Homework 1
MACM 101-D1
September 15, 2018
Date due: September 24, 2018 in the Tutorial.

1 Practice Problems (Not to be handed in)

1. Problems (pages 12-18) 5, 23, 24, 31, 34, 36
2. Problems (pages 24-26) 4, 8, 12, 17, 20, 22, 30

2 Homework Problems (To be handed in)

1. How many bit strings contain exactly five 0 and 14 1, if every 0 must be immediately followed by 2 successive 1.

   Ans:: Every 0 appears as the first members of a subsequence 011. Our words can be viewed as words in the alphabet {011,1}. Any permutation of repeated copies of these primitive words can be analyzed to determine uniquely the numbers of the two subwords used. In the present case we are permuting 5 copies of 011 and 4 copies of 1. The number of permutations is $\frac{(5+4)!}{5!.4!}$

2. Determine how many strings of n lowercase letters from the English alphabet contain

   (a) the letter a.

   (b) the letters a and b.

   (c) the letters a and b in consecutive positions with a preceding b, with all letters of the string distinct.

   (d) the letters a and b, where a is somewhere to the left of b in the string, with all letters distinct.

   Ans: (a) The total number of strings, without restriction, is 26^n**. We subtract from this number the number of strings constructed from the alphabet** $\{b, c, ..., z\}$**, i.e.** 25^n**, leaving a balance of** $26^n - 25^n$**. (b) We first count the strings which do not contain both a and b. Those containing no a number** 25^n**; the same is the number of strings containing no b. We can add these two numbers, but the sum will count the strings containing neither a nor b hose number**

is 24^n wice; hence the number of strings which contain both a and b is $26^n - 2^n + 24^n$.

(c) Consider the 2-letter string ab as one object, to be permuted with $n-2$ other objects, all distinct. The $n-2$ other objects may be chosen from the set of 24 letters {c, d, ..., z}; the number of choices is $\binom{24}{n-2}$. We then permute n objects: these $n-2$ letters and the 2-letter object ab. The number of such permutations is $(n-1)!$. By the product rule, the number of strings is $(n-1)!\binom{24}{n-2}$.

(d) The total number of n-letter words containing a and b, in which all letters are distinct, is $n!.\binom{24}{n-2}$: we apply the product rule after selecting the $n-2$ letters distinct from a and b. Since there is no essential difference between the objects a and b, half of these strings have a to the left of b.

3. Five rooms of a house are to be painted in such a way that rooms with an interconnecting door have different colors. If there are n colors available, how many different color schemes are possible when the rooms in the house are arranged in the following way?

   (a) Connected rooms form a linear order with one door interconnecting two adjacent rooms. **Ans: The first room can be colored in n different ways, the second room can be colred in n-1 dfferent ways, the third room and the fourth room can be colored in $n-1$ different ways and the last room can also be colored in n-1 different ways. Total number of ways is $n(n-1)^4$.**

   (b) Connected rooms form a linear order with one door interconnecting two adjacent rooms. The first and last rooms must be colored differently. **Ans: If we are not careful, we probably would answer n(n-1)(n-1)(n-1)(n-2). The number of colors available for the last room is not always n-2. If room 4 is painted with the same color as that of room 1, room 5 can be colored in n-1 ways. Now the question is how many ways can you color the first four rooms such that first room and the last room are colored the same. If the first and the fourth room are colored the same, observe that the second and the third rooms cannot be colored with the color of the first room. Therefore, the number of ways to color the first four rooms such that the first and the last rooms have the same color is $A = n.(n-1)(n-2)$. Therefore, the number of ways to color the first four rooms such that the first and the last rooms have different colors $B = n(n-1)(n-1)(n-1) - n(n-1)(n-2)$. Thus, the answer to**

question (b) is $C = A.(n-1) + B.(n-2)$.

(c) Connected rooms form a circular order with one door interconnecting two adjacent rooms.

Ans: Each valid circular arrangement will appear five times in the arrangements valid for question (b). Therefore, the total number is $\frac{C}{5}$.

4. (a) How many terms are there in the expansion of $(1+x)^{25}$?

   (b) Determine the coefficients of x^3 and x^{10}.

   (c) Determine the largest coefficient.

   Ans: Using the binomial expansion, we get

$$(1+x)^{25} = \sum_{i=0}^{25} \binom{25}{i} x^i.$$

Therefore,there are 26 terms in the binomial expansion. The coefficients of x^3 and x^{10} are $\binom{25}{3}$ and $\binom{25}{10}$ respectively. The maximum coefficient is $\binom{25}{12}$.

5. Construct a truth table for each statement.

   (a) $p \to \neg q$

   (b) $[p \wedge (p \to q)] \to q$

   (c) $[p \to (q \wedge \neg q)] \leftrightarrow \neg p$

   Ans: This question is easy to answer.