Homework 1
MACM 101-D1
September 23, 2019
Date due: October 2-3, 2019 in the Tutorial.

1 Practice Problems (Not to be handed in)

1. Exercise Problems of sections 1.4
2. Problems (Exercises 1.4) $2, 8, 16, 18$
3. Problems (Exercises 1.6) $2, 3, 4, 8, 10, 14, 20, 22$
4. Problems (Exercises 2.1) $4, 5, 8, 12$
5. Find the number of integral solutions to the equation $x_1+x_2+x_3+x_4+x_5+x_6 = 21$ subject to the conditions
   (a) $x_i \geq 0, i = 1, 2, 3, 4, 5, 6.$ **C(21+6-1,6-1)**
   (b) $x_3 \geq 5, x_i \geq 1, i = 1, 2, 4, 5, 6.$ **C(11+6-1)**
   (c) $0 \leq x_1 \leq 3, x_i \geq 1, i = 2, 3, 4, 5, 6.$ **Ans = number of integral solutions to** $x_1+x_2+x_3+x_4+x_5+x_6 = 21, x_1 \geq 0, x_i \geq 1, i = 2, 3, 4, 5, 6$ **minus the number of integral solutions to** $x_1+x_2+x_3+x_4+x_5+x_6 = 21, x_1 \geq 4, x_i \geq 1, i = 2, 3, 4, 5, 6 = C(16+6-1, 6-1)-C(12+6-1, 6-1)$.

2 Homework Problems (To be handed in)

1. In how many ways can we place r red balls and w white balls in n boxes so that each box contains one ball of each color?
   Solution: Here one has to assume that the values of w, r and n could be arbitrary. If $(n < w) \vee (n < r)$ is true, there is no solution to the problem. Therefore, the answer is zero. If $\neg((n < w) \vee (n < r))$ is true, i.e. $n \geq w$ and $n \geq r$, the number of ways to distribute w balls to n boxes is the number of integral solutions to:

$$x_1 + x_2 + \ldots + x_n = w, x_i \geq 1, i = 1, 2, \ldots, n.$$

   This number is the same as the number of integral solutions to:

$$y_1 + y_2 + \ldots + y_n = w - n, y_i \geq 0, i = 1, 2, \ldots, n.$$

which is $\binom{w-n+n-1}{n-1} = \binom{w-1}{n-1}$. Similarly, we can distribute r red balls in $\binom{r-1}{n-1}$ ways. The answer to the question is, therefore, $\binom{w-1}{n-1} \times \binom{r-1}{n-1}$.

2. (a) How many sequences (lists) of m 0s and n 1s are there?

   Solution: This problem can be viewed as permutation with repetitions. We have $(m+n)$ objects. There are m objects of type 1 (0s), and n objects of type 2 (1s). The number of ways to arrange the objects is $\frac{(m+n)!}{m!.n!}$.

   Another approach: In any sequence, there are n ones in an arrangement. Yherefore, there are $(n+1)$ places where m zeroes can go. One of the places is to the left of the first 1. The other places are: to the right of the first 1 and to the left of the second 1; to the right of the second 1 and to the left of the third 1; and so on. Let x_i denote the number of zeros to the right of the $i-1^{th}$ 1 and to the left of the i^{th} ones, $i = 2, 3, \ldots, n$. x_1 and x_{n+1} represent the number of zeros to the left of x_1 and to the right of x_{n+1}, respectively. The number of sequences of m 0s and n 1s is the same as the number of integral solutions to the following:

   $$x_1 + x_2 + \ldots + x_{n+1} = m \quad where \ \ x_i \geq 0, i = 1, 2, \ldots, n+1$$

   The answer to the above is $C(m+(n+1)-1, (n+1)-1) = C(m+n, n)$.

   (b) How many sequences are there in which each 1 is separated by at least two 0s? (Assume that for this part $m \geq 2(n-1)$.)

   Solution: From the second formulation in the solution of question 2(a), the answer is the number of integral solution to:

   $$x_1 + x_2 + \ldots + x_{n+1} = m \quad where \ x_1 \geq 0, x_2 \geq 2, x_3 \geq 2, \ldots x_n \geq 2, x_{n+1} \geq 0$$

   You can show that the answer is the same as the number of integral solutions to

   $$x_1 + x_2 + \ldots + x_{n+1} = m - 2(n-1) \quad where \ \ x_i \geq 0, i = 1, 2, \ldots, n+1$$

   The above problem makes sense only when $m - 2(n-1) \geq 0$, i.e. $m \geq 2(n-1)$.

3. We are given a red box, a blue box and a green box. We are also given 10 red balls, 10 blue balls, and 10 green balls. Balls of the same colour are indistinguishable. Consider the following constraints:

1 No box contains a ball that has the same colour as the box.

2 No box is empty.

Determine the number of ways in which we can put 30 balls into boxes so that:

(a) No constraint has to be satisfied. Every combination is allowed.
Solution: We cannot distribute 30 balls in 3 boxes in one shot. This implies that the balls are identical, which is not the case. We can distribute 10 red balls into three boxes in $C(10+3-1, 3-1)$ ways. Similarly, we can distribute 10 blue balls and 10 green balls in $C(12, 2)$ ways each. Therefore, the total number of distributing 10 red balls, 10 blue balls and 10 green balls in $C(12, 12)^3$ ways.

(b) Constraint 1 is satisfied.
Solution: We solve it in the same way as in problem (a). When distributing 10 red balls, we have only two boxes to place the red balls. We cannot place red balls in red box. The solution is, therefore, $C(10+2-1, 2-1)^3$.

(c) Constraint 2 is satisfied.
Solution: Let A be the number of ways to distribute 30 balls into boxes with no constraints (Problem (a)). From the solution of problem (a) we know that $A = C(12, 2)^3$. We identify the combinations which are not valid for problem (c). We can represent a combination of problem (a) by a 3-tuple (n_1, n_2, n_3) where n_1 balls are in red box, n_2 balls in blue box and n_3 balls in green box. Remember that any coloured balls can be present in any box. All the invalid combinations will be of the following types:

Type 1: $(0, n_2, n_3), (n_1, 0, n_3), (n_1, n_2, 0)$

Type 2: $(0, 0, n_3), (0, n_2, 0), (n_1, 0, 0)$

There are exactly 3 combinations of Type 2 where $n_1 = n_2 = n_3 = 30$. In this case only one box is available for all the balls. Note that the set of Type 1 combinations also contains combinations of Type 2. Therefore, the answer to Problem (c) is (A - # of Type 1 solutions + # of Type 2) solutions. The number of solutions of the type $(0, n_2, n_3)$ is $C(10+2-1, 2-1)^3$ (why?). Therefore, the solution to Problem (c) is $C(12, 2)^3 - 3.C(11, 1)^3 + 3$.

(d) Constraints 1 and 2 are satisfied.

Solution: The approach to the solution is similar to the answer given for problem (c). We start with the combinations valid for problem (b).

Each combination guarantees that a colored ball is not placed in the same coloured box. Therefore, from $C(11, 10)^3$ we first subtract the number of combinations of type 1. For the type $(0, n_2, n_3)$, the red balls can be distributed in $(C(11, 1)$ way. However for blue balls, red box and blue box are off limit. Therefore, in $C(10 + 1 - 1, 1 - 1)$ ways we can distribute 10 blue balls in green box (only left). This is also the same for the distribution of green balls. Therefore, the number of type 1 solutions is $3.C(11, 1).C(10, 0).C(10, 0)$. Now consider the case when the valid combinations of problem (b) are of the type $((0, 0, n_3)$. Consider the case when green balls are being distributed. They cannot be placed in red box and blue box (we are looking for the the (0,0,*) type). Moreover, they cannot be placed in green box(condition 1). There is no place for the green balls to go. Thus, type 2 combinations cannot exist. The answer to problem d is $C(11, 1)^3 - 3.C(11, 1).C(10, 0)^2$.

4. Construct a truth table for each statement.

   (a) $p \to \neg q$

   (b) $[p \wedge (p \to q)] \to q$

   (c) $[p \to (q \wedge \neg q)] \leftrightarrow \neg p$

   Ans: Easy