n+1)(n+2)$ is divisible by 3. This can be done by showing that any integer of the type $n = 3t+u, u = 0, 1, 2$ is divisible by 3 (use arguments similar to the one described above). Since 2 and 3 do not have a common factor, therefore $n(n+1)(n+2)$ is divisible by $2 \cdot 3$.

Combinatorial approach for $n \geq 3$ We know that the the number of ways to select 3-subsets of a set of $n+2$ elements is $C(n+2,3) = \frac{(n+2)(n+1)n}{3!}$. Since $C(n+2,3)$ is an integer, 3! divides $n(n+1)(n+2)$

when $n \geq 3$. We can exhaustively show that the statement is also true for $n = 0, 1, 2$. For $n < 0$, the arguments are very similar to that for $n > 0$ (ignoring the sign). This completes the for arbitrary n.
Also note that the above arguments can be applied to show that $n(n+1)(n+2)\dots(n+k-1)$ id divisible by $k!$.

5. (a) Prove that $\sqrt{7}$ is an irrational number.

   Solution: We can prove this by contradiction. Suppose $\neg p$ is true, i.e. $\sqrt{7}$ is rational. Therefore, we can use the fact that $\sqrt{7}$ can be expressed as $\sqrt{7} = \frac{a}{b}$ where integers a and b have no common factors. We can write $a^2 = 7b^2$. This implies that a^2 is divisible by 7. **Since 7 is a prime number, 7 divides a^2 implies 7 divides a.** Thus $a = 7 \cdot t$ for some integer t. Now $a^2 = 7b^2$ can be written as $49t^2 = 7b^2$. This means that 7 divides b^2 as well. Since 7 is a prime number, 7 divides b. We now arrive at a contradiction: We started with the fact that a and b have no common factor. We then showed that 7 is a common factor of a and b. This leads to the conclusion that $\neg p$ is false. This implies that $\sqrt{7}$ is an irrational number.

   (b) Show where your arguments in (a) get violated if you want to show in a similar manner that $\sqrt{9}$ is an irrational number.

   Solution: The arguments used above cannot be applied for the case of $\sqrt{9}$ since the highlighted statement above is not true for 9, since 9 is not a prime number. (9 divides 6^2 doesn't mean that 9 divides 6.)

   (c) Find a counterexample to the statement that every positive integers can be written as the sum of the squares of three integers. What is the smallest integer for which it is a counterexample.

   Solution: We see that

   - $1 = 1^2 + 0^2 + 0^2$
   - $2 = 1^2 + 1^2 + 0^2$
   - $3 = 1^2 + 1^2 + 1^2$
   - $4 = 2^2 + 0^2 + 0^2$
   - $5 = 2^2 + 1^2 + 0^2$
   - $6 = 2^2 + 1^2 + 1^2$

   We are unable to express 7 as the sum of the squares of three integers. Therefore, $n = 7$ is the smallest integer for which it is a counterexample.